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Controller synthesis to achieve robust stability against bicoprime factor uncertainty: an LMI approach

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Abstract: In this paper, a Linear Matrix Inequality approach is presented for synthesizing controllers that robustly stabilize a plant against Bicoprime Factor uncertainty. Following the development of the general case, non-normalized results, the usefulness of normalized Bicoprime Factorizations is studied in this context and shown to be beneficial in deducing the existence of a robustly stabilizing controller for given robust stability margin. Finally, a numerical example is provided to demonstrate the practical applicability of the developed methodology.

Keywords: Robust control, H-infinity control, Coprime factorization, Uncertainty, Uncertain linear systems, Iterative methods

1. INTRODUCTION

Bicoprime factorizations (BCFs) of the plant are rooted in the Polynomial Matrix Descriptions (PMDs) studied by Rosenbrock (1970) in the mid 20^{th} century. As a stable factorization of the plant, they were first introduced by Vidyasagar (2011), but were minimally explored. Though they received some attention in the late 80's (Desoer and Gündeş, 1988; Gündeş and Desoer, 1990), their study was quickly abandoned in favour of coprime factorizations of which they are a generalization.

Recent work (Tsiakkas and Lanzon, 2017, 2015) has shown that it is possible to capture and generalize earlier coprime factor results into a more complete BCF theory. Several BCF results were presented therein mostly pertaining to internal stability, state space parametrization of BCFs and robustness analysis. Robust control synthesis against BCf uncertainty was presented by Tsiakkas and Lanzon (2019) where it was shown that the use of BCFs can incur computational benefits in the synthesis of robustly stabilizing controllers. Results were also extended to the time varying case by Yu (2019). The notion of normalization was adapted to the BCF case by Tsiakkas and Lanzon (2018) closely resembling (though not coinciding with) normalized coprime factorizations of the plant as defined by Vidyasagar (1988).

Just like the classical case of left and right coprime factorizations (LCFs and RCFs respectively), every plant in $\mathscr{R}$ admits a BCF over $\mathscr{RH}_\infty$. The ordered quad $\{N, M, L, K\}$ is said to be bicoprime (BC) in $\mathscr{RH}_\infty$ if $\{L, M\}$ is left coprime (LC) in $\mathscr{RH}_\infty$, $\{N, M\}$ is right coprime (RC) in $\mathscr{RH}_\infty$ and $K \in \mathscr{RH}_\infty$. Furthermore, it is a BCF of $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if $P = NM^{-1}L + K$.

The uncertainty structure induced by BCFs is appealing as it captures a wide range of modelling errors (Lanzon and Papageorgiou, 2009). It was shown in Tsiakkas and Lanzon (2017) that the left and right coprime factor uncertainty structures are in fact special structured cases of their BCF counterpart. Furthermore, the BCF uncertainty closely resembles the four-block structure commonly studied by the robust control community and known to be appropriate for many practical situations. It is demonstrated herein by way of example that in certain cases BCF uncertainty is superior to the simpler coprime factor uncertainty.

Coprime factor theory has played an important role robust control synthesis. Specifically, imposing the normalization property on the coprime factors of the plant can lead to two main benefits. First is the fact that the optimal robust stability margin can be directly computed without the need for iteratively solving Algebraic Riccati Equations (AREs) while the second is that the induced robust stability margin matches that of the standard four block problem (McFarlane and Glover, 1990).

The $\mathscr{H}_\infty$ robust control synthesis methodology developed by Doyle et al. (1989) requires the solution of two sign-indefinite AREs while under various technical assumptions. These assumptions were relaxed by Gahinet and Apkarian (1994) by reformulating the $\mathscr{H}_\infty$ optimal control problem in a Linear Matrix Inequality (LMI) framework.

This paper exploits the above-mentioned LMI formulation to develop stabilizing controllers that are robust with re-

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spect to the BCF uncertainty structure. These results are then specialized to the case of normalized BCFs (NBCFs) as defined by Tsiakkas and Lanzon (2018); it is shown that using a NBCF of the plant leads to simplifications in the synthesis a robustly stabilizing controller as is the case for coprime factors. A numerical example is provided at the end of the paper comparing the robust performance of the proposed BCF based controller with its classical, coprime factor based, counter part.

2. PRELIMINARIES

2.1 Linear Algebra

Let $A \in \mathbb{R}^{m\times n}$. Then $\operatorname{Ker} A$ denotes the kernel (or null space) of A while $\operatorname{Im}(A)$ denotes its image (or range). The largest singular value of A is given by $\overline{\sigma}(A)$. The pseudo-inverse of A is given by $A^\dagger$. If A is square (i.e. $A \in \mathbb{R}^{n\times n}$), $\Lambda(A)$ represents its spectrum while $\underline{\lambda}(A)$ and $\rho(A)$ denote its smallest eigenvalue and spectral radius respectively. Furthermore, A is said to be positive definite (resp. semi-definite), denoted $A > 0$ (resp. $A \geq 0$) if it is Hermitian and all its eigenvalues are strictly positive (resp. non-negative).

The following shorthand notation is introduced for compactness. Let $A \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$. Then $\mathcal{L}(A, X)$ will be used to denote

$$\mathcal{L}(A, X) = AX + XA^*.$$

2.2 Systems Theory

The set of all real, rational, proper transfer matrices is denoted $\mathscr{R}$. The subset of $\mathscr{R}$ containing all stable transfer matrices is given by $\mathscr{RH}_\infty$.

Let $H \in \mathscr{R}$ and $\Delta \in \mathscr{R}$, then the lower and upper linear fractional transformations (LFTs) of H with respect to Δ are given by $\mathcal{F}_l(H, \Delta)$ and $\mathcal{F}_u(H, \Delta)$ respectively. See (Zhou et al., 1996, Chapter 10) for further details.

Let $P \in \mathscr{R}$, then $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is shorthand for $P = C(sI - A)^{-1}B + D$.

Left and right coprime factorizations (L/RCFs) are invaluable tools in control theory with uses ranging from distance measures (Lanzon and Papageorgiou, 2009) to control synthesis (Vidyasagar and Kimura, 1986; Georgiou and Smith, 1990; McFarlane and Glover, 1992; Vinnicombe, 1993). A pair $\{L, M\}$ is said to be LC over $\mathscr{RH}_\infty$ if $L, M \in \mathscr{RH}_\infty$ and there exist $X, Y \in \mathscr{RH}_\infty$ such that $MX + LY = I$. Furthermore, the pair is a LCF of a plant $P \in \mathscr{R}$ if it is LC in $\mathscr{RH}_\infty$, M is square with $\det M(\infty) \neq 0$ and $P = M^{-1}L$. Similarly, the pair $\{N, M\}$ is RC over $\mathscr{RH}_\infty$ if $N, M \in \mathscr{RH}_\infty$ and there exist $X, Y \in \mathscr{RH}_\infty$ such that $XM + YN = I$. A RCF is defined dually to a LCF with the pair being a RCF of a plant $P \in \mathscr{R}$ if it is RC in $\mathscr{RH}_\infty$, M is square with $\det M(\infty) \neq 0$ and $P = NM^{-1}$.

2.3 BCF Fundamentals

BCFs first appeared in literature in (Vidyasagar, 2011) where their existence was acknowledged with no significant results given. In the original definition, BCFs of a plant were presented as a quad of objects in $\mathscr{RH}_\infty$; this definition follows.

Definition 1. (Vidyasagar (2011)). The ordered quad $\{N, M, L, K\}$ is bicoprime (BC) in $\mathscr{RH}_\infty$ if $\{L, M\}$ is LC in $\mathscr{RH}_\infty$, $\{N, M\}$ is RC in $\mathscr{RH}_\infty$ and $K \in \mathscr{RH}_\infty$. Furthermore, the quad is a BCF of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if it is BC in $\mathscr{RH}_\infty$, M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}L + K$.

Formulae for computing L/RCFs of a system using state space data were first given by Nett et al. (1984). The following theorem presents a similar result for constructing a BCF of a plant.

Theorem 1. (Tsiakkas and Lanzon (2017) Theorem 9). Let $P \in \mathscr{R}^{p\times q}$ have a stabilizable and detectable state space realization $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Furthermore, suppose that $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ are such that $A + QR$ is Hurwitz. Finally, let $D_N \in \mathbb{R}^{p\times r}$ and $D_L \in \mathbb{R}^{r\times q}$ be arbitrarily chosen matrices and define

$$\left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] = \left[\begin{array}{c|c:c} A + QR & Q & B + QD_L \\ \hline R & I & D_L \\ \hdashline C + D_N R & D_N & D + D_N D_L \end{array}\right]. \quad (1)$$

Then $\{N, M, L, K\}$ is a BCF of P.

The BCF parametrization presented in Theorem 1 with the restrictions $D_N = 0$ and $D_L = 0$ will henceforth be referred to as a QR-BCF parameterization, as it is purely parameterized by the matrices Q and R.

The BCF uncertainty structure was first proposed by Tsiakkas and Lanzon (2015). Following coprime factor convention, BCF uncertainty is defined by additive perturbations on the BC factors. With the resulting perturbed plant given by

$$P_\Delta = (N+\Delta_N)(M+\Delta_M)^{-1}(L+\Delta_L)+(K+\Delta_K) \quad (2)$$

where $\{N, M, L, K\}$ is a BCF of P and Δ_N, Δ_M, Δ_L, $\Delta_K \in \mathscr{RH}_\infty$. Similarly to L/RCF uncertainty where a perturbed plant is admissible only if coprimeness of the factors is preserved (Glover and McFarlane, 1989, Remark 4.4), the condition that bicoprimeness is preserved under the perturbations is imposed herein. That is $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\}$ is BC in $\mathscr{RH}_\infty$.

A block diagram representation of the BCF uncertainty structure given by (2) can be found in (Tsiakkas and Lanzon, 2017) where it can be observed that this generalizes many of the uncertainty structures studied in the past, for example by Lanzon and Papageorgiou (2009).

A generalized plant and uncertainty matrix for this uncertainty structure can be defined as

$$\Pi = \left[\begin{array}{cc:c} M^{-1} & 0 & M^{-1}L \\ 0 & 0 & I \\ \hdashline NM^{-1} & I & P \end{array}\right] \text{ and} \quad (3)$$

$$\Delta = \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix}. \quad (4)$$

It is straightforward to confirm that using the above Π and Δ yields $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$.

Before developing a robust control synthesis result, a state space realization of the BCF generalized plant is needed.

Let $P \in \mathscr{R}$ have a stabilizable and detectable state space realization $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Combining the QR-BCF parametrisation in (1) with the generalized plant Π in (3), the BCF generalized plant can be expressed in state space form as

$$\Pi = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc|c} A & Q & 0 & B \\ \hline -R & I & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline C & 0 & I & D \end{array}\right]. \tag{5}$$

2.4 LMI Synthesis

We end the preliminaries section with a robust control synthesis result. The following theorem was developed by Gahinet and Apkarian (1994) and provides a LMI conditions for the existence of a robustly stabilizing controller for a given robust stability margin. When used in combination with a LMI solver, this could be used as a stepping stone to synthesize an optimal (in terms of $\mathscr{H}_\infty$ norm) controller.

Theorem 2. (Gahinet and Apkarian (1994) Theorem 4.3). Let $\Pi \in \mathscr{R}^{(p_1+p_2)\times(q_1+q_2)}$ have a minimal state space realization

$$\Pi = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Define

$$\hat{B}_2 = B_2 D_{12}^\dagger,\ \hat{A} = A - \hat{B}_2 C_1,\ \hat{B}_1 = B_1 - \hat{B}_2 D_{11},$$
$$\hat{C}_1 = (I - D_{12}D_{12}^\dagger)C_1,\ \hat{D}_{11} = (I - D_{12}D_{12}^\dagger)D_{11} \tag{6}$$

and

$$\tilde{C}_2 = D_{21}^\dagger C_2,\ \tilde{A} = A - B_1\tilde{C}_2,\ \tilde{C}_1 = C_1 - D_{11}\tilde{C}_2,$$
$$\tilde{B}_1 = B_1(I - D_{21}^\dagger D_{21}),\ \tilde{D}_{11} = D_{11}(I - D_{21}^\dagger D_{21}). \tag{7}$$

Furthermore, let W_{12} and W_{21} be such that $\operatorname{Im} W_{12} = \operatorname{Ker}(I - D_{12}^\dagger D_{12})B_2^*$ and $\operatorname{Im} W_{21} = \operatorname{Ker}(I - D_{21}D_{21}^\dagger)C_2$.

Then there exists a stabilizing controller $C_\infty \in \mathscr{R}^{q_2\times p_2}$ such that $\mathcal{F}_l(\Pi, C_\infty) < \gamma$ if and only if

(*i*) $\gamma > \max(\bar{\sigma}(\hat{D}_{11}), \bar{\sigma}(\tilde{D}_{11}))$,

(*ii*) $\exists S > 0,\ T > 0 :\ \underline{\lambda}(ST) \geq 1$,

$$W_{12}^* \left\{ \begin{array}{l} \mathcal{L}(\hat{A}, S) - \gamma \hat{B}_2 \hat{B}_2^* \\ + \begin{bmatrix} \hat{C}_1 S \\ \hat{B}_1^* \end{bmatrix}^* \begin{bmatrix} \gamma I & -\hat{D}_{11} \\ -\hat{D}_{11}^* & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \hat{C}_1 S \\ \hat{B}_1^* \end{bmatrix} \end{array} \right\} W_{12} < 0, \tag{8}$$

$$W_{21}^* \left\{ \begin{array}{l} \mathcal{L}(\tilde{A}^*, T) - \gamma \tilde{C}_2^* \tilde{C}_2 \\ + \begin{bmatrix} \tilde{B}_1^* T \\ \tilde{C}_1 \end{bmatrix}^* \begin{bmatrix} \gamma I & -\tilde{D}_{11} \\ -\tilde{D}_{11}^* & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{B}_1^* T \\ \tilde{C}_1 \end{bmatrix} \end{array} \right\} W_{21} < 0. \tag{9}$$

The above theorem only gives a test for the existence of a (possibly suboptimal) controller achieving a specific robust stability margin but no method of constructing it. However, this controller can be constructed from the matrices S and T satisfying (8) and (9) by following the procedure outlined in (Gahinet and Apkarian, 1994, Section 7).

3. BCF LMI SYNTHESIS

A robust control synthesis theorem based on the two-ARE solution of Doyle et al. (1989) was given by Tsiakkas and Lanzon (2019) for the BCF generalized plant given by (5). In this section we adapt the results of Theorem 2 to produce an LMI based alternative. First, in Subsection 3.1, the general case is considered (i.e. no restrictions are imposed on the QR-BCF used) while in Subsection 3.2 we study the case where the normalization property is imposed on the factorization.

3.1 General Case

We begin by directly applying Theorem 2 to the BCF generalized plant given by (5).

Theorem 3. Let $P \in \mathscr{R}^{p\times q}$ have the stabilizable and detectable state space realization $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $A \in \mathbb{R}^{n\times n}$. Furthermore suppose that $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ are such that $A + QR$ is Hurwitz. Then there exists a stabilizing controller $C_\infty \in \mathscr{R}^{q\times p}$ such that $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if $\gamma > 1$ and there exists a pair of matrices $S > 0$ and $T > 0$ such that

a) $$\mathcal{L}(A + \epsilon QR, S) + \gamma\epsilon SR^*RS + \gamma\,(\epsilon QQ^* - BB^*) < 0, \tag{10}$$

b) $$\mathcal{L}((A + \epsilon QR)^*, T) + \gamma\epsilon TQQ^*T + \gamma\,(\epsilon R^*R - C^*C) < 0, \tag{11}$$

c) $$\underline{\lambda}(ST) \geq 1 \tag{12}$$

where $\epsilon = (\gamma^2 - 1)^{-1}$.

Proof. By direct substitution, the definitions of (6) and (7) take the form

$$\hat{B}_2 = [0\ B],\ \hat{A} = A,\ \hat{B}_1 = [Q\ 0],$$
$$\hat{C}_1 = \begin{bmatrix} -R \\ 0 \end{bmatrix},\ \hat{D}_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{C}_2 = \begin{bmatrix} 0 \\ C \end{bmatrix},\ \tilde{A} = A,$$
$$\tilde{C}_1 = \begin{bmatrix} -R \\ 0 \end{bmatrix},\ \tilde{B}_1 = [Q\ 0],\ \tilde{D}_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

We can now make the following simplifying observations

$$\hat{A} = \tilde{A} = A,\ \hat{B}_1 = \tilde{B}_1 = B_1 = [Q\ 0],$$
$$\hat{C}_1 = \tilde{C}_1 = C_1 = \begin{bmatrix} -R \\ 0 \end{bmatrix},\ \hat{D}_{11} = \tilde{D}_{11} = D_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Now let W_{12} and W_{21} denote bases for the null spaces of $(I - D_{12}^\dagger D_{12})B_2^*$ and $(I - D_{21}D_{21}^\dagger)C_2$ respectively. It can easily be confirmed that $(I - D_{12}^\dagger D_{12})B_2^* = 0$ and $(I - D_{21}D_{21}^\dagger)C_2 = 0$ which implies that $W_{12} = I$ and $W_{21} = I$.

From condition (*i*) of Theorem 2 we have that a stabilizing controller exists only if $\gamma > \max(\bar{\sigma}(\hat{D}_{11}), \bar{\sigma}(\tilde{D}_{11}))$. In this case, $\bar{\sigma}(\hat{D}_{11}) = \bar{\sigma}(\tilde{D}_{11}) = 1$ and therefore γ must satisfy $\gamma > 1$. Following through with the conditions of the LMI synthesis theorem, there exists a stabilizing controller with $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if $\gamma > 1$ and there exists a pair of positive definite matrices $S > 0$ and $T > 0$

satisfying (8) and (9). Using some simple linear algebra manipulations, we obtain

$$\begin{bmatrix} \gamma I & -\tilde{D}_{11}^* \\ -\tilde{D}_{11} & \gamma I \end{bmatrix}^{-1} = \begin{bmatrix} \gamma\epsilon I & 0 & -\epsilon I & 0 \\ 0 & \frac{1}{\gamma} I & 0 & 0 \\ -\epsilon I & 0 & \gamma\epsilon I & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I \end{bmatrix}.$$

Then substituting for each variable, (8) is transformed to

$$\mathcal{L}(A,S) - \gamma BB^* + \gamma\epsilon SR^*RS + \epsilon QRS + \epsilon S(QR)^* + \gamma\epsilon QQ^* < 0,$$

which can be further simplified to

$$\mathcal{L}(A+\epsilon QR, S) + \gamma\epsilon SR^*RS + \gamma(\epsilon QQ^* - BB^*) < 0.$$

In a similar manner, we can show that (9) reduces to

$$\mathcal{L}(A+\epsilon QR, S) + \gamma\epsilon TQQ^*T + \gamma\left(\epsilon R^*R - C^*C\right) < 0$$

which concludes the proof. □

Remark 4. Using standard techniques such as the Schur complement method of (Laub, 2005, Theorem 10.33) it can be shown that conditions (10) and (11) of the above theorem can be equivalently expressed as

$$(a_1) \begin{bmatrix} \mathcal{L}(A,S) - \gamma BB^* & -SR^* & Q \\ -RS & -\gamma I & I \\ Q^* & I & -\gamma I \end{bmatrix} < 0, \tag{13}$$

$$(b_1) \begin{bmatrix} \mathcal{L}(A^*,T) - \gamma C^*C & TQ & -R^* \\ Q^*T & -\gamma I & I \\ -R & I & -\gamma I \end{bmatrix} < 0. \tag{14}$$

This can be proven by noting that (8) holds if and only if

$$\begin{bmatrix} \mathcal{L}(\hat{A},S) - \gamma \hat{B}_2\hat{B}_2^* & S\hat{C}_1^* & \hat{B}_1 \\ \hat{C}_1 S & -\gamma I & \hat{D}_{11} \\ \hat{B}_1^* & \hat{D}_{11}^* & -\gamma I \end{bmatrix} < 0$$

which gives (13) upon substituting for the various parameters. The second LMI given by (14) can be obtained from (9) in a similar manner. ♢

3.2 Application to the NBCF Case

The result presented by Theorem 3 in conjunction with standard LMI tools is sufficient for synthesizing robustly stable BCF controllers. In this section however, we will examine the application of this procedure in some special cases, namely, when restricting the selection of Q and R. Specifically, we will utilise the normalization property as defined by (Tsiakkas and Lanzon, 2018, Definition 3) to obtain some computational benefits. This provides a significant advantage in the robust control synthesis procedure which parallels the results obtained in the classical case of normalized coprime factor synthesis as demonstrated by Glover and McFarlane (1989). Namely, one no longer needs to solve (10) and (11) but instead closed form solutions can be obtained. Then the problem of robust control synthesis reduces to a simple scalar inequality that can be solved easily even with a simple line search algorithm. It must be noted that imposing the normalization property on the BC factors of the plant restricts the robustness potential of the closed loop system; in a manner similar to the coprime factor case examined by Engelken et al. (2011).

Suppose that Q and R are chosen according to the conditions laid out by (Tsiakkas and Lanzon, 2018, Theorem 2), satisfying $Q + XR^* = 0$ and $R + Q^*Y = 0$ where $X \geq 0$ and $Y \geq 0$ are the stabilising solutions to the AREs

$$\mathcal{L}(A,X) - XR^*RX + BB^* = 0 \text{ and} \tag{15}$$

$$\mathcal{L}(A^*,Y) - YQQ^*Y + C^*C = 0. \tag{16}$$

Then the induced QR-BCf is normalized in the sense of (Tsiakkas and Lanzon, 2018, Definition 3).

It is useful to note that the above can be transformed to

$$\mathcal{L}(A+QR,X) + QQ^* + BB^* = 0 \text{ and} \tag{17}$$

$$\mathcal{L}((A+QR)^*,Y) + R^*R + C^*C = 0. \tag{18}$$

The above constraints can now be used to deduce the existence of a robustly stabilizing controller as in the following theorem.

Theorem 5. Let $P \in \mathscr{R}^{p\times q}$ have the stabilizable and detectable state space realization $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $A \in \mathbb{R}^{n\times n}$. Furthermore suppose that $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ are such that the induced QR-BCF is normalized. Finally, let $\tilde{X} \geq 0$ and $\tilde{Y} \geq 0$ be the solutions to the Lyapunov equations

$$\mathcal{L}(A+QR,\tilde{X}) + QQ^* + \delta I = 0 \text{ and} \tag{19}$$

$$\mathcal{L}((A+QR)^*,\tilde{Y}) + R^*R + \delta I = 0 \tag{20}$$

for some $\delta \in \mathbb{R}_+$.

Then there exists a stabilizing controller $C_\infty \in \mathscr{R}^{q\times p}$ such that $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if

a) $\gamma > 1$,

b) $(1+\epsilon)\rho(X\tilde{Y}) < 1$, (21)

c) $(1+\epsilon)\rho(Y\tilde{X}) < 1$, (22)

d) $\rho\left\{(\gamma\epsilon)^2(I-(1+\epsilon)\tilde{X}Y)^{-1}\tilde{X}\tilde{Y}(I-(1+\epsilon)X\tilde{Y})^{-1}\right\} \leq 1$, (23)

where $X > 0$ and $Y > 0$ are the solutions to (15) and (16) respectively and $\epsilon = (\gamma^2-1)^{-1}$.

Proof. First note that since $A + QR$ is Hurwitz and $\left(A+QR, [Q\ \sqrt{\delta}I]\right)$ and $\left(\begin{bmatrix} R \\ \sqrt{\delta}I \end{bmatrix}, A+QR\right)$ are controllable and observable respectively for any non-zero δ, it follows from (Zhou et al., 1996, Lemma 3.18) that $\tilde{X} > 0$ and $\tilde{Y} > 0$ hence $\tilde{X}^{-1}$ and $\tilde{Y}^{-1}$ exist.

Using the Schur complement method, (13) yields

$$\begin{bmatrix} \mathcal{L}(A+QR,S) - \gamma(BB^*+QQ^*) & \gamma Q - SR^* \\ \gamma Q^* - RS & -(\gamma\epsilon)^{-1}I \end{bmatrix} < 0.$$

Now let $S = (\gamma\epsilon\tilde{Y})^{-1} - \gamma X$ which when using (17) and $Q + XR^* = 0$ yields

$$\begin{aligned} &(\gamma\epsilon)^{-1}\begin{bmatrix} \mathcal{L}(A+QR,\tilde{Y}^{-1}) & -\tilde{Y}^{-1}R^* \\ -\tilde{Y}^{-1} & -I \end{bmatrix} \\ &\quad - \gamma\begin{bmatrix} \mathcal{L}(A+QR,X) + (BB^*+QQ^*) & -(Q+XR^*) \\ -(Q+XR^*)^* & 0 \end{bmatrix} \\ &\quad = (\gamma\epsilon)^{-1}\begin{bmatrix} \mathcal{L}(A+QR,\tilde{Y}^{-1}) & -\tilde{Y}^{-1}R^* \\ -R\tilde{Y}^{-1} & -I \end{bmatrix} < 0 \\ &\quad \Leftrightarrow \begin{bmatrix} \mathcal{L}((A+QR)^*,\tilde{Y}) & -R^* \\ -R & -I \end{bmatrix} < 0 \\ &\quad \Leftrightarrow \mathcal{L}((A+QR)^*,\tilde{Y}) + R^*R = -\delta I < 0. \end{aligned}$$

Hence (10) is guaranteed given the above selection of S. In a similar manner, with $T = (\gamma\epsilon\tilde{X})^{-1} - \gamma Y$ and (18), we have that (11) is satisfied.

The above assignments for S and T do not guarantee that they will be positive definite as required by Theorem 3. Hence additional conditions must be imposed; these can be derived as follows.

$$
\begin{aligned}
S > 0 &\Leftrightarrow (\gamma\epsilon\tilde{Y})^{-1} - \gamma X > 0 \\
&\Leftrightarrow I - \gamma^2 \epsilon \tilde{Y}^{\frac{1}{2}} X \tilde{Y}^{\frac{1}{2}} > 0 \\
&\Leftrightarrow 1 - (1+\epsilon)\rho(\tilde{Y}^{\frac{1}{2}} X \tilde{Y}^{\frac{1}{2}}) > 0 \\
&\Leftrightarrow (1+\epsilon)\rho(X\tilde{Y}) < 1.
\end{aligned}
$$

Similarly, it can be shown that $T > 0$ if and only if (7) holds.

Finally, (23) follows by simply substituting $S = (\gamma\epsilon\tilde{Y})^{-1} - \gamma X$ and $T = (\gamma\epsilon\tilde{X})^{-1} - \gamma Y$ into (12).

$$
\begin{aligned}
ST &= ((\gamma\epsilon\tilde{Y})^{-1} - \gamma X)((\gamma\epsilon\tilde{X})^{-1} - \gamma Y) \\
&= (\gamma\epsilon)^{-2}(I - (1+\epsilon)X\tilde{Y})(\tilde{X}\tilde{Y})^{-1}(I - (1+\epsilon)\tilde{X}Y).
\end{aligned}
$$

Then $\underline{\lambda}(ST) \geq 1$ if and only if

$$
\begin{aligned}
&\rho\left\{(ST)^{-1}\right\} \leq 1 \\
&\Leftrightarrow \rho\left\{(\gamma\epsilon)^2(I-(1+\epsilon)\tilde{X}Y)^{-1}\tilde{X}\tilde{Y}(I-(1+\epsilon)X\tilde{Y})^{-1}\right\} \leq 1
\end{aligned}
$$

which concludes the proof. □

The above theorem provides a controller existence check based on a set of three scalar inequalities over the domain $(1, \infty)$. Since a solution is guaranteed to exist due to the stabilizability and detectability of the plant, iterative methods such as the bisection algorithm, can be used to obtain the the smallest achievable γ.

Remark 6. Note that as $\delta \rightarrow 0$ in Theorem 5, $\tilde{X}$ and $\tilde{Y}$ tend to the controllability and observability Gramians of $[M^* \; N^*]^*$ and $[M \; -L]$ respectively. As a result, (21) and (22) can be expressed using the Hankel norm as $\rho(X\tilde{Y}) = \|[M \; -L]\|_H^2$ and $\rho(Y\tilde{X}) = \left\|[M^* \; N^*]^*\right\|_H^2$ (see (Zhou et al., 1996, Theorem 8.1) for a definition of the Hankel norm). It can be shown that this gives a lower bound on the achievable robust stability margin when using a NBCF of the plant. The proof for this fact relies on the following.

Let $\tilde{X}(\delta)$ be the solution to (19) for a given value of δ. For some positive δ

$$
\begin{aligned}
&\mathcal{L}(A + QR, \tilde{X}(0) + (\tilde{X}(\delta) - \tilde{X}(0))) + QQ^* + \delta I = 0 \\
&\Rightarrow \mathcal{L}(A + QR, \tilde{X}(\delta) - \tilde{X}(0)) + \delta I = 0 \\
&\Rightarrow \tilde{X}(\delta) > \tilde{X}(0) \quad \forall \delta > 0.
\end{aligned}
$$

We know that $\tilde{X}(\delta) > \tilde{X}(0)$ and not $\tilde{X}(\delta) \geq \tilde{X}(0)$ since $A + QR$ is Hurwitz and $\delta I > 0$ (Zhou et al., 1996, Lemma 3.18). ◇

4. NUMERICAL EXAMPLE

In this section we present a numerical example and compare with results obtained using classical coprime factor theory. For this example we consider the robust stabilization of a flexible booster rocket studied by Enns (1991). The dynamics of the system under consideration can be modelled as

$$
P = \left[\begin{array}{ccccccc|cc}
-0.003 & -0.218 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.04 & -4.5 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 2.5 & 0.02 & 0.8 & 0 & 0 & 0 & 0 & 0.1 \\
0 & 0 & 0 & 0 & 0.5 & 0.004 & 1.125 & 0.1 & 0
\end{array}\right],
$$

mapping the thrust vectoring control input to the pitch rate measurements.

A normalized BCF of the plant was constructed using the algorithm developed by Tsiakkas and Lanzon (2018).Then, exploiting the results of Theorem 5 with $\delta = 10^{-6}$, a lower bound for the robust stability margin was obtained as 3.242. The solutions to (13) and (14) were then constructed as demonstrated in the theorem proof. Finally, the associated controller was constructed as outlined in (Gahinet and Apkarian, 1994, Section 7), with the resulting robust stability margin given by $\gamma_{\text{BCF}} = 3.246$.

In addition to the above BCF based controller, a second controller was synthesized using classical coprime factor theory; specifically, using a normalized LCF (NLCF) of the plant resulting in a robust stability margin of $\gamma_{\text{NCF}} = 2.640$.

Three simulations were executed, the first considering the nominal plant and the other two the plant BC and LC factors were perturbed by an uncertainty of magnitude $\|\Delta\|_\infty = 0.3$. The simulation results are shown in Figures 1 through 3. For these simulations, the designed controllers were tasked with stabilizing the system to the origin, starting from non-zero initial conditions. In both the nominal and uncertain cases, the initial output vector of the plant was set to $[0.3 \; -0.1]^*$. Figures 1 and 2 show the nominal and perturbed responses of the system. It can be seen that both controllers successfully and robustly stabilize the system. Figure 3 shows the performance degradation associated with each controller between the nominal and perturbed cases. This is quantified simply by the norm of the difference of the output in the nominal and perturbed plants.

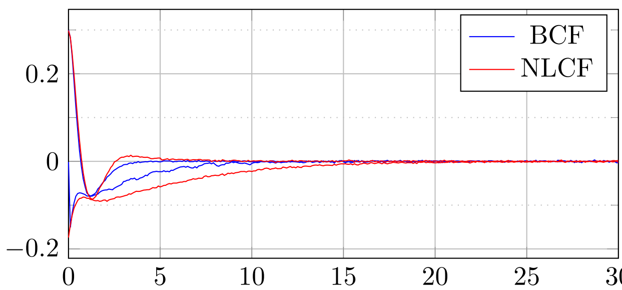
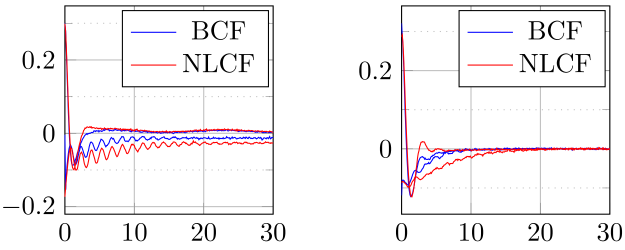


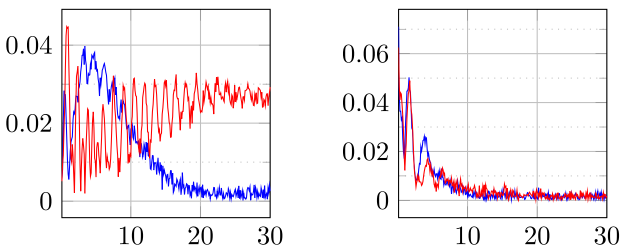
Fig. 1. Nominal performance of the synthesized controllers.

Although Figures 1 and 2 indicate that both controllers robustly stabilize the plant with respect to both BCF and NLCF uncertainty, Figure 3 demonstrates how, in this scenario, the robust performance of the BCF based controller is superior to that of its classical counterpart. It can be observed that the BCF controller successfully counters the perturbations at steady state without deviations from the nominal case while the NLCF controller only does so



(a) BCF uncertainty. (b) NLCF uncertainty.

Fig. 2. Robust performance of the synthesized controllers.



(a) BCF uncertainty. (b) NLCF uncertainty.

Fig. 3. Performance degradation of BCF (——) and NLCF (——) based controllers in the presence of BC and LC factor perturbations.

successfully in the case of LC factor perturbations. This seems to indicate that a BCF based controller is more robust to ill-selected uncertainty models.

5. CONCLUSION

In this paper, a LMI based robust control synthesis result is presented using BCF theory. The developed methodology is obtained by adapting the work of Gahinet and Apkarian (1994) to the BCF generalized plant as proposed by Tsiakkas and Lanzon (2017). These results are then further specialized to the case of normalized BCFs defined by Tsiakkas and Lanzon (2018) and shown to yield interesting results. Finally, a numerical example is presented to demonstrate the practical applicability of the developed results.

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