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# Robustness Against Indirect Invasions in Asymmetric Games ⋆

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**Abstract:** The concept of robustness against indirect invasions is well-known for symmetric games. We are concerned with the technical aspects and relevance of this concept for asymmetric games with continuous strategy space. For such games, we show that the set of all indirect neutral mutants of a robust profile is equivalent to a minimal evolutionarily stable set. It is also proved that a globally strong uninvadable profile is robust and the set of its indirect neutral mutants is a singleton. The results are illustrated using examples.

*Keywords:* Game theory, Stability robustness, Evolutionary stability

## 1. INTRODUCTION

Evolutionary game theory has been one of the significant branches of game theory. Evolutionary games can be categorized as symmetric and asymmetric.

Symmetric games are those wherein the roles of players do not have any influence on their strategies and payoffs. Maynard Smith and Price (1973) initiated evolutionary approach to game theory by introducing the concept of an evolutionarily stable strategy (ESS) for symmetric games. Further, the concept of an evolutionarily stable set (ES set) was discussed in Thomas (1985) and Balkenborg and Schlag (2001). Significant majority of the research done in evolutionary game theory is regarding symmetric games; see Bomze (1991), Hofbauer and Sigmund (1998), Oechssler and Riedel (2001), Oechssler and Riedel (2002), Cressman (2003), Shaiju and Bernhard (2009), Van Veelen (2012), Hingu et al. (2018) and the references therein.

Numerous game theoretical models in bargaining theory (see for example Qin et al. (2019)), oligopoly theory (see for example Leininger and Moghadam (2018)), war of attrition theory (see for example Hammerstein and Parker (1982)) are asymmetric but there has been very limited research done concerning the evolutionary aspects of these problems. Selten (1980) has been the first to study asymmetric evolutionary games followed by Samuelson (1991), Samuelson and Zhang (1992), Ritzberger and Weibull (1995), Cressman (2009), Cressman (2010). Further, Mendoza-Palacios and Hernández-Lerma (2015) introduced the concept of strong uninvadable profile (SUP) for asymmetric games having continuous strategy space. Moving in a similar direction, Narang and Shaiju (2019a) established that a polymorphic SUP has to be necessarily monomorphic. In addition, they extended the definition of strong uninvadability to sets of profiles and proved related results. To study more general profiles/sets of profiles than the polymorphic profiles, Narang and Shaiju (2019b) introduced the concepts of globally strong uninvadable profiles/sets of profiles.

The concept of ESS has a rather weaker form known as neutrally stable strategy (NSS). For neutrally stable strategies in the games where neutral mutants are natural, it is better to check if these neutral mutants that do not have an edge over evolutionary natural selection themselves have a possibility of being harmful or not when they open way for other mutants that do have an edge over evolutionary natural selection in case the fraction of those neutral mutants in the existing population is high enough. Such different situations are better understood by the notion of robustness against indirect invasions.

Van Veelen (2012) introduced the concept of robustness against indirect invasions for finite symmetric games and proved the equivalence of a 'Balkenborg and Schlag ES set' and a 'Thomas ES set' which in turn is used to establish the equivalence of the set of all (indirect) neutral mutants of a strategy (which is robust against indirect invasions (RAII)) and a minimal (Balkenborg and Schlag) ES set. This enabled to derive various dynamic stability results for the set of (indirect) neutral mutants of a RAII strategy. He also proved that if a strategy is RAII, then the set of its (indirect) neutral mutants is a Balkenborg and Schlag ES set and hence, robustness against indirect invasions comes out as a simple check to explore if a strategy is an element of a Thomas ES set. These results of Van Veelen (2012) were further generalized by Hingu (2017) for continuous symmetric games.

In the present paper, we extend these results for continuous asymmetric games. As our main result, we prove that if $\bar{\mu}$ is a RAII profile, then $S_{NM}(\bar{\mu})$ is equivalent to a minimal Balkenborg and Schlag ES set where $S_{NM}(\bar{\mu})$ is the set containing $\bar{\mu}$ and its (indirect) neutral mutants. We also

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show that a globally strong uninvadable profile $\bar{\mu}$ is RAII and in this case, $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$.

The paper is structured as follows. Section 2 introduces the preliminary notations and definitions. Section 3 discusses the auxiliary and main results followed by three illustrative examples in Section 4. We end the paper with some concluding remarks in Section 5.

## 2. PRELIMINARIES: NOTATIONS AND DEFINITIONS

Let us consider a model (introduced by Mendoza-Palacios and Hernández-Lerma (2015)) of asymmetric evolutionary games involving $n$-player contests. Here, the set of players is denoted by $I := \{1, 2, \ldots, n\}$. The set of pure strategies for player $i \in I$ is a Polish space denoted by $A_i$. A generic pure strategy of player $i$ is notated as $a_i \in A_i$. For $i \in I$, each $a = (a_1, \ldots, a_n) \in A := A_1 \times \cdots \times A_n$ can be written as $(a_i, a_{-i})$ where $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i} := A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$. The Borel $\sigma$- algebra of $A_i$ is denoted by $\mathcal{B}(A_i)$. Moreover, $\mathbb{P}(A_i)$ is the set of mixed strategies for player $i$ where $\mu_i \in \mathbb{P}(A_i)$ denotes a typical mixed strategy and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is a mixed strategy profile. The notations $\mu = (\mu_i, \mu_{-i})$ and $\mu_{-i} := (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n) \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_{i-1}) \times \mathbb{P}(A_{i+1}) \times \cdots \times \mathbb{P}(A_n)$ are used in the case of mixed strategies like the case of pure strategies.

The payoff function $J_i : \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \to \mathbb{R}$ for player $i$ is defined as

$$J_i(\mu_1, .., \mu_n) = \int_{A_1} .. \int_{A_n} U_i(a_1, .., a_n)\ \mu_n(da_n)..\mu_1(da_1) \tag{1}$$

where $U_i : A_1 \times \cdots \times A_n \to \mathbb{R}$ is assumed to be measurable and bounded.

The present game model can be expressed as

$$\Gamma = [I, \{\mathbb{P}(A_i)\}_{i \in I}, \{J_i(\cdot)\}_{i \in I}]. \tag{2}$$

To define stable sets of profiles, we also require the $\|\cdot\|_\infty$-norm defined as

$$\|\mu\|_\infty = \|(\mu_1, \ldots, \mu_n)\|_\infty := \max_{i \in I} \|\mu_i\| \tag{3}$$

where $\|\mu_i\|$ is the variational or strong norm of the strategy $\mu_i$ (see for example p. 360 of Shiryaev (1995), Narang and Shaiju (2019a)).

Based on the asymmetric evolutionary game model (2) with continuous strategy space, we now introduce a strong Thomas ES set (analogous to the definition of a strong Thomas ES set in symmetric games by Thomas (1985)).

*Definition 1.* (Strong Thomas ES Set). A set of population profiles $\Pi = \Pi_1 \times \cdots \times \Pi_n \subseteq \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is said to be a strong Thomas ES set if it is nonempty and satisfies the following three properties:

(a) $\Pi_i$ is closed for all $i \in I$,
(b) Each $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) \in \Pi$ is a Nash equilibrium of $\Gamma$ (that is, $J_i(\bar{\mu}_i, \bar{\mu}_{-i}) \geq J_i(\mu_i, \bar{\mu}_{-i})$ for all $i \in I$ and for all $\mu_i \in \mathbb{P}(A_i)$),
(c) Each $\bar{\mu} \in \Pi$ has some neighborhood $N(\bar{\mu})$ (w.r.t. $\|\cdot\|_\infty$-norm defined in (3)) such that whenever there are $\mu \in N(\bar{\mu})$ and $i \in I$ satisfying $J_i(\bar{\mu}_i, \bar{\mu}_{-i}) = J_i(\mu_i, \bar{\mu}_{-i})$, we have $J_i(\bar{\mu}_i, \mu_{-i}) \geq J_i(\mu_i, \mu_{-i})$ with strict inequality for $\mu_i \notin \Pi_i$.

Now, analogous to the definition of a Balkenborg and Schlag ES set in symmetric games by Balkenborg and Schlag (2001), we define a Balkenborg and Schlag ES set in asymmetric games.

*Definition 2.* (Balkenborg and Schlag ES Set). A set of population profiles $\Pi = \Pi_1 \times \cdots \times \Pi_n \subseteq \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is said to be a Balkenborg and Schlag ES set if it is nonempty and satisfies the following three properties:

(a) $\Pi_i$ is closed for all $i \in I$,
(b) Each $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) \in \Pi$ is a Nash equilibrium of $\Gamma$,
(c) If for $\bar{\mu} \in \Pi$, we have $J_i(\bar{\mu}_i, \bar{\mu}_{-i}) = J_i(\mu_i, \bar{\mu}_{-i})$ for some $i \in I$ and $\mu_i \in \mathbb{P}(A_i)$, then $J_i(\bar{\mu}_i, \mu_{-i}) \geq J_i(\mu_i, \mu_{-i})$ where strict inequality holds for $\mu_i \notin \Pi_i$.

Before moving on to define the robustness of a profile against indirect invasions, we need to define the following for a profile $\bar{\mu} \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$:

(a) $S_E(\bar{\mu}) = \{\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) : J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) = J_i(\mu_i, \mu_{-i})\ \forall i \in I\}$, the set of (evolutionarily) equal performers against $\bar{\mu}$.
(b) $S_B(\bar{\mu}) = \{\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) : \exists i \in I$ such that '$J_i(\mu_i, \bar{\mu}_{-i}) > J_i(\bar{\mu}_i, \bar{\mu}_{-i})$' or '$J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) < J_i(\mu_i, \mu_{-i})$'$\}$, the set of (evolutionarily) better performers against $\bar{\mu}$.

The set $S_E(\bar{\mu})$ in (a) above can also be written as $S_E(\bar{\mu}) = \{\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) : \mu_i \in S_E(\bar{\mu}_i)\ \forall i \in I\}$ where $S_E(\bar{\mu}_i) = \{\mu_i \in \mathbb{P}(A_i) : J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) = J_i(\mu_i, \mu_{-i})\}$ is the set of (evolutionarily) equal performers against $\bar{\mu}_i$. We can call the elements of $S_E(\bar{\mu})$ as the neutral mutants of $\bar{\mu}$. Moreover, if $\mu \in S_E(\bar{\mu})$, then $\bar{\mu} \in S_E(\mu)$.

We use the sets $S_E(\bar{\mu})$ and $S_B(\bar{\mu})$ to define the robustness against indirect invasions. A strategy which is RAII is a midway between ESS and NSS. Such a strategy is a slightly relaxed form of ESS as it permits neutral mutants but a more rigid form of NSS as it does not permit those neutral mutants which indirectly pave way for some other mutants which have an edge over evolutionary natural selection, if the proportion of these mutants is high enough in the population. We now define the robustness of a profile against indirect invasions in an asymmetric evolutionary game.

*Definition 3.* (Robustness Against Indirect Invasions). A profile $\bar{\mu} \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is robust against indirect invasions (RAII) if

(a) $S_B(\bar{\mu}) = \phi$ and,
(b) there does not exist $\mu^1, \ldots, \mu^m$, $m \geq 2$ such that $\mu^1 \in S_E(\bar{\mu})$, $\mu^j \in S_E(\mu^{j-1})$, $\mu^m \in S_B(\mu^{m-1})$, $2 \leq j \leq m-1$.

Now, we group a RAII profile $\bar{\mu}$ with all its (indirect) neutral mutants into a set $S_{NM}(\bar{\mu})$ as given below.

$$S_{NM}(\bar{\mu}) = \{\, \mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) : \exists \mu^1, \ldots, \mu^m, \; m \geq 1 \text{ such that } \mu^1 \in S_E(\bar{\mu}),\; \mu^j \in S_E(\mu^{j-1}),\; \mu \in S_E(\mu^m),\; 2 \leq j \leq m\}.$$

*Remark 4.* From the above definitions, it is clear that if $\bar{\mu}$ is RAII and $\mu \in S_{NM}(\bar{\mu})$, then $\mu$ is also RAII and $S_{NM}(\bar{\mu}) = S_{NM}(\mu)$. Hence, the profile $\bar{\mu}$ does not play any distinctive role in $S_{NM}(\bar{\mu})$.

We next define a similar set for a component $\bar{\mu}_i$ of a RAII profile $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n)$:

$$S_{NM}(\bar{\mu}_i) = \{\, \mu_i \in \mathbb{P}(A_i) : \exists \mu_i^1, \ldots, \mu_i^m, \; m \geq 1 \text{ such that } \mu_i^1 \in S_E(\bar{\mu}_i),\; \mu_i^j \in S_E(\mu_i^{j-1}),\; \mu_i \in S_E(\mu_i^m),\; 2 \leq j \leq m\}.$$

It is obvious that $S_{NM}(\bar{\mu}) = S_{NM}(\bar{\mu}_1) \times \cdots \times S_{NM}(\bar{\mu}_n)$. We assume $S_{NM}(\bar{\mu}_i)$ to be closed for all $i \in I$ in the remainder of the paper (Note that the continuity of the payoff functions $J_i$ in $\mu_i$ guarantees the closedness of $S_{NM}(\bar{\mu}_i)$). This is required for establishing the results in the next section.

We end this section with the definition of a globally strong uninvadable profile (Narang and Shaiju (2019b)). We prove in Theorem 11 that such a profile $\bar{\mu}$ is RAII and in this case, $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$.

*Definition 5.* (Globally Strong Uninvadable Profile). A profile $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is said to be globally strong uninvadable if $J_i(\bar{\mu}_i, \mu_{-i}) \geq J_i(\mu_i, \mu_{-i})$; $i \in I,\; \mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ with strict inequality whenever $\mu_i \neq \bar{\mu}_i$.

## 3. MAIN RESULTS

In this section, we first establish three auxiliary results needed to prove the main result (Theorem 10) regarding the equivalence of $S_{NM}(\bar{\mu})$ (the set containing a RAII profile $\bar{\mu}$ and all its (indirect) neutral mutants) and a minimal Balkenborg and Schlag ES set. We also present a result concerning the RAII property of globally strong uninvadable profiles.

*Lemma 6.* Let $\Pi = \Pi_1 \times \cdots \times \Pi_n \subseteq \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$. Then $\Pi$ is a strong Thomas ES set iff it is a Balkenborg and Schlag ES set.

**Proof.** Let $\Pi$ be a Balkenborg and Schlag ES set. By taking $N(\bar{\mu}) = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ in Definition 1, for each $\bar{\mu} \in \Pi$, it follows that $\Pi$ is a strong Thomas ES set.

For the converse, let $\Pi$ be a strong Thomas ES set. This implies that $\Pi_i$ is closed for all $i \in I$ and each profile in $\Pi$ is a Nash equilibrium of $\Gamma$ which verify conditions (a) and (b) of a Balkenborg and Schlag ES set.
If possible, let condition (c) (in Definition 2) be not satisfied. This would imply that there exists $\bar{\mu} \in \Pi$ for which, either

(i) $\exists j \in I$ and $\mu_j \in \mathbb{P}(A_j)$ such that $J_j(\mu_j, \bar{\mu}_{-j}) = J_j(\bar{\mu}_j, \bar{\mu}_{-j})$ and $J_j(\bar{\mu}_j, \mu_{-j}) < J_j(\mu_j, \mu_{-j})$, or,
(ii) $\exists j \in I$ and $\mu_j \notin \Pi_j$ such that $J_j(\mu_j, \bar{\mu}_{-j}) = J_j(\bar{\mu}_j, \bar{\mu}_{-j})$ and $J_j(\bar{\mu}_j, \mu_{-j}) = J_j(\mu_j, \mu_{-j})$.

Let $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$ be such that

$$\hat{\mu}_i = \hat{\mu}_i(\alpha) := \alpha \bar{\mu}_i + (1-\alpha)\mu_i,\; i \in I, \tag{4}$$

where $\alpha \in [0, 1]$.

If (i) holds true, then by using (4), it is clear that $J_j(\bar{\mu}_j, \bar{\mu}_{-j}) = J_j(\hat{\mu}_j, \bar{\mu}_{-j})$ and $J_j(\hat{\mu}_j, \hat{\mu}_{-j}) > J_j(\bar{\mu}_j, \hat{\mu}_{-j})$. For any given neighborhood $N(\bar{\mu})$ of $\bar{\mu}$ (w.r.t. $\|\cdot\|_\infty$-norm defined in (3)), one can always choose $\alpha$ close to 1 so that $\hat{\mu} \in N(\bar{\mu})$. This results in a violation of condition (c) of Definition 1 (of a strong Thomas ES set).

If (ii) holds true, then we have $J_j(\hat{\mu}_j, \bar{\mu}_{-j}) = J_j(\bar{\mu}_j, \bar{\mu}_{-j})$ and $J_j(\bar{\mu}_j, \hat{\mu}_{-j}) = J_j(\hat{\mu}_j, \hat{\mu}_{-j})$. Since $\Pi_j$ is closed, $\bar{\mu}_j \in \Pi_j$ (as $\bar{\mu} \in \Pi$), and $\mu_j \notin \Pi_j$, it follows that there exists at least one $\tilde{\alpha}$ such that $\tilde{\mu}_j := \hat{\mu}_j(\tilde{\alpha}) \in \Pi_j$ but there is no strong neighborhood $N(\tilde{\mu}_j)$ for which all strategies $\hat{\mu}_j(\alpha)$ that are in $N(\tilde{\mu}_j)$ are also elements of $\Pi_j$. This contradicts the fact that $\Pi$ is a strong Thomas ES set.

Now, both (i) and (ii) lead to contradictions to the assumption that $\Pi$ is a strong Thomas ES set. Hence, $\Pi$ is a Balkenborg and Schlag ES set. □

*Lemma 7.* If $\bar{\mu}$ is RAII, then $S_{NM}(\bar{\mu})$ is a Balkenborg and Schlag ES set.

**Proof.** Let $\bar{\mu}$ be RAII. Since $S_{NM}(\bar{\mu}_i)$ is closed for all $i \in I$, condition (a) of Definition 2 is satisfied. As mentioned in Remark 4, any $\mu \in S_{NM}(\bar{\mu})$ is also RAII and hence, by Definition 3, $S_B(\mu) = \phi$ for all $\mu \in S_{NM}(\bar{\mu})$. This implies that $S_{NM}(\bar{\mu})$ contains only Nash equilibria thereby satisfying condition (b) of Definition 2.

If $S_{NM}(\bar{\mu})$ were not a Balkenborg and Schlag ES set, then we would have the violation of condition (c) in Definition 2 which in turn yields, either

(i) $\exists i \in I$ and $\mu_i \in \mathbb{P}(A_i)$ such that $J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) < J_i(\mu_i, \mu_{-i})$, or
(ii) $\exists i \in I$ and $\mu_i \notin S_{NM}(\bar{\mu}_i)$ such that $J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) = J_i(\mu_i, \mu_{-i})$.

If (i) holds true, then $\mu \in S_B(\bar{\mu})$ which is a contradiction to the fact that $S_B(\bar{\mu}) = \phi$ (as $\bar{\mu}$ is RAII).

If (ii) holds true, then there exists $i \in I$ such that $\mu_i \in S_E(\bar{\mu}_i)$ which in turn implies that $\mu_i \in S_{NM}(\bar{\mu}_i)$ (by the definition of $S_{NM}(\bar{\mu}_i)$). This contradicts the fact that $\mu_i \notin S_{NM}(\bar{\mu}_i)$.

As (i) and (ii) both lead to contradictions, we conclude that $S_{NM}(\bar{\mu})$ is a Balkenborg and Schlag ES set. □

*Remark 8.* The above lemma shows that if a profile is RAII, then the set containing this RAII profile and its (indirect) neutral mutants is a ES set which is an extension of the stability of ESS in the set form. An evolutionary game with no RAII profile will have no (Thomas) or (Balkenborg and Schlag) ES set.

*Lemma 9.* If $\Pi = \Pi_1 \times \cdots \times \Pi_n \subseteq \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is a Balkenborg and Schlag ES set and $\bar{\mu} \in \Pi$, then the profile $\bar{\mu}$ is RAII.

**Proof.** Let $\Pi$ be a Balkenborg and Schlag ES set, $\bar{\mu} \in \Pi$ and $\mu \in S_E(\bar{\mu})$. We now claim that all profiles of the form $\mu(\alpha) = \alpha\bar{\mu} + (1-\alpha)\mu$ also belong to $\Pi$ for $\alpha \in [0,1]$. If this is not true, then using the closedness of $\Pi$, there exists $\alpha \in [0,1]$ for which $\mu(\alpha) \in \Pi$ but every neighborhood $N$ (w.r.t. the norm in (3)) of $\mu(\alpha)$ contains a profile $\mu(\beta) \notin \Pi$ such that $J_i(\mu_i(\alpha), \mu_{-i}(\beta)) = J_i(\mu_i(\beta), \mu_{-i}(\beta))$ for all $i \in I$. This contradicts the fact that $\Pi$ is a strong Thomas ES set (by the equivalence of a 'Balkenborg and Schlag ES set' and a 'strong Thomas ES set' as established in Lemma 6) and hence the claim.
We can repeatedly apply this argument to prove that $S_{NM}(\bar{\mu}) \subseteq \Pi$.

Now, assume that $\bar{\mu}$ is not RAII. This implies that either $S_B(\bar{\mu}) \neq \phi$ or $\exists \mu^1, \ldots, \mu^m,\ m \geq 2$ such that $\mu^1 \in S_E(\bar{\mu}),\ \mu^j \in S_E(\mu^{j-1}),\ \mu^m \in S_B(\mu^{m-1})$ where $2 \leq j \leq m-1$. We look at the case when $S_B(\bar{\mu}) \neq \phi$ as the latter is similar because in that case $\mu^{m-1} \in \Pi$ (since $S_{NM}(\bar{\mu}) \subseteq \Pi$ and $\mu^{m-1}$ is an indirect neutral mutant of $\bar{\mu}$) and we would then focus on $S_B(\mu^{m-1}) \neq \phi$ instead.

If $S_B(\bar{\mu}) \neq \phi$, then there exists $\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ for which either there is at least one $i \in I$ such that $J_i(\mu_i, \bar{\mu}_{-i}) > J_i(\bar{\mu}_i, \bar{\mu}_{-i})$, or
there exists $i \in I$ such that $J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) < J_i(\mu_i, \mu_{-i})$.
Both cases lead to the contradiction that $\Pi$ is not a Balkenborg and Schlag ES set (The former and latter violate conditions (b) and (c) of Definition 2 respectively). □

We now present our main result (for a RAII profile $\bar{\mu}$) concerning the equivalence of $S_{NM}(\bar{\mu})$ and a minimal Balkenborg and Schlag ES set. This is important because being RAII makes the profile to be a member of a minimal ES set and opens doors to relate this stability concept to nice dynamic properties.

*Theorem 10.* The following statements are equivalent for a set $\Pi = \Pi_1 \times \cdots \times \Pi_n \subseteq \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$:

(i) $\Pi = S_{NM}(\bar{\mu})$ for some $\bar{\mu} \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and $\bar{\mu}$ is RAII.
(ii) $\Pi$ is a minimal Balkenborg and Schlag ES set.

**Proof.** Assume (i). Now, as $\bar{\mu}$ is RAII, by Lemma 7, $S_{NM}(\bar{\mu})$ is a Balkenborg and Schlag ES set. The definition of $S_{NM}(\bar{\mu})$ makes $\Pi$ a minimal Balkenborg and Schlag ES set.

Conversely, let (ii) be true. As $\Pi$ is a Balkenborg and Schlag ES set, by Lemma 9, any $\bar{\mu} \in \Pi$ is RAII. As $\bar{\mu}$ is RAII, $S_{NM}(\bar{\mu})$ is a Balkenborg and Schlag ES set by Lemma 7. Since $\Pi$ is minimal, $\Pi \subseteq S_{NM}(\bar{\mu})$. Moreover, as in the proof of Lemma 9, $S_{NM}(\bar{\mu}) \subseteq \Pi$. This implies that $\Pi = S_{NM}(\bar{\mu})$ for any $\bar{\mu} \in \Pi$. □

We move on to our next result regarding the robustness of a globally strong uninvadable profile.

*Theorem 11.* If $\bar{\mu}$ is a globally strong uninvadable profile, then it is RAII and $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$.

**Proof.** Let $\bar{\mu}$ be a globally strong uninvadable profile. This implies that,

$$J_i(\bar{\mu}_i, \mu_{-i}) \geq J_i(\mu_i, \mu_{-i});\ i \in I,\ \mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \quad (5)$$

with strict inequality whenever $\mu_i \neq \bar{\mu}_i$.

Also, a globally strong uninvadable profile is a Nash equilibrium of $\Gamma$ (see Theorem 5.7 of Mendoza-Palacios and Hernández-Lerma (2015)). Therefore, by the definition of Nash equilibrium, we have

$$J_i(\bar{\mu}_i, \bar{\mu}_{-i}) \geq J_i(\mu_i, \bar{\mu}_{-i});\ i \in I,\ \mu_i \in \mathbb{P}(A_i). \quad (6)$$

Now, (5) and (6) imply that $S_B(\bar{\mu}) = \phi$ which proves the first condition in the definition of a RAII profile.

If possible, let there be $\mu^1, \ldots, \mu^m,\ m \geq 2$ such that $\mu^1 \in S_E(\bar{\mu}),\ \mu^j \in S_E(\mu^{j-1}),\ \mu^m \in S_B(\mu^{m-1})$ where $2 \leq j \leq m-1$.

It is clear that $\mu^1 \in S_E(\bar{\mu})$ implies for all $i \in I$,

$$J_i(\mu_i^1, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i}), \quad (7)$$

and

$$J_i(\bar{\mu}_i, \mu_{-i}^1) = J_i(\mu_i^1, \mu_{-i}^1). \quad (8)$$

By (5) and (8), we have that $\mu_i^1 = \bar{\mu}_i$ for all $i \in I$ which in turn gives $\mu^1 = \bar{\mu}$. Similarly, $\mu^2 \in S_E(\mu^1) = S_E(\bar{\mu})$ implies that $\mu^2 = \bar{\mu}$. Applying this argument repeatedly, we get $\mu^{m-1} = \bar{\mu}$. Now, we know that $\mu^m \in S_B(\mu^{m-1})$ i.e. $\mu^m \in S_B(\bar{\mu})$. This is a contradiction to the already established fact that $S_B(\bar{\mu}) = \phi$. Hence, $\bar{\mu}$ is RAII.

We next prove that $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$. If $\mu \in S_{NM}(\bar{\mu})$, then there exists $\mu^1, \ldots, \mu^m, m \geq 1$ such that $\mu^1 \in S_E(\bar{\mu}),\ \mu^j \in S_E(\mu^{j-1}),\ \mu \in S_E(\mu^m)$ where $2 \leq j \leq m$. As in the first part of this proof, $\mu^1 \in S_E(\bar{\mu})$ implies that $\mu^1 = \bar{\mu}$ (because $\bar{\mu}$ is globally strong uninvadable). Applying this argument repeatedly, we get $\mu^m = \bar{\mu}$. Now, $\mu \in S_E(\mu^m)$ gives $\mu \in S_E(\bar{\mu})$ which in turn yields $\mu = \bar{\mu}$. Therefore, $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$. □

## 4. ILLUSTRATIVE EXAMPLES

We provide three illustrative examples below.

*Example 1* Consider an $n$-person game where the strategy sets $A_1, \ldots, A_n$ are Polish spaces and the payoff functions are $U_i(a_i, a_{-i}) = r_i(a_i) + s_i(a_{-i}),\ i \in I = \{1, 2, \ldots, n\}$. Here $r_i : A_i \to \mathbb{R}$ and $s_i : A_{-i} \to \mathbb{R}$ are bounded and measurable functions. Let $r_i$ achieves its maximum exactly at $x_i \in A_i$ for all $i \in I$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) = (\delta_{x_1}, \ldots, \delta_{x_n})$.

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and $i \in I$, we have

$$\begin{aligned}J_i(\bar{\mu}_i, \mu_{-i}) &= \int_{A_i}\int_{A_{-i}} U_i(a_i, a_{-i})\ \mu_{-i}(da_{-i})\ \bar{\mu}_i(da_i)\\ &= \int_{A_i} r_i(a_i)\ \bar{\mu}_i(da_i) + \int_{A_{-i}} s_i(a_{-i})\ \mu_{-i}(da_{-i})\\ &= \int_{A_i} r_i(a_i)\ \delta_{x_i}(da_i) + \int_{A_{-i}} s_i(a_{-i})\ \mu_{-i}(da_{-i})\\ &= r_i(x_i) + \int_{A_{-i}} s_i(a_{-i})\ \mu_{-i}(da_{-i})\\ &\geq \int_{A_i} r_i(a_i)\ \mu_i(da_i) + \int_{A_{-i}} s_i(a_{-i})\ \mu_{-i}(da_{-i})\\ &= \int_{A_i}\int_{A_{-i}} (r_i(a_i) + s_i(a_{-i}))\ \mu_{-i}(da_{-i})\ \mu_i(da_i)\\ &= \int_{A_i}\int_{A_{-i}} U_i(a_i, a_{-i})\ \mu_{-i}(da_{-i})\ \mu_i(da_i)\\ &= J_i(\mu_i, \mu_{-i}). \qquad (9)\end{aligned}$$

It is clear that strict inequality holds in (9) when $\mu_i \neq \bar{\mu}_i = \delta_{x_i}$, and hence $\bar{\mu}$ is a globally strong uninvadable profile which in turn, by Theorem 11, is RAII and $S_{NM}(\bar{\mu}) = \{\bar{\mu}\}$ which is a Balkenborg and Schlag ES set by Lemma 7.

*Example 2* Consider the previous example with the change that $r_i : A_i \to \mathbb{R}$ achieves its maximum exactly at $x_i, y_i \in A_i$ for all $i \in I$. We show that $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) = (\delta_{x_1}, \ldots, \delta_{x_n})$ is a RAII profile.

We use the notations $\bar{\nu} = (\bar{\nu}_1, \ldots, \bar{\nu}_n) := (\delta_{y_1}, \ldots, \delta_{y_n})$ and $\mu_{\text{mix}} := (\alpha_1\delta_{x_1} + (1-\alpha_1)\delta_{y_1}, \ldots, \alpha_n\delta_{x_n} + (1-\alpha_n)\delta_{y_n})$ where $\alpha_i \in [0,1]$ for all $i \in I$.

In order to prove that $\bar{\mu}$ is a RAII profile, we need to show the following:

(i) $S_B(\bar{\mu}) = \phi$ and,
(ii) there does not exist $\mu^1, \ldots, \mu^m,\ m \geq 2$ such that $\mu^1 \in S_E(\bar{\mu}),\ \mu^j \in S_E(\mu^{j-1}),\ \mu^m \in S_B(\mu^{m-1})$ where $2 \leq j \leq m-1$.

If possible, let $S_B(\bar{\mu}) \neq \phi$. This (by the definition of $S_B(\bar{\mu})$) would give rise to two possibilities. In the first case, there would exist $\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and $i \in I$ such that $J_i(\mu_i, \bar{\mu}_{-i}) > J_i(\bar{\mu}_i, \bar{\mu}_{-i})$. This implies that $\int_{A_i} r_i(a_i)\ \mu_i(da_i) > r_i(x_i)$ which is a contradiction because $r_i$ achieves its maximum at $x_i$.
In the second case, there would be $\mu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and $i \in I$ such that $J_i(\mu_i, \bar{\mu}_{-i}) = J_i(\bar{\mu}_i, \bar{\mu}_{-i})$ and $J_i(\bar{\mu}_i, \mu_{-i}) < J_i(\mu_i, \mu_{-i})$. This also leads to a contradiction by similar arguments. Hence, $S_B(\bar{\mu}) = \phi$. Similarly, we can also prove that $S_B(\mu_{\text{mix}}) = \phi$.
To prove (ii), if possible, let there be $\mu^1, \ldots, \mu^m,\ m \geq 2$ such that $\mu^1 \in S_E(\bar{\mu}),\ \mu^j \in S_E(\mu^{j-1}),\ \mu^m \in S_B(\mu^{m-1})$ where $2 \leq j \leq m-1$. As $\mu^1 \in S_E(\bar{\mu}),\ \mu^1 = \mu_{\text{mix}}$. Similarly, $\mu^2 \in S_E(\mu^1)$ leads to $\mu^2 \in S_E(\mu_{\text{mix}})$ which in turn implies that $\mu^2 = \mu_{\text{mix}}$. Applying the same argument repeatedly, we get $\mu^{m-1} = \mu_{\text{mix}}$ and $\mu^m \in S_B(\mu^{m-1})$ which is a contradiction to the fact that $S_B(\mu_{\text{mix}}) = \phi$. This proves (ii) and hence the fact that $\bar{\mu}$ is a RAII profile.

Now, $S_{NM}(\bar{\mu})$ is the collection of all profiles $\mu_{\text{mix}}$ where $(\alpha_1, \ldots, \alpha_n)$ runs over $[0,1]^n$. Note that $S_{NM}(\bar{\mu})$ can also be viewed as the Cartesian product of line segments joining $\bar{\mu}_i$ and $\bar{\nu}_i$; $i = 1, 2, \ldots, n$. As $\bar{\mu}$ is RAII, by Lemma 7, the set $(S_{NM}(\bar{\mu}))$ of its indirect neutral mutants is a Balkenborg and Schlag ES set. It is worth mentioning that as $\bar{\mu}$ is RAII then so is $\mu_{\text{mix}}$ and $S_{NM}(\bar{\mu}) = S_{NM}(\mu_{\text{mix}})$ using Remark 4.

*Example 3* Consider the $n$-person game discussed in previous examples with a change that $r_i : A_i \to \mathbb{R}$ achieves its maximum exactly at $x_i^1, x_i^2, \ldots, x_i^{k_i} \in A_i$ for all $i \in I$. By similar arguments as Example 2, it is clear that $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n) = (\delta_{x_1^1}, \delta_{x_2^1}, \ldots, \delta_{x_n^1})$ is a RAII profile and $S_{NM}(\bar{\mu})$ is the set of all profiles $\mu^{\text{mix}}$ where $(\mu^{\text{mix}})_i := \alpha_i^1\delta_{x_i^1} + \cdots + \alpha_i^{k_i}\delta_{x_i^{k_i}},\ \alpha_i^j \in (0,1]$ for $j \in \{1, \ldots, k_i\}$, $\sum_{j=1}^{k_i} \alpha_i^j = 1$ and $i \in I$. Now, by Remark 4, $\mu^{\text{mix}} \in S_{NM}(\bar{\mu})$ is RAII and $S_{NM}(\bar{\mu}) = S_{NM}(\mu^{\text{mix}})$.

In continuation of this example, we consider the problem of the use of common property resources as discussed in Example 2.12 of Aliprantis and Chakrabarti (2012) and Example 6.2.3 of Hingu (2017) with respect to a common utility function of the players involved but here we assume asymmetry in the utility functions of the players. One of the problems of the use of common property resources is the imbalance in ecology and the depletion of fish stocks and other marine species due to the overfishing of the fishing grounds of the world (a common resource property) by different countries. Assume that there are $n$ countries/players who have the permit to fish in the open seas of the world and every country attains some of this common resource. This situation can be modelled by a $n$-person normal form continuous game. Let $\hat{A} = \sum_{i=1}^n a_i$ be the total attained resource where $a_i$ is the resource attained by the $i$th country and $v_i : (0,\infty) \to (0,\infty)$ be the utility function (which is assumed to be concave) for the $i$th country. The cost of attaining a certain $a_i$ amount of the resource by the $i$th country depends on the attained amount of resource $a_i$ and also on the amount of resource attained by other countries which is given by $\hat{A} - a_i$. For simplification, the cost function $c : (0,\infty) \times (0,\infty) \to (0,\infty)$ is assumed to take the form:

$$c(a_i, \hat{A} - a_i) = h(a_i) + g(\hat{A} - a_i)$$

where $h, g : (0,\infty) \to (0,\infty)$ are both convex. In addition, we assume that the marginal cost at zero is strictly less than the marginal utility at zero. This relationship is given by

$$\lim_{a_i \to 0^+} \frac{d}{da_i} h(a_i) < \lim_{a_i \to 0^+} \frac{d}{da_i} v_i(a_i).$$

The profit of the country $i$ by the attainment of $a_i$ amount of the common resource is the quantity $v_i(a_i) - h(a_i)$ and the payoff function is defined as $U_i(a_i, a_{-i}) = v_i(a_i) - h(a_i) - g(\hat{A} - a_i)$. In the absence of a restricting or governing body, the countries would want to attain as much resource as required to gain maximum/desired profit but this may lead to a situation in which some countries may not get enough resource or the resource may be over depleted to ever get replenished. In order to escape such a scenario, the governing body can put some restrictions on the amount of the resource to be attained or on the profit gained by attaining a certain amount of the resource.

Let us consider a special case of this situation where only two countries are involved in this competition, i.e. $i \in I = \{1, 2\}$. The utility functions for both the players are given as $v_1(a_1) = \sqrt{a_1}+a_1$ and $v_2(a_2) = \sqrt{a_2}+0.5$. The cost functions are $h(a_i) = g(a_i) = a_i^2$. It is clear that for $i \in I$, $v_i$ is concave function, $h, g$ are convex functions and the marginal cost at zero is strictly less than the marginal utility at zero for these functions. The payoffs for both the countries/players are as follows:

$$U_1(a_1, a_2) = \sqrt{a_1} + a_1 - a_1^2 - a_2^2,$$
$$U_2(a_2, a_1) = \sqrt{a_2} + 0.5 - a_2^2 - a_1^2.$$

Note that this game is a part of Example 2 where $r_i(a_i) = v_i(a_i) - h(a_i)$, $s_i(a_{-i}) = -g(\hat{A} - a_i)$. The maximum value of the profit gained by each country is 0.8 due to the restriction by the government. Now, the maximum value 0.8 of $r_1$ is achieved at the points $x_1^1 = 0.3337, x_1^2 = 1.2553$ and the maximum value 0.8 of $r_2$ is achieved at the points $x_2^1 = 0.0956, x_2^2 = 0.7538$. Let $\mu^{\text{mix}}$ be of the form discussed in the beginning of this example for $k_i \in \{1, 2\}$ and $i \in I = \{1, 2\}$, then $\mu^{\text{mix}}$ is RAII. Moreover, $S_{NM}(\mu^{\text{mix}})$ is the set of those profiles whose $i$th components are convex combinations of $x_i^1$ and $x_i^2$. By Lemma 7, $S_{NM}(\mu^{\text{mix}})$ is a Balkenborg and Schlag evolutionarily stable set.

## 5. CONCLUSION

The paper adopts the concepts of evolutionarily equal and better performers and show that they can be used to define RAII profiles for asymmetric games with continuous strategy space. The equivalence between 'strong Thomas' and 'Balkenborg and Schlag' ES sets is shown to be true for these games. The main result (Theorem 10) establishes the fact that a set of profiles is a minimal evolutionarily stable set iff it is the set of indirect neutral mutants of a RAII profile. Furthermore, a globally strong uninvadable profile is proved to be RAII and in this case the set of indirect neutral mutants is singleton. These results are illustrated using three examples. The main theorem (Theorem 10) motivates a deeper study of the dynamic stability properties of ES sets in asymmetric games with continuous strategy space as these properties would provide more insight about RAII profiles and their indirect neutral mutants.

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