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H_∞-synthesis and control of uncertain fractional-order systems of commensurate type ⋆

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Abstract: New Linear-Matrix-Inequality (LMI) conditions are proposed for H_∞ analysis and synthesis of *uncertain* fractional-order systems where the non-integer order of differentiation belongs to the set $]0\ 2[$. The developed conditions are *extended* LMI conditions involving additional LMI variables needed for numerical calculation of the feedback gains. The stability conditions are embedded with the necessary H_∞ LMI conditions leading to new formulation of the bounded-real-lemma result. The stabilizability conditions with H_∞ performance are subsequently derived and tested with static-pseudo-state feedbacks and static-output feedbacks as well.

Keywords: Fractional uncertain systems; H_∞ control; Stability and stabilizability; Static-output feedback.

1. INTRODUCTION

Stability and stabilizability of fractional-order systems constitute one of the major concerns for scientists and engineers alike. In the past decades, great efforts have been devoted to the analysis and synthesis of fractional-order models of different types, see e.g., Manabe (1960), Oustaloup (1983), Oustaloup (1995); Matignon (1996); Farges et al. (2010); Trigeassou et al. (2011); Oustaloup (2014); Sabatier et al. (2007). Additionally, the use of convex-optimization tools in characterization, analysis, and synthesis of non-integer models has tremendously contributed to the development of straightforward numerical methods that enhance the theory and the practice related to this special class of dynamical systems, see e.g., Farges et al. (2010) and the references therein.

Robust stability and stabilizability of dynamical systems subject to unknown parameters and external-input uncertainties have received a great deal of interest by control theorists and practitioners, see e.g., Zhang et al. (2013), Lan et al. (2012), Lan and Zhou (2013), Lan and Zhou (2011), Ibrir and Bettayeb (2015), Ibrir (2017). The H_∞ analysis method is one of those methods that provides a good characterization of the H_∞ norms of transfer functions relating some inputs or disturbances to some specific outputs. By minimizing these norms, we utterly aim to measure and improve the system robustness against eventual undesirable external inputs. As a matter of fact, even the H_∞ analysis and synthesis methods are well developed for integer-order dynamical models, there are some difficulties in its generalization to fractional-order systems of commensurate and non-commensurate type. Preliminary results on this subject could be traced in the references Mose et al. (2008), Fadiga et al. (2011), Farges et al. (2013), Shen and Lam (2014).

A nice introduction to H_∞ analysis and synthesis of fractional-order systems of *uncertain* commensurate type are provided in Mose et al. (2008), Fadiga et al. (2011), Farges et al. (2013), Shen and Lam (2014). The list is not exhaustive, and the reader may find other contributions that concern other classes of systems. To the best of our knowledge, the H_∞ analysis and synthesis of uncertain fractional-order systems with stability requirement has been discussed for all possible values of the non-integer differentiation orders. In this note, new extended LMI conditions are proposed for H_∞ analysis and synthesis of fractional-order systems. The developed LMI conditions, are subsequently exploited for the design of numerically tractable conditions for controller design with H_∞ performance. The stabilizability results are presented with both static pseudo-state feedbacks and static-output feedbacks as well. The presented results are seen as a generalization of the bounded-real-lemma result when uncertainties are randomly distributed in the state and the output matrices. Thanks to the extended nature of the proposed LMI conditions, the simultaneous check of stability along with the estimation of the H_∞ bound are made possible with decoupled-variable conditions and hence, more flexibility is introduced in the design of H_∞ controllers for this specific class of fractional-order systems.

2. PRELIMINARY RESULTS

Throughout this paper, we note by $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{C}$ the set of real numbers, the set of positive real numbers, and the set of complex numbers, respectively. In all the paper statements, the parameter "r" is denoting the complex $r = e^{j(1-\alpha)\frac{\pi}{2}}$ where α is a non-integer positive real.

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The notation $A > 0$, with A being a Hermitian matrix (respectively, $A < 0$), means that the matrix A is positive definite (respectively, negative definite). $P^\star$ denotes the conjugate transpose of the complex matrix P. The star entry in a matrix stands for the element induced by the corresponding conjugate transpose. The notation $\mathrm{Her}(P)$ denotes the sum $P + P^\star$. A' is the matrix transpose of A. The $\mathscr{L}_2$ norm of a square integrable function $\varphi(t)$ over the interval $[0\ t]$, noted $\|\varphi\|_{\mathscr{L}_2}$, is defined as $\int_0^t \varphi^2(s)ds$. $\|G\|_\infty$ denotes the H_∞ norm of the transfer function $G(s)$. The $\mathrm{spec}(A)$ denotes the set of the eigenvalues of the matrix A. We note by I and $\mathbf{0}$ the identity matrix of appropriate dimension and the null matrix of appropriate dimension, respectively. $\Re(Z)$ stands for the real part of the complex matrix/number while $\Im(Z)$ denotes the imaginary part of the complex matrix/number. The notation $\bar{X}$ stands for the conjugate of the complex X. Recall that a Hermitian matrix $Q > 0$ iff the following holds true:

$$\begin{pmatrix} \Re(Q) & \Im(Q) \\ -\Im(Q) & \Re(Q) \end{pmatrix} > 0. \tag{1}$$

2.1 Problem statement

Consider the fractional-order system represented by the pseudo-state formulation:

$$\begin{aligned} {}_a^{RL}D_t^\alpha\, x &= A(\theta)\, x + B_1\, u + B_2 \xi, \\ y &= C(\theta) x + D\, \xi, \end{aligned} \tag{2}$$

where $x = x(t)$ is the pseudo-state vector, $u = u(t) \in \mathbb{R}^m$ is the control input, $\xi = \xi(t) \in \mathbb{R}^q$ is an external bounded input that may not be measured, and $y = y(t) \in \mathbb{R}^p$ is the system measured output. The operator ${}_a^{RL}D_t^\alpha$ stands for the Riemann-Liouville fractional differentiation with respect to time. Recall that the fractional integral of a continuously differentiable function $f(t)$ is defined by Samko et al. (1987):

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a \tag{3}$$

where, $\alpha \in \mathbb{R}_{>0}$ denotes the fractional-integration order, and

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1}\, dx. \tag{4}$$

The order α Riemann-Liouville fractional derivative of a function $f(t)$, with $\alpha \in \mathbb{R}_{>0}$, is consequently defined by:

$$\begin{aligned} {}_a^{RL}D_t^\alpha\, f(t) &= \frac{d^{\bar{m}}}{dt^{\bar{m}}} \left(I^{\bar{m}-\alpha} f(t) \right) \\ &= \frac{1}{\Gamma(\bar{m}-\alpha)} \left(\frac{d}{dt} \right)^{\bar{m}} \int_a^t (t-\tau)^{\bar{m}-\alpha-1} f(\tau)\, d\tau; \quad t > a. \end{aligned} \tag{5}$$

where $\bar{m}$ is the smallest integer verifying "$\bar{m} - 1 \leq \alpha < \bar{m}$." Similarly, the Caputo-fractional derivative of order α is defined as

$${}_a^{C}D_t^\alpha\, f(t) = \frac{1}{\Gamma(\bar{m}-\alpha)} \int_a^t (t-\tau)^{\bar{m}-\alpha-1} \left(\frac{d}{dt} \right)^{\bar{m}} f(\tau)\, d\tau. \tag{6}$$

In the sequel and for simplicity of notation, the Riemann-Liouville fractional derivative operator is simply noted D^α. The real-valued matrices $A(\theta) \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times q}$, $C(\theta) \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times q}$ are dependent on a vector of parameters θ that is assumed to be constant and bounded. Additionally, it is assumed that $A(\theta) \in \mathscr{A}$, $C(\theta) \in \mathscr{C}$ where $\mathscr{A}$ and $\mathscr{C}$ are polytopes of $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times p}$ defined as follows:

$$\begin{aligned} \mathscr{A} = \Big\{ A(\theta) \in \mathbb{R}^{n\times n} :\ & A(\theta) = \sum_{i=1}^{N} \tau_i\, A_i, \\ & \tau_i \geq 0, \forall i \in \{1, \cdots, N\}, \quad \sum_{i=1}^{N} \tau_i = 1 \Big\}, \\ \mathscr{C} = \Big\{ C(\theta) \in \mathbb{R}^{p\times n} :\ & C(\theta) = \sum_{i=1}^{M} \sigma_j\, C_j, \\ & \sigma_j \geq 0, \forall i \in \{1, \cdots, M\}, \quad \sum_{j=1}^{M} \sigma_j = 1 \Big\}. \end{aligned} \tag{7}$$

It is important to mention that the initial condition $x(t_0)$ is not sufficient to evaluate the future state of the system. Therefore, the vector $x = x(t)$ does not strictly represent the true state of the system. As a consequence, the nomenclature of a pseudo-state is utilized.

2.2 H_∞ norm of uncertain commensurate fractional-order systems

For $u = 0$, the transfer function of the system from the input ξ to the output y is given by $G(s) = C(\theta)(s^\alpha I - A(\theta))^{-1} B_2 + D$ and the H_∞ norm of $G(j\omega)$ is defined as

$$\|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)), \tag{8}$$

where $\bar{\sigma}(G(j\omega))$ denotes the largest singular value of $G(j\omega)$ at the frequency ω. In time domain,

$$\|G\|_\infty = \sup_{\xi(t) \neq 0} \frac{\|y(t)\|_{\mathscr{L}_2}}{\|\xi(t)\|_{\mathscr{L}_2}}. \tag{9}$$

In case where the system is not presenting any uncertainty, a linear-matrix-inequality approach to the determination of an upper bound of the H_∞ norm is discussed in Mose et al. (2008). This result is known as the bounded real lemma for fractional-order systems. Based on this result, a generalization of this Lemma for uncertain systems is summarized in the following result.

Lemma 1. The $\mathscr{L}_2$-gain of the fractional-order system (2) is bounded by γ for $u = 0$, that is, $\|G\|_\infty < \gamma$ if there exists a Hermitian matrix $P = P^\star \in \mathbb{C}$, such that the set of the following LMIs hold:

$$\begin{aligned} \Upsilon_{i,j} = \begin{pmatrix} \mathrm{Her}(rPA_i) & PB_2 & \bar{r}C_j' \\ \star & -\gamma I & D' \\ \star & \star & -\gamma I \end{pmatrix} < 0, \quad \forall i, \forall j, \\ 1 \leq i \leq N, \quad 1 \leq j \leq M. \end{aligned} \tag{10}$$

Proof. The proof of this Lemma is straightforward by showing $\sum_{i=1}^{N} \sum_{j=1}^{M} \tau_i \sigma_j \Upsilon_{i,j} < 0$, and therefore, the result provided in Mose et al. (2008) becomes fulfilled. This ends the proof. ■

As a matter of fact, the bounded real Lemma, provided in Mose et al. (2008), characterises only the upper bound of the H_∞ norm of the transfer function and it does not guarantee the system stability in the general case. Consequently, the stabilizability synthesis using the result of

Lemma 1 may not be possible for certain stable fractional-order systems. Recall that depending on the value of the fractional-differentiation order "α", several stability theorems have been stated, see the results in Matignon (1996) for $0 < \alpha < 1$ and Sabatier et al. (2010) for $1 < \alpha < 2$. We would rather recall these results for sake of clarity.

Theorem 2. Moze et al. (2005) Let $A \in \mathbb{R}^{n\times n}$ be a real matrix. Then, the fractional-order system:

$$D^{\alpha}x(t) = A\,x(t), \;\; 1 < \alpha < 2, \tag{11}$$

is asymptotically stable, that is, $|\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2}$ if and only if there exists a symmetric and positive definite matrix P verifying

$$\begin{bmatrix} (AP+PA')\sin(\psi) & (AP-PA')\cos(\psi) \\ \star & (AP+PA')\sin(\psi) \end{bmatrix} < 0 \tag{12}$$

where $\psi = (1-\frac{\alpha}{2})\pi$.

According to this result, it can be inferred that the pseudo-state system: $D^{\alpha}x = A\,x$; $(1 \le \alpha < 2)$ is asymptotically stable iff there exists a real matrix $P > 0$ such that

$$rPA' + \bar{r}AP < 0, \quad r = e^{j(1-\alpha)\frac{\pi}{2}}. \tag{13}$$

Condition (13) is equivalent to condition (12). This can be easily proved using (1). In Farges et al. (2010), it is shown that the stability of commensurate fractional-order systems of the form: $D^{\alpha}x = A\,x$; $0 < \alpha < 1$, is dependent on the solution of the following convex optimization problem:

$$X = X^{\star} \in \mathbb{C}, X > 0, \quad A(rX+\bar{r}\bar{X}) + (rX+\bar{r}\bar{X})'A' < 0. \tag{14}$$

Note that the matrix $rX+\bar{r}\bar{X}$ is always real and it is not necessarily positive definite. Based on these results, the H_∞ synthesis with stability requirements can be generalized for uncertain polytopic fractional-order systems. We summarize the second result of this paper in the following statement.

Theorem 3. The fractional-order system (2) is stable for $u = 0$, $\xi = 0$ and the norm of the transfer function, from the input ξ to the output y, is less than γ for $u = 0$ and non-null ξ if the following hold true:

- For $0 < \alpha < 1$: $\exists\; X > 0$; $X \in \mathbb{C}^{n\times n}$ and $\exists\, P = P^{\star} \in \mathbb{C}^{n\times n}$ such that

$$\begin{pmatrix} \mathrm{Her}(rPA_i) & PB_2 & \bar{r}C_j' \\ \star & -\gamma I & D' \\ \star & \star & -\gamma I \end{pmatrix} < 0, \quad \forall i, \forall j, \qquad 1 \le i \le N, \quad 1 \le j \le M. \tag{15}$$

$$X > 0, \; A_i(rX+\bar{r}\bar{X}) + (rX+\bar{r}\bar{X})'A_i' < 0, \qquad 1 \le i \le N.$$

- For $1 \le \alpha < 2$: $\exists\, Q = Q' \in \mathbb{R}^{n\times n}$ and $\exists\, P = P^{\star} \in \mathbb{C}^{n\times n}$ such that:

$$\begin{pmatrix} \mathrm{Her}(rPA_i) & PB_2 & \bar{r}C_j' \\ \star & -\gamma I & D' \\ \star & \star & -\gamma I \end{pmatrix} < 0, \quad \forall i, \forall j, \qquad 1 \le i \le N, \quad 1 \le j \le M. \tag{16}$$

$$Q > 0, \; rQA_i' + \bar{r}A_iQ < 0, \quad 1 \le i \le N.$$

The LMIs (15) and (16) could be restrictive in the general case because the matrices P and Q should be found to satisfy all the conditions for the different convex-hull matrices A_i; $1 \le i \le N$, C_j; $1 \le j \le M$. The objective of this paper is three folds. The first is to set new LMI conditions that are less restrictive and the second one is to decouple the state matrix $A(\theta)$ from the matrices P and Q so as we can design state feedbacks with real gains. The last fold is to gather the stability conditions with the H_∞ LMI conditions in one unified condition. Since the system is uncertain, it is not possible to evaluate explicitly the true value of $\|G\|_\infty$; therefore, it is always desirable to know the minimum value of the upper bound of $\|G\|_\infty$ and hence, an estimate of the true value may be found. For instance, we start by giving the sufficient conditions for the open-loop system stability with H_∞ performance.

Theorem 4. Consider the commensurate fractional-order system (2) with $u = 0$ and $\alpha \in [1\;2[$. Let $G(s)$ be the transfer function from the input ξ o the output y. The system is stable with $\|G\|_\infty < \gamma$ if there exist a set of $n \times n$ symmetric and positive-definite matrices $P_1, \cdots, P_N$, a complex hermitian matrix $\mathbb{F} \in \mathbb{C}^{n\times n}$ and a real matrix $\mathbb{G} \in \mathbb{R}^{n\times n}$ such that the optimization problem (17) (see next page) has a solution for some $\gamma > 0$.

Proof. When conditions (17) are all satisfied then, $\sum_{i=1}^{N}\sum_{j=1}^{M}\tau_i\sigma_j\mathscr{W}_{i,j} < 0$. This immediately implies that the matrix:

$$\begin{pmatrix} rA(\theta)\mathbb{F} + \bar{r}\mathbb{F}^{\star}A'(\theta) & r\sum_{i=1}^{N}\tau_i\,P_i - \bar{r}\mathbb{F}^{\star} + A(\theta)\mathbb{G} \\ \star & -\mathbb{G}-\mathbb{G}' \end{pmatrix} \tag{18}$$

is negative definite. Consequently,

$$\begin{pmatrix} I & A(\theta) \end{pmatrix} \begin{pmatrix} rA(\theta)\mathbb{F} + \bar{r}\mathbb{F}^{\star}A'(\theta) & r\sum_{i=1}^{N}\tau_i\,P_i - \bar{r}\mathbb{F}^{\star} + A(\theta)\mathbb{G} \\ \star & -\mathbb{G}-\mathbb{G}' \end{pmatrix} \begin{pmatrix} I \\ A'(\theta) \end{pmatrix} < 0. \tag{19}$$

The last inequality is nothing but

$$\mathrm{Her}\Big(rP_\tau\,A'(\theta)\Big) < 0; \quad P_\tau = \sum_{i=1}^{N}\tau_i\,P_i, \tag{20}$$

which translates the stability condition of the fractional-order system for $1 \le \alpha < 2$. The second part of the proof consists in showing that $\|G\|_\infty < \gamma$. Starting from the fact that $\mathscr{W}_{i,j} < 0$; $\forall i, \forall j$. Then,

$$\begin{aligned} &\sum_{i=1}^{N}\sum_{j=1}^{M}\tau_i\sigma_j \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix}' \mathscr{W}_{i,j} \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M}\tau_i\sigma_j \begin{pmatrix} \mathrm{Her}(rA_i\mathbb{F}) & B_2 & \bar{r}\mathbb{F}C_j' \\ \star & -\gamma I & D' \\ \star & \star & -\gamma I \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Her}(rA(\theta)\mathbb{F}) & B_2 & \bar{r}\mathbb{F}C'(\theta) \\ \star & -\gamma I & D' \\ \star & \star & -\gamma I \end{pmatrix} < 0. \end{aligned} \tag{21}$$

The last inequality is a direct consequence of the result of the bounded real Lemma for fractional-order systems that

$$\min_{P_1,\cdots,P_N,\mathbb{F},\mathbb{G}} \gamma$$
$$\text{subject to: } \mathscr{W}_{i,j} = \begin{pmatrix} rA_i\mathbb{F} + \bar{r}\mathbb{F}^\star A_i' & rP_i - \bar{r}\mathbb{F}^\star + A_i\mathbb{G} & B_2 & \bar{r}\mathbb{F}C_j' \\ \star & -\mathbb{G} - \mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{17}$$

has been largely discussed in Mose et al. (2008). This ends the proof. ■

The result given by Theorem 4 is seen as an alternative to the result of the bounded real Lemma discussed Mose et al. (2008). In case where the non-integer differentiation order α belongs to the set $]0\ 1[$, a similar result is formulated in the following statement.

Theorem 5. Consider the commensurate fractional-order system (2) with $u = 0$ and $\alpha \in]0\ 1[$ and define $G(s)$ as the transfer function from the input ξ o the output y. The system is stable with $\|G\|_\infty < \gamma$ if there exist a set of positive definite hermitian matrices $X_1, \cdots, X_N$, a complex hermitian matrix $\mathbb{F} \in \mathbb{C}^{n\times n}$ and a real matrix $\mathbb{G} \in \mathbb{R}^{n\times n}$ such that the optimization problem (22) (see next page) has a solution for some $\gamma > 0$.

Proof. Assume that the set of LMIs (22) are fulfilled. Then, for all possible values of i, the matrix:

$$\begin{pmatrix} rA_i\mathbb{F} + \bar{r}\mathbb{F}^\star A_i' & (rX_i + \bar{r}\bar{X}_i)' - \bar{r}\mathbb{F}^\star + A_i\mathbb{G} \\ \star & -\mathbb{G} - \mathbb{G}' \end{pmatrix} \tag{23}$$

is negative definite. As a consequence,

$$\begin{aligned} &\left(I \ A(\theta) \right) \\ &\sum_{i=1}^N \tau_i \begin{pmatrix} rA_i\mathbb{F} + \bar{r}\mathbb{F}^\star A_i' & (rX_i + \bar{r}\bar{X}_i)' - \bar{r}\mathbb{F}^\star + A_i\mathbb{G} \\ \star & -\mathbb{G} - \mathbb{G}' \end{pmatrix} \\ &\begin{pmatrix} I \\ A'(\theta) \end{pmatrix} \\ &= A(\theta)(rX_\tau + \bar{r}\bar{X}_\tau) + (rX_\tau + \bar{r}\bar{X}_\tau)'A'(\theta) < 0. \end{aligned} \tag{24}$$

where $X_\tau = \sum_{i=1}^N \tau_i X_i$. The fulfillment of inequality (24) implies the stability of the system for $u = 0$. By following the same proof outline of Theorem 5, one can conclude that the H_∞ norm of $G(s)$ is less than γ when the set of conditions (22) are met simultaneously. This is the end of the proof. ■

3. STABILIZABILITY WITH H_∞ ANALYSIS

3.1 Stabilization with static pseudo-state feedbacks

In the general case, the fractional-order system may not be stable. In this case, a pseudo-state feedback is needed to not only stabilize the system but to ensure certain robustness against additive disturbances. The previously developed conditions are utilized to set stabilizing controllers with the best H_∞ performance index. The design is summarized in the following statement.

Theorem 6. Consider the fractional-order model given by (2) where α may take non-integer values in the set $]0\ 2[$ and $G(s)$ being the transfer function from the input ξ to the output y. Then, for $0 < \alpha < 1$, the closed-loop system is stable under the feedback $u = Y\mathbb{G}^{-1}x$ for $\xi = 0$ and $\|G\|_\infty < \gamma$ for $\xi \neq 0$ if the optimization problem (25) (see next page) is solvable for some hermitian positive definite matrices $X_i \in \mathbb{C}^{n\times n}$; $1 \le i \le N$ and two real matrices $\mathbb{G} \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$. Furthermore, for $1 < \alpha < 2$, the fractional-order system (2) is stabilizable by the same feedback $u = Y\mathbb{G}^{-1}x$ with $\|G\|_\infty < \gamma$ for non-null ξ if the convex optimization problem (26) is solvable for a set of real positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$; $1 \le i \le N$, and real matrices $Y \in \mathbb{R}^{m\times n}$ and $\mathbb{G} \in \mathbb{R}^{n\times n}$.

Proof. The proof becomes straightforward by replacing in the LMI conditions (17) and (22) the state matrix $A(\theta)$ by the closed-loop matrix and the complex matrix $\mathbb{F}$ by $\mathbb{G}$. This ends the proof. ■

3.2 Stabilization with static output feedbacks

The static-output-feedback problem is generally stated as a non-convex optimization problem where numerically tractable solutions to this problem are often related to sufficient conditions that raise different types of conservatism. More precisely, the static-output stabilization problem is often formulated as bilinear matrix inequalities due to the coupling nature of matrix variables. Elegant and straightforward solutions to the static-output-feedback problems have been proposed for both continuous-time and discrete-time-integer-order linear systems, see e.g., Boyd et al. (1994), Boyd and Vandenberghe (2004), Ebihara et al. (2015), Dong and Yang (2013), Prempain and Postlethwaite (2001), Crusius and Trofino (1999), Ghaoui et al. (1997) and the references therein. The static-output feedback problem continuous to be a challenging issue when other additional requirement are imposed. In this part of the paper, the H_∞ control of uncertain fractional-order systems is investigated with simple static feedbacks. The complete design is clarified in the following statement.

Theorem 7. Consider the fractional-order model given by (2) where α belongs to $]0\ 2[$ and $G(s)$ being the transfer function defined from the input ξ to the output y. Then, for $0 < \alpha < 1$, the closed-loop system is stable under the output feedback $u = Y\boldsymbol{N}^{-1}y$ for $\xi = 0$ with guaranteed bound $\|G\|_\infty < \gamma$ for $\xi \neq 0$ if the optimization problem (27) is solvable for some hermitian positive definite matrices $X_i \in \mathbb{C}^{n\times n}$; $1 \le i \le N$ and three real matrices $\mathbb{G} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times p}$ and $\boldsymbol{N} \in \mathbb{R}^{p\times p}$. Furthermore, for $1 < \alpha < 2$, the fractional-order system (2) is stabilizable by the same feedback $u = Y\boldsymbol{N}^{-1}y$ with $\|G\|_\infty < \gamma$ for non-null ξ if the convex optimization problem (28) (see next page) is solvable for a set of real positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$; $1 \le i \le N$, and arbitrary real matrices $Y \in \mathbb{R}^{m\times p}$, $\mathbb{G} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{N} \in \mathbb{R}^{p\times p}$.

Proof. The outline of the proof of this Theorem is similar to the proof of Theorem 6 after imposing the equality constraint $C\mathbb{G} = \boldsymbol{N}C$. This ends the proof. ■

$$\min_{X_1,\cdots,X_N,\mathbb{F},\mathbb{G}} \gamma$$
$$\text{subject to: } \begin{pmatrix} rA_i\mathbb{F}+\bar{r}\mathbb{F}^\star A_i' & (rX_i+\bar{r}\bar{X}_i)'-\bar{r}\mathbb{F}^\star+A_i\mathbb{G} & B_2 & \bar{r}\mathbb{F}C_j' \\ \star & -\mathbb{G}-\mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{22}$$

$$\min_{X_1,\cdots,X_N,\mathbb{G},Y} \gamma$$
$$\text{subject to: } \begin{pmatrix} \mathrm{Her}(rA_i\mathbb{G}+rB_1Y) & (rX_i+\bar{r}\bar{X}_i)'-\bar{r}\mathbb{G}'+A_i\mathbb{G}+B_1Y & B_2 & \bar{r}\mathbb{G}C_j' \\ \star & -\mathbb{G}-\mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{25}$$

$$\min_{P_1,\cdots,P_N,\mathbb{G},Y} \gamma$$
$$\text{subject to: } \begin{pmatrix} \mathrm{Her}(rA_i\mathbb{G}+rB_1Y) & rP_i-\bar{r}\mathbb{G}'+A_i\mathbb{G}+B_1Y & B_2 & \bar{r}\mathbb{G}C_j' \\ \star & -\mathbb{G}-\mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{26}$$

$$\min_{X_1,\cdots,X_N,\mathbb{G},Y} \gamma$$
$$\text{subject to: } C\mathbb{G} = \boldsymbol{N}C,$$
$$\begin{pmatrix} \mathrm{Her}(rA_i\mathbb{G}+rB_1YC) & (rX_i+\bar{r}\bar{X}_i)'-\bar{r}\mathbb{G}'+A_i\mathbb{G}+B_1YC & B_2 & \bar{r}\mathbb{G}C_j' \\ \star & -\mathbb{G}-\mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{27}$$

$$\min_{P_1,\cdots,P_N,\mathbb{G},Y} \gamma$$
$$\text{subject to: } C\mathbb{G} = \boldsymbol{N}C,$$
$$\begin{pmatrix} \mathrm{Her}(rA_i\mathbb{G}+rB_1YC) & rP_i-\bar{r}\mathbb{G}'+A_i\mathbb{G}+B_1YC & B_2 & \bar{r}\mathbb{G}C_j' \\ \star & -\mathbb{G}-\mathbb{G}' & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma I & D' \\ \star & \star & \star & -\gamma I \end{pmatrix} < 0, \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{28}$$

3.3 Illustrative example

Consider the uncertain fractional-order system:

$$\begin{aligned} D^{1.15}x &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & \theta_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xi, \\ y &= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & \theta_2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xi, \quad x(0) \neq 0. \end{aligned} \tag{29}$$

where θ_1 and θ_2 are constant uncertain parameters having the following bounds: $-3.18 \le \theta_1 \le -3$ and $1 \le \theta_2 \le 1.2$. The objective here is to illustrate the design given by the statement of Theorem 7 to set up a static-output feedback that stabilizes the unstable fractional-order system (29) with a minimum H_∞ bound γ. The LMIs (28) are found feasible for the domains of variation of the parameters θ_1 and θ_2 where

$$\begin{aligned} P_1 = P_2 &= \begin{pmatrix} 1253/2857 & -317/3401 & -148/491 \\ -317/3401 & 1218/1619 & 438/637 \\ -148/491 & 438/637 & 1956/1043 \end{pmatrix}, \\ \mathbb{G} &= \begin{pmatrix} 1150/3169 & 0 & 0 \\ \star & 1150/3169 & 0 \\ \star & \star & 1150/3169 \end{pmatrix}, \\ \boldsymbol{N} &= \begin{pmatrix} 1150/3169 & 0 \\ 0 & 1150/3169 \end{pmatrix}, \\ Y &= \begin{pmatrix} -2103/1675 & -1297/1001 \end{pmatrix}, \gamma = 1646/501. \end{aligned} \tag{30}$$

The linear matrix inequalities subject to equality constraints continue to be fulfilled with $\alpha = 1.25$ and wider ranges of variations of the uncertain parameters θ_1 and θ_2; that are: $-4.18 \le \theta_1 \le -3$ and $1 \le \theta_2 \le 1.3$. However, the minimum value of γ verifying (28) has increased which is not surprising because the H_∞ norm of the system transfer function may increase with the non-integer differentiation order. Of course, there is a limitation for the feasibility of the LMIs when the parameter variations are notably high.

4. CONCLUSION

For different values of the non-integer differentiation order, new LMI conditions for H_∞ control and analysis of uncertain fractional-order systems of commensurate type are developed. The newly developed conditions permit the estimation of the upper bound of the system H_∞ norm when uncertainties are distributed in the state and the output matrices. The extended nature of the proposed LMI conditions permit latterly to shape numerically tractable conditions for stabilizability by means of static state and output feedbacks.

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