Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Nonsmooth stabilization and its computational aspects

Pavel Osinenko, Patrick Schmidt, Stefan Streif *

* *Technische Universität Chemnitz, Automatic Control and System Dynamics Laboratory, Germany (e-mail: p.osinenko@gmail.com, {patrick.schmidt, stefan.streif}@etit.tu-chemnitz.de)*

Abstract: This work has the goal of briefly surveying some key stabilization techniques for general nonlinear systems, for which, as it is well known, a smooth control Lyapunov function may fail to exist. A general overview of the situation with smooth and nonsmooth stabilization is provided, followed by a concise summary of basic tools and techniques, including general stabilization, sliding-mode control and nonsmooth backstepping. Their presentation is accompanied with examples. The survey is concluded with some remarks on computational aspects related to determination of sampling times and control actions.

Keywords: Stabilization, nonsmooth analysis, control Lyapunov function

1. INTRODUCTION

Nonsmooth tools play central role in nonlinear system stabilization theory as the smooth ones are prone to limitations posed by the celebrated work of Brockett (1983). He demonstrated that even such simple systems, as a three-wheel robot, do not admit a smooth control Lyapunov function (CLF). A particular consequence of this fact is that there can be no continuous control law, which depends only on the system's state and which parks the robot into the desired position. The alternatives are either to consider a *time-varying* (or dynamical) control law or to give up the continuity condition. The former approach received great attention in the 80s and 90s (Aeyels, 1985; Kawski, 1989; Coron and d'Andrea Novel, 1991; Samson, 1991; Pomet, 1992; Coron, 1992; Coron and Pomet, 1993; Coron and Rosier, 1994; Coron, 1995; Khaneja and Brockett, 1999; Morin et al., 1999). Although the design of time-varying control laws may happen to be somewhat involved, the great advantage of this approach is that it requires the usual analysis tools and the closed-loop system trajectory is a classical Carathéodory solution which enjoys uniqueness properties. Contrary to this approach, if the continuity condition of the control law is omitted, care must be taken when defining what a system trajectory actually is. That is where alternative solutions to the respective initial value problems with discontinuous right-hand side come into play. One of the most well-known is the Filippov solution (Filippov, 1988). In brief, it is an absolutely continuous function that satisfies the said initial value problem with the right-hand side interpreted in the sense of a *differential inclusion* of the kind $\dot{x} \in F(x,t)$, where F is a *set-valued map*. Sliding-mode control (SMC) makes wide use of Filippov solutions (Slotine and Li, 1991; Young et al., 1996; Fridman and Levant, 1996; Fridland and Levant, 1999; Perruquetti and Barbot, 2002). A particular drawback of this kind of a solution in the context of stabilization is that the very same Brockett's conditions apply just as if one were to limit to the classical solutions (Coron and Rosier, 1994; Ryan, 1994). Further solutions include the ones in the sense of Hermes, Krasovskii, Sentis etc. whose good overview was done by Cortes (2008).

Of particular interest for this survey is the setting of *sample-and-hold* (S&H) solutions which are very simple to interpret. For the original nonlinear system

$$\dot{x} = f(x,u), x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{Sys}$$

if one designs a discontinuous static control law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, instead of applying κ literally, one can consider its *digital implementation* with a sampling time $\delta > 0$ in the form

$$\begin{aligned} &\dot{x} = f(x, u_k), \\ &t \in [k\delta, (k+1)\delta], u_k \equiv \kappa(x(k\delta)), k \in \mathbb{N}. \end{aligned} \tag{Sys-SH}$$

In this case, if f is locally Lipschitz w. r. t. x, classical trajectories of (Sys-SH) exist (at least locally) and are unique, since f is also measurable in t, precisely due to the sample nature of the control. The only property one has to give up in general when stabilizing (Sys) in the S&H mode, is asymptotic stabilizability of the closed loop. Instead, one has *practical stabilizability* in the sense of

Definition 1. (Practical stabilizability). A control law κ is said to practically stabilize (Sys) in the sense of S&H, i. e., (Sys-SH) if, for any data $R > r > 0$, there is a sufficiently small $\delta > 0$ such that any closed-loop trajectory $x(t), t \geq 0, x(0) \in \mathcal{B}_R$ is bounded and enters and stays in the ball $\mathcal{B}_r$ within a time T depending uniformly on R, r.

So, by properly selecting the sampling time, one can achieve any desirable precision of stabilization. Thus, S&H scenario of stabilization can be justified from both the implementation and usefulness sides.

A brief summary of the discussed stabilization methods can be found in Table 1. The focus of this survey is set to discontinuous static control laws in their S&H realization due to the aforementioned usefulness and practicability. In general, as said earlier, a nonlinear system does not admit a smooth CLF, so one has to widen the perspective to include nonsmooth tools, such as nonsmooth CLFs, gener-

Table 1. Overview of stabilization methods

	Cont., static	**Cont., time-varying**	**Discont., static**
pros	Classical trajectories, rel. simple design	Classical trajectories	Rel. simple design
cons	Application limited	Sophisticated design	Nonstandard trajectories (however, S&H)

alized derivatives, subgradients etc., whose brief overview is given in the next section. Particular stabilization techniques are surveyed in Section 3. In Section 4, we overview a very powerful technique of nonsmooth backstepping with application to dynamically actuated three-wheel robot and dynamical Artstein's circles. A brief survey of SMC in the S&H setting is given in Section 5. The work is concluded with a discussion on computational aspects of stabilization techniques and a related case study in Section 6.

In the following, $\|x\|$ describes the Euclidean norm of x and $\overline{\text{co}}(\mathbb{X})$ is defined as the closure of the convex hull of a set $\mathbb{X}$. Furthermore, $\mathcal{B}_R(x)$ denotes a ball with radius R at x, i. e., $\mathcal{B}_R(x) := \{x \in \mathbb{R}^n : \|x\| \leq R\}$ and $\mathcal{B}_R$ means the same with $x = 0$. Finally, $\rightrightarrows$ denotes a set-valued mapping.

2. BASIC NONSMOOTH TOOLS

Perhaps, the most central difference between the machinery of continuous and discontinuous stabilization lies at the level of a CLF. As mentioned earlier, a general nonlinear system does not admit a smooth CLF, so one is forced to consider nonsmooth alternatives. For this sake, let us first consider

Definition 2. (Lower directional generalized derivative). For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, the function $\mathcal{D}_\bullet V(\bullet) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\mathcal{D}_\vartheta V(x) \triangleq \liminf_{\mu \to 0^+} \frac{V(x + \mu\vartheta) - V(x)}{\mu}. \tag{Der}$$

is called *lower directional generalized derivative* (further, just "LDGD").

Example 1. Consider a function

$$V(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x < 0, \\ 0, & x = 0, \\ 2x^2 \sin\frac{1}{x}, & x > 0, \end{cases}$$

LDGD at zero along one is -2.

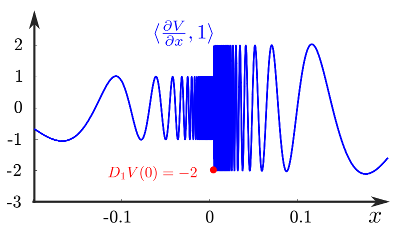

Fig. 1. Graph of the derivative of V

Using the introduced LDGD, we can now consider one type of a nonsmooth CLF:

Definition 3. (CLF in LDGD sense). For the system (Sys), a locally Lipschitz, proper and positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ is called *CLF in LDGD sense* if there exists a continuous function $w : \mathbb{R}^n \to \mathbb{R}, x \neq 0 \implies w(x) > 0$ satisfying a *decay condition*: for any compact set $\mathbb{X} \subseteq \mathbb{R}^n$, there exists a compact set $\mathbb{U}_\mathbb{X} \subseteq \mathbb{U}$ such that

$$\forall x \in \mathbb{X} \inf_{\vartheta \in \overline{\text{co}}(f(x, \mathbb{U}_\mathbb{X}))} \mathcal{D}_\vartheta V(x) \leq -w(x). \tag{Dec}$$

The function w is also called *decay function.*

The condition (Dec) effectively means that V is an *upper minimax solution* of the Hamilton-Jacobi equation

$$\inf_{u \in \mathbb{U}_\mathbb{X}} \langle \nabla V, f(x, u) \rangle + w(x) = 0 \tag{HJ}$$

on the respective domain (Camilli et al., 2008; Subbotin, 2013). There exist several techniques of practical stabilization of (Sys) using an LDGD CLF in the sense of Definition 3, some of which are reviewed in Section 3. Here, it has to be clarified what "∇V" means in the nonsmooth setting. There are several substitutes for gradients with different contexts, some of which are overviewed here. First of all, ∇V for a smooth function V is a unique vector at each point. If V is nonsmooth, there is no unique vector which describes a descent direction of V, so one speaks of a set of those, summarizing them in a *subdifferential.* Here is the first such subdifferential, which is useful in practical stabilization (Clarke et al., 1997):

Definition 4. (Proximal subdifferential). A vector $\zeta \in \mathbb{R}^n$ is called *proximal subgradient* of V at $x \in \mathbb{R}^n$ if there exists a ball $\mathcal{B}_r(x)$ and $\sigma > 0$ s. t.

$$\forall y \in \mathcal{B}_r(x) \; V(y) \geq V(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2. \tag{Prox}$$

The set of all such vectors is called *proximal subdifferential* and is denoted by $\partial_P V(x)$.

It follows straight from the definition of an LDGD, that for any vector ϑ and any proximal subgradient ζ, it holds that

$$\langle \zeta, \vartheta \rangle \leq \mathcal{D}_\vartheta V(x). \tag{1}$$

Therefore, a decay condition in the spirit of (Dec) can be formulated as

$$\forall \zeta \in \partial_P V(x) \inf_{u \in \mathbb{U}_\mathbb{X}} \langle \zeta, f(x, u) \rangle \leq -w(x). \tag{PDec}$$

The condition (PDec) means that V is a *proximal supersolution* of (HJ) (Clarke et al., 1995), or, equivalently, a *viscosity supersolution* thereof (Crandall and Lions, 1983).

Other subdifferentials exist, for instance:

Definition 5. (Limiting subdifferential). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a function. The set

$$\begin{aligned} \partial_L V(x) := \{ &\zeta \in \mathbb{R}^n : \zeta = \text{w-lim } \zeta_i, \\ &\zeta_i \in \partial_P V(x_i), x_i \to x, V(x_i) \to V(x) \} \end{aligned} \tag{2}$$

is called *limiting subdifferential* of V at x, where w-lim is the weak limit.

Here, a sequence $\{\zeta_i\}_{i=1,2,\ldots} \subset \mathbb{R}^n$ is said to converge weakly to $\zeta \in \mathbb{R}^n$, if $\langle \zeta_i, \theta \rangle \to \langle \zeta, \theta \rangle$ for all $\theta \in \mathbb{R}^n$. Such limiting constructions are used, e. g., in the nonsmooth practical stability analysis of SMC (Clarke and Vinter, 2009). An overview is presented in Section 5.

Definition 6. (Clarke subdifferential). Let $\partial_L V(x)$ be the limiting subdifferential of V at x. Then, the *Clarke subdifferential* $\partial_C V(x)$ is defined as the closed convex hull of (2), i.e.

$$\partial_C V(x) = \overline{\text{co}}\left(\partial_L V(x)\right). \tag{3}$$

Example 2. Consider $V(x) = \begin{cases} x^2, & \text{if } x < 1 \\ (x-1)^2 + 1, & \text{else} \end{cases}$. The subdifferentials defined in Definitions 4, 5 and 6 are given as

$$\partial_P V(1) = \varnothing,\ \partial_L V(1) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$
$$\partial_C V(1) = \overline{\text{co}} \left(\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \right).$$

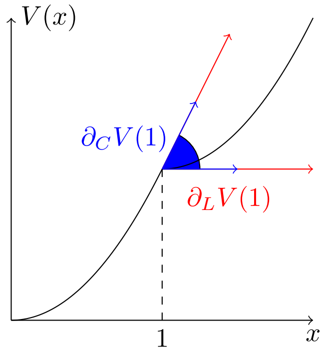

Fig. 2. Graph of V and the three subdifferentials

An important property of many practical stabilizing techniques is *semiconcavity.* In fact, semiconcavity is a ubiquitous property of CLFs (Clarke, 2011; Cannarsa and Sinestrari, 2004).

Definition 7. (Locally semiconcave function). A function $V : \mathbb{X} \to \mathbb{R}$ with $\mathbb{X} \subset \mathbb{R}^n$ being convex, is called *locally semiconcave with linear modulus*, if it is continuous in $\mathbb{X}$ and there exists $C \geq 0$ such that the following inequality holds for all $x, y \in \mathbb{X}$

$$V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \leq C\left\|x - y\right\|^2. \quad \text{(SemiConc)}$$

The following theorem states that any locally semiconcave function can be represented as the infimum of a family of $\mathcal{C}^2$ functions. The proof can be found in (Cannarsa and Sinestrari, 2004).

Theorem 1. Let $V : \mathbb{X} \to \mathbb{R}, \mathbb{X} \subseteq \mathbb{R}^n$ be a locally semiconcave function with linear modulus according to Definition 7. Then, for each compact subset $\mathbb{K} \subset \mathbb{X}$, there exists a compact set $\Theta \subset \mathbb{R}^{2n}$ and a continuous function $F : \mathbb{K} \times \Theta \to \mathbb{R}$, s. t. $F(\bullet; \theta)$ is $\mathcal{C}^2$ for any $\theta \in \Theta$, the gradients $\nabla_x F(\bullet; \theta)$ are equicontinuous, and V can be expressed as

$$V(x) = \min_{\theta \in \Theta} F(x; \theta), \quad (4)$$

for all $x \in \mathbb{K}$. If there exists such a representation of V, then V is called *marginal function* (Cannarsa and Sinestrari, 2004).

Theorem 1 will be useful in presenting a particular technique of nonsmooth backstepping in Section 4. Marginal functions are also used to define

Definition 8. (F-disassembled subdifferential). Let $V : \mathbb{X} \to \mathbb{R}$, $\mathbb{X} \subseteq \mathbb{R}^n$ be a locally semiconcave function and let $F : \mathbb{K} \times \Theta \to \mathbb{R}$, where $\mathbb{K} \subset \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^{2n}$ are compact sets. The set-valued map $\partial_D^F V : \mathbb{X} \rightrightarrows \mathbb{R}^n$ defined as

$$\partial_D^F V(x) \triangleq \left\{ \frac{\partial F(x;\theta)}{\partial x} : \theta \in \arg\min_{\theta \in \Theta} F(x;\theta) \right\} \quad (5)$$

is called *F-disassembled subdifferential.* A single element of $\partial_D^F V(x)$ is called *F-disassembled subgradient.*

Such a subdifferential was used in (Cannarsa and Sinestrari, 2004) and was named in (Nakamura et al., 2013).

Using disassembled subdifferentials, the following type of a nonsmooth CLF can be introduced (Nakamura et al., 2013)

Definition 9. (F-disassembled CLF). A proper, positive-definite, locally semiconcave function $V : \mathbb{R}^n \to \mathbb{R}$ is called *F-disassembled CLF* for (Sys), if there exists a continuous, positive definite function $w : \mathbb{R}^n \to \mathbb{R}$ satisfying the following decay condition: for any compact subset $\mathbb{X} \subseteq \mathbb{R}^n$, there exists a compact set $\mathbb{U}_{\mathbb{X}} \subseteq \mathbb{U}$, such that

$$\forall x \in \mathbb{X}\ \exists \zeta \in \partial_D^F V(x) : \min_{u \in \mathbb{U}_{\mathbb{X}}} \langle \zeta, f(x,u) \rangle \leq -w(x). \quad \text{(DisDec)}$$

An immediate relation between F-disassembled and proximal subdifferentials can be stated in the following lemma.

Lemma 2. Let $\mathbb{X} \subseteq \mathbb{R}^n$ be open and let $\Theta \subset \mathbb{R}^{2n}$ be compact. Let V be a locally semiconcave function given as (4) for F continuous in $\mathbb{X} \times \Theta$. If ξ is a proximal subgradient of V at x, then it is also an F-disassembled subgradient of V at x, i. e.,

$$\partial_P V(x) \subseteq \partial_L V(x) \subseteq \partial_D^F V(x). \quad (6)$$

Proof. The first inclusion follows from Definition 5. The second one is a part of the proof of Theorem 3.4.4 in (Cannarsa and Sinestrari, 2004). ■

The following lemma shows the relation between the two different kinds of CLFs.

Lemma 3. Consider (Sys) and a related F-disassembled CLF $V : \mathbb{R}^n \to \mathbb{R}$ with an arbitrary F. Then, V is also a CLF in LDGD sense.

Proof. Choose a function $F : \mathbb{X} \times \Theta \to \mathbb{R}$, $F \in \mathcal{C}^2(\mathbb{X} \times \Theta)$ satisfying (4) and let $\partial_D^F V(x)$ be the corresponding F-disassembled differential. Define for all $x \in \mathbb{X}$

$$\Phi(x, \vartheta, \varepsilon; \theta) := \frac{F(x + \varepsilon\vartheta; \theta) - F(x; \theta)}{\varepsilon}$$

Obviously,

$$\begin{aligned} &\liminf_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta) \\ &\leq \lim_{\varepsilon \to 0} \Phi(x, \vartheta, \varepsilon; \theta) = \left\langle \frac{\partial F(x;\theta)}{\partial x}, \vartheta \right\rangle, \end{aligned} \quad (7)$$

holds for all $x \in \mathbb{X}$, which follows from the definition of lim inf. Since Φ is differentiable w.r.t. x and θ, the equality in (7) holds for all $\theta \in \Theta$. In particular, for $\theta^\star \in \arg\min_{\theta \in \Theta} F(x;\theta)$ it holds that

$$\lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^\star) = \left\langle \left. \frac{\partial F(x;\theta)}{\partial x} \right|_{\theta = \theta^\star}, \vartheta \right\rangle = \langle \zeta, \vartheta \rangle \quad (8)$$

for some $\zeta \in \partial_D^F V(x)$ corresponding to $\theta^\star$ at x. In turn, one obtains using (1)

$$\lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^\star) = \langle \zeta, \vartheta \rangle \leq \mathcal{D}_\vartheta V(x). \quad (9)$$

Since $\lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^\star)$ is independent of the choice of F, (DisDec) $\implies$ (Dec) holds with (9). And since F was chosen arbitrary, (9) holds for all F that satisfy (4). ■

Before proceeding to concrete stabilizing techniques, the *inf-convolution* (InfC) should be recalled (Clarke et al., 2008):

Definition 10. (Inf-convolution). Let $V : \mathbb{X} \to \mathbb{R}, \mathbb{X} \subseteq \mathbb{R}^n$. For $\alpha \in (0, 1)$, the *inf-convolution* of V at x is defined by

$$V_\alpha(x) \triangleq \inf_{y\in\mathbb{R}^n} \left\{ V(y) + \frac{1}{2\alpha^2} \|y - x\|^2 \right\}. \quad \text{(InfC)}$$

In classical convex analysis, V_α is known as Yoreau-Mosida regularization of a (convex) function V. Furthermore, if V is a lower semicontinuous function and bounded from below, then V_α is locally Lipschitz and an approximation of V in the sense of $\lim_{\alpha\to 0} V_\alpha(x) = V(x)$ (Clarke et al., 1997).

3. STABILIZATION TECHNIQUES

In the following section, some stabilization techniques are presented. Some of them are also described in (Braun et al., 2017).

3.1 Steepest descent

A steepest descent control law $\kappa : \mathbb{R}^n \to \mathbb{U} \subseteq \mathbb{R}^m$ at $x \neq 0$ is computed via

$$\kappa(x) \in \arg\min_{u\in\mathbb{U}} \mathcal{D}_{f(x,u)}V(x). \quad (10)$$

It is shown in (Braun et al., 2017) that $\kappa(x)$, computed by (10) with a semiconcave LDGD CLF $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, practically asymptotically stabilizes the origin of (Sys). The semiconcavity of V is crucial to guarantee practical stabilizability by steepest descent (Braun et al., 2017).

3.2 Dini Aiming

A control law at the state $x \in \mathbb{R}^n$ by Dini Aiming (Kellett and Teel, 2000) is computed in two steps based on a nondecreasing continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

1 Identify a direction $\vartheta^\star$ by means of minimizing the LDGD CLF V over a neighborhood of x for a given $r > 0$, i. e.,
$$\vartheta^\star \in \arg\min_{s\in\bar{\mathcal{B}}_r(x)} V(s). \quad (11)$$
2 Compute an admissible control $\kappa(x) \in \mathbb{U} \cap \bar{\mathcal{B}}_{\sigma(\|x\|+r)}$, $\mathbb{U} \cap \bar{\mathcal{B}}_{\sigma(\|x\|+r)} = \{u \in \mathbb{U} : \|u\| \leq \sigma(\|x\| + r)\}$ via
$$\kappa(x) = \arg\min_{u\in\mathbb{U}\cap\bar{\mathcal{B}}_{\sigma(\|x\|+r)}} \frac{\langle x - \vartheta^\star, f(x, u)\rangle}{\|x - \vartheta^\star\|} \quad (12)$$

If the sampling time δ is chosen small enough (in accordance with σ and r), the control computed in (12) practically asymptotically stabilizes the origin of (Sys) (Kellett et al., 2004; Braun et al., 2017). Such a function σ always exists, since V is an LDGD CLF. Furthermore, $\|x\| \to 0 \implies \|u\| \to 0$ has to hold. In contrast to the steepest descent, V does not necessarily need to be semiconcave.

3.3 Optimization-based control

In optimization-based control, the given LDGD is directly minimized over the set of admissible constant inputs (Braun et al., 2017), i. e.,

$$\min_{u\in\mathbb{U}} \int_0^\delta \mathcal{D}_{f(x,u)}V(\varphi(t, x, u))\, \mathrm{d}t \quad (13)$$

or, equivalently,

$$\min_{u\in\mathbb{U}} |V(\varphi(\delta, x, u)) - V(x)|. \quad (14)$$

At every step, $\varphi(\delta, x, u)$, as a solution of (Sys), is computed over the sampling time period $[0, \delta]$. The one step optimization-based control can be defined as

$$\kappa_\delta(x) \in \arg\min_{u\in\mathbb{U}} V(\varphi(\delta, x, u)). \quad (15)$$

This method combines the two steps (11) and (12) in one single optimization problem. In comparison to other techniques, $\kappa_\delta(x)$ computed in (15), explicitly depends on δ.

3.4 Inf-convolution-based stabilization

In this technique, a minimizer of (InfC), i. e.,

$$y_\alpha(x) \in \arg\inf_{y\in\mathbb{R}^n} \left\{ V(y) + \frac{1}{2\alpha^2} \|y - x\|^2 \right\} \quad (16)$$

is computed to define a proximal subgradient $\zeta_\alpha(x) := \frac{x-y_\alpha(x)}{\alpha^2}$ (Clarke et al., 1997). A control law is obtained by

$$\kappa(x) = \arg\inf_{u\in\mathbb{U}_\mathbb{Y}} \langle\zeta_\alpha(x), f(y_\alpha(x), u)\rangle \quad (17)$$

where $\mathbb{Y}$ is a compact set which contains $y_\alpha(x)$. Numerical studies with the above described methods can be found in (Braun et al., 2017; Osinenko et al., 2018a). The next section discusses specifically the use of F-disassembled subdifferentials and CLFs for nonsmooth backstepping.

4. NONSMOOTH BACKSTEPPING

In this section, a variant of nonsmooth backstepping on the example of three-wheel robot with dynamical actuators and dynamical Artstein's circles is presented based on F-disassembled subdifferentials and CLFs.

4.1 Three-wheel robot with dynamical actuators

A three-wheel robot with dynamical control of the driving and steering torques can be described as follows:

$$\begin{aligned} \dot{x}_1 &= \eta_1, & \dot{\eta}_1 &= u_1, \\ \dot{x}_2 &= \eta_2, & \dot{\eta}_2 &= u_2, \\ \dot{x}_3 &= \eta_1 x_2 - x_1\eta_2. \end{aligned} \quad \text{(ENDI)}$$

The system (ENDI), $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \eta \end{bmatrix} = f_{\mathrm{ENDI}}(x, \eta, u)$, is also called *extended nonholonomic dynamical integrator* (ENDI) (Abbasi et al., 2017; Sankaranarayanan and Mahindrakar, 2009; Pascoal and Aguiar, 2002). It is essentially the Brockett's *nonholonomic integrator* (NI) with additional integrators before the control inputs. The former reads, accordingly, as

$$\dot{x} = f_{\mathrm{NI}}(x, u) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix}}_{=:g_1(x)} u_1 + \underbrace{\begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}}_{=:g_2(x)} u_2. \quad \text{(NI)}$$

The following functions are nonsmooth LDGD CLFs for (NI) (Braun et al., 2017; Clarke, 2011):

$$V_1(x) = x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2},$$
$$V_2(x) = x_1^2 + x_2^2 + 2x_3^2 + |x_3|(10 - 2(|x_1| + |x_2|)).$$

For the ENDI, nonsmooth backstepping based on (Matsumoto et al., 2015) may be utilized. To this end, consider the following function (Kimura et al., 2015):

$$F(x;\theta) := x_1^4 + x_2^4 + \frac{|x_3|^3}{(x_1\cos(\theta) + x_2\sin(\theta) + \sqrt{|x_3|})^2}. \tag{18}$$

So, $V(x) := \min_{\theta\in\Theta} F(x;\theta)$ is an F-disassembled CLF for (NI). Choose a minimizer $\theta^\star \in \Theta^\star := \arg\min_{\theta\in\Theta} F(x;\theta)$ and compute $\zeta(x;\theta) = \nabla_x F(x;\theta)$. Then, $\zeta(x;\theta^\star) \in \partial_D^F V(x) = \nabla_x F(x;\theta)\mid_{\theta\in\Theta^\star}$ holds.

A corresponding stabilizing controller for (NI) can be chosen using the Sontag's formula as (Nakamura et al., 2013)

$$\kappa(x;\theta) = -\begin{pmatrix}\langle\zeta(x;\theta), g_1(x)\rangle\\ \langle\zeta(x;\theta), g_2(x)\rangle\end{pmatrix}. \tag{19}$$

Then, for all $x \in \mathbb{X}\setminus\{0\}$ and $\theta\in\Theta$, a decay of the CLF is ensured due to

$$\begin{aligned}&\langle\zeta(x;\theta), f_{\mathrm{NI}}(x,u)\rangle\\ &= \langle\zeta(x;\theta), g_1(x)\rangle\, u_1 + \langle\zeta(x;\theta), g_2(x)\rangle\, u_2\\ &= -\langle\zeta(x;\theta), g_1(x)\rangle^2 - \langle\zeta(x;\theta), g_2(x)\rangle^2 < 0.\end{aligned} \tag{20}$$

Observe, that for all $\theta\in\Theta$, it holds formally

$$\langle\zeta(x;\theta), G(x)\kappa(x;\theta)\rangle \le 0 \tag{21}$$

with $G(x) = [g_1(x)\ g_2(x)] = \begin{pmatrix}1 & 0\\ 0 & 1\\ -x_2 & x_1\end{pmatrix}$.

Now, the F-disassembled CLF (18) of (NI) is augmented in the spirit of backstepping as follows

$$V_c(x,\eta) = \min_{\theta\in\Theta}\left\{F(x;\theta) + \frac{1}{2}\|\eta - \kappa(x;\theta)\|^2\right\}. \tag{22}$$

The F-disassembled differential for $\theta_c^\star \in \Theta_c^\star$, where $\Theta_c^\star := \arg\min_{\theta\in\Theta}\{F(x;\theta) + 1/2\,\|\eta-\kappa(x;\theta)\|^2\}$ is given by

$$\begin{aligned}\partial_D^F V_c(x,\eta) &= \left.\frac{\partial F(x;\theta) + \frac{1}{2}\|\eta-\kappa(x;\theta)\|^2}{\partial\begin{bmatrix}x\\ \eta\end{bmatrix}}\right|_{\theta\in\Theta_c^\star}\\ &= \left.\begin{bmatrix}\frac{\partial F(x;\theta)}{\partial x} - \frac{\partial\kappa(x;\theta)}{\partial x}(\eta-\kappa(x;\theta))\\ \eta-\kappa(x;\theta)\end{bmatrix}\right|_{\theta\in\Theta_c^\star}.\end{aligned} \tag{23}$$

Let $\zeta_c(x,\eta;\theta) = \nabla_{(x,\eta)} F_c(x,\eta;\theta)$, then it follows that

$$\begin{aligned}&\langle\zeta_c(x,\eta;\theta), f_{\mathrm{ENDI}}(x,\eta,u)\rangle\\ &= \left\langle\begin{bmatrix}\zeta(x;\theta) - \nabla_x\kappa(x;\theta)z\\ z\end{bmatrix}, \begin{bmatrix}G(x)\eta\\ u\end{bmatrix}\right\rangle\\ &= \langle\zeta(x;\theta) - \nabla_x\kappa(x;\theta)z, G(x)\eta\rangle + \langle z,u\rangle\\ &= \langle\zeta(x;\theta), G(x)\eta\rangle - \langle\nabla_x\kappa(x;\theta)z, G(x)\eta\rangle + \langle z,u\rangle\\ &= \langle\zeta(x;\theta), G(x)z\rangle + \langle\zeta(x;\theta), G(x)\kappa(x;\theta)\rangle\\ &\quad - \langle\nabla_x\kappa(x;\theta)z, G(x)\eta\rangle + \langle z,u\rangle =: S(x,z;\theta).\end{aligned} \tag{24}$$

Now, consider the following cases.

Case 1: $z = 0$. Since $z = 0$, $S(x,z;\theta)$ in (24) reduces to $S(x,z;\theta_c^\star) = \langle\zeta(x;\theta_c^\star), G(x)\kappa(x;\theta_c^\star)\rangle \le 0$ for $\theta_c^\star\in\Theta_c^\star$. In this case, $\Theta^\star = \Theta_c^\star$ holds and $\kappa(x;\theta^\star)$, $\theta^\star\in\Theta^\star$ is the control for both, (NI) and (ENDI).

Case 2: $z \ne 0$. Since

$$\begin{aligned}&S(x,z;\theta_c^\star)\\ &= \left\langle z, G(x)^{\mathrm{T}}\zeta(x;\theta_c^\star)\right\rangle + \langle\zeta(x;\theta_c^\star), G(x)\kappa(x;\theta_c^\star)\rangle\\ &\quad - \left\langle z, \nabla_x\kappa(x;\theta_c^\star)^{\mathrm{T}} G(x)\eta\right\rangle + \langle z,u\rangle\end{aligned} \tag{25}$$

holds, a suitable choice for u is, e. g.,

$$u = \nabla_x\kappa(x;\theta_c^\star)^{\mathrm{T}} G(x)\eta - G(x)^{\mathrm{T}}\zeta(x;\theta_c^\star) - Kz \tag{26}$$

with $K > 0$. This yields a decay

$$\langle\zeta(x;\theta_c^\star), G(x)\kappa(x;\theta_c^\star)\rangle - K\|z\|^2 < 0 \tag{27}$$

for all $(x,\eta)\in\mathbb{R}^5\setminus\{0\}$. The resulting control law $\kappa_c(x,\eta;\theta_c^\star)$ is obtained by choosing $\theta_c^\star\in\Theta_c^\star$ in (26). Thus, the origin of (ENDI) is practically stabilized by the described control law. As a recap of Section 4.1, first compute the set $\Theta^\star$ by minimizing $F(x;\theta)$. Second, compute a control law for the NI-subsystem according to (19). Third, the F-disassembled CLF $F(x;\theta)$ is augmented with $z = \eta - \kappa(x;\theta)$ to a new function $F(x;\theta) + 1/2\,\|z\|^2$. The resulting control law for the whole system is given either via (19), or, respectively (26), depending on z.

4.2 Dynamical Artstein's circles

The same method as described in the previous section can be applied to the following system, a dynamic extension of the Artstein's Circle:

$$\begin{aligned}\dot{x}_1 &= (-x_1^2 + x_2^2)x_3,\\ \dot{x}_2 &= -2x_1x_2x_3,\\ \dot{x}_3 &= u,\end{aligned} \tag{AC}$$

which can be written as

$$\begin{aligned}\dot{v} &= g(v)w\\ \dot{w} &= u\end{aligned} \tag{28}$$

with $v := (x_1\ x_2)^{\mathrm{T}} \in\mathbb{R}^2$ and $w := x_3\in\mathbb{R}$.

A nonsmooth LDGD CLF for $\dot{v} = g(v)u$ is given as (Braun et al., 2018):

$$V(v) = \sqrt{3x_1^2 + 4x_2^2} - |x_1|. \tag{29}$$

Referring to (18), the following marginal function gives rise to an F-disassembled CLF:

$$F(v;\theta) = \sqrt{3}x_1\cos(\theta) + 2x_2\sin(\theta) - |x_1|. \tag{30}$$

Obviously, for (29) and (30), (4) holds with $\Theta = [0,2\pi]$.

After computing the F-disassembled differential $\partial_D^F V(v) = \nabla_v F(v;\theta)\ \mid_{\theta\in\Theta^\star}$, choose $\zeta(v;\theta^\star)\in\partial_D^F V(v)$, where $\theta^\star\in\Theta^\star := \arg\min_{\theta\in\Theta} F(v;\theta)$. Then, with $\kappa(v;\theta) = -\langle\zeta(v;\theta), g(v)\rangle$ as a feedback, it follows

$$\langle\zeta(v;\theta), g(v)\kappa(v;\theta)\rangle = -\langle\zeta(v;\theta), g(v)\rangle^2 < 0 \tag{31}$$

which holds also for all $\theta\in\Theta$ and all $v\in\mathbb{R}^2\setminus\{0\}$, and practically stabilizes $\dot{v} = g(v)w$. Now, (30) can be extended via backstepping to

$$V_c(v,w) = \min_{\theta\in\Theta_c}\left\{F(v;\theta) + \frac{1}{2}\|w - \kappa(v;\theta)\|^2\right\}. \tag{32}$$

Define $z := w - \kappa(v;\theta)$. Since the disassembled subdifferential is given similarly to (23), one can show, that for $\zeta_c(v,w;\theta^\star)\in\partial_D^F V_C(v,w)$ with $\theta^\star\in\Theta_c^\star$, $\Theta_c^\star$ defined as in Section 4.1, the following equality holds:

$$\begin{aligned}&\left\langle\begin{bmatrix}\zeta(v;\theta) - \nabla_v\kappa(v;\theta)z\\ z\end{bmatrix}, \begin{bmatrix}gw\\ u\end{bmatrix}\right\rangle\\ &= \langle\zeta(v;\theta) - \nabla_v\kappa(v;\theta)z, gw\rangle + \langle z,u\rangle\\ &= \langle\zeta(v;\theta), gw\rangle - \langle\nabla_v\kappa(v;\theta)z, gw\rangle + \langle z,u\rangle\\ &= \langle\zeta(v;\theta), gz\rangle + \langle\zeta(v;\theta), g\kappa(v;\theta)\rangle\\ &\quad - \langle\nabla_v\kappa(v;\theta)z, gw\rangle + \langle z,u\rangle.\end{aligned} \tag{33}$$

The corresponding control law can be derived in a similar manner as in Section 4.1.

5. SLIDING MODE

Consider a system

$$\begin{aligned}\dot{x} &= f(x,u), x \in \mathbb{R}^n,\\ u &= \kappa(x) + s(x)\sigma(x),\end{aligned} \tag{34}$$

where κ, s are continuous and

$$\sigma(x) = \begin{cases}1, & \chi(x) > 0,\\ -1, & \chi(x) < 0\end{cases} \tag{35}$$

describes the discontinuous part of the controller around a sliding surface $\Sigma = \{x : \chi(x) = 0\}$. To process (34), unlike in the usual case, two Lyapunov-like functions V_1, V_2 are employed that satisfy, for some $\bar{w}_1 > 0$,

V_1 and $V_1 + V_2$ are proper, $V_1(x) = 0 \iff x \in \Sigma$,
V_1 is continuously differentiable on $\mathbb{R}^n \setminus \Sigma$ and
$\langle \nabla V_1(x), f(x, \kappa(x) + s(x)\sigma(x))\rangle \le -\bar{w}_1, x \in \mathbb{R}^n \setminus \Sigma$;
$V_2(0) = 0, w_2(0) = 0$,
$x \in \Sigma \setminus \{0\} \implies V_2(x) > 0, w_2(x) > 0$,
V_2 is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$ and
$\sup_{v \in \partial^f_L(x)} \langle \nabla V_2(x), v\rangle \le -w_2(x), x \in \Sigma \setminus \{0\}$,

where $\partial^f_L(x) := \{\lim_{i\to\infty} f(x, \kappa(x) + s(x)\sigma_i) : x_i \to x, \sigma_i \in \sigma(x_i)\}$ describes the limiting behavior of possible system velocities (cf. the construction in Definition 5). Let $f_1(x) := f(x, \kappa(x) + s(x))$ and $f_2(x) := f(x, \kappa(x) - s(x))$, and $\mathbb{X}_1 := \{\chi(x) > 0\}, \mathbb{X}_2 := \{\chi(x) < 0\}$. In general, (34) in SMC mode is usually treated in the sense of a differential inclusion (Perruquetti and Barbot, 2002; Slotine and Li, 1991):

$$\dot{x} \in F(x) = \begin{cases} f_1(x), & x \in \mathbb{X}_1,\\ (1-\alpha)f1(x) + \alpha f_2(x), \alpha \in [0,1], & x \in \Sigma,\\ f_2(x), & x \in \mathbb{X}_2.\end{cases} \tag{36}$$

This is a particular example of *Filippov regularization* (Cortes, 2008) and, in this case, is upper semi-continuous, has closed and convex images and thus admits a Filippov solution (Zabczyk, 2009), which is an absolutely continuous function whose derivative satisfies the differential inclusion almost everywhere. A Filippov solution is an idealized construction describing a perfect sliding mode in this context. In practice, since the control is usually realized digitally, S&H analysis of SMC comes in handy (Clarke and Vinter, 2009). The general idea thereof is to chose a sampling time bound small enough that the system approach a specially chosen vicinity of Σ where V_2 still retains some of it decay rate w_2 (by a continuity argument). The attraction to this vicinity is in turn ensured by retaining some of the decay rate $\bar{w}_1$. Combining the two decay properties together ensures practical stabilization of (34) in the S&H mode (Clarke and Vinter, 2009).

6. COMPUTATIONAL ASPECTS

The described practical stabilization methods of the previous sections rely on computation of primarily two things: (1) allowed sampling time for the desired stabilization precision; (2) control actions. General techniques of Section 3 rely heavily on various optimizations. It was shown that optimization accuracy greatly influences stabilization precision (Osinenko et al., 2018a). Moreover, the involved LDGD CLF must satisfy certain regularity properties. In particular, the following: For all compact sets $\mathbb{Y}, \mathbb{F} \subset \mathbb{R}^n$ and for all $\nu, \chi > 0$ there exist $\tilde{\mathbb{Y}} \subseteq \mathbb{Y}, \mu \ge 0$ such that:

(1) For each $\tilde{y} \in \tilde{\mathbb{Y}}, \theta \in \mathbb{F}$ and $\forall \mu' \in (0, \mu]$ it holds that

$$\left| \frac{V(\tilde{y} + \mu'\theta) - V(\tilde{y})}{\mu'} - \mathcal{D}_\theta V(\tilde{y})\right| \le \nu; \tag{hom}$$

(2) For each $y \in \mathbb{Y}$ there exists $\tilde{y} \in \tilde{\mathbb{Y}}$ such that

$$\|y - \tilde{y}\| \le \chi. \tag{npt}$$

Under these conditions, one can find bounds on the optimization accuracy of the inf-convolution-based method so as to achieve the desired stabilization precision. Furthermore, it was shown in (Osinenko et al., 2018b) what regularity properties of the involved Lyapunov-like functions have to be satisfied to effectively compute required bounds on the sampling time. The respective machinery was addressed on the example of SMC. In contrast to (Clarke and Vinter, 2009), an actual numerical example of practical SMC stabilization for vehicle slip control was shown. The computed sampling time bounds were satisfactory for the considered application.

Here, we give a short case study that demonstrates the effect of the described computational uncertainty. Namely, for (ENDI) and its CLF (22), a control law was computed in the S&H framework using (16) and (17) with optimization accuracy ε and η, respectively. The initial condition is set to $x_0 = (-1\ 0.5\ 0.01\ 0.05\ 0.075)^{\mathrm{T}}$ and the set of admissible controls is given as $\mathbb{U} = [-3, 3]$. Furthermore, we set α in (InfC) to $\alpha = 0.1$ and the sampling time δ to $\delta = 0.005$.

The influence of the optimization accuracy on the state and CLF behavior can be seen in Fig. 3 for different values of ε and η, namely $\varepsilon = \eta \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}, 10^{-8}\}$. It can be observed that insufficient accuracy ($\varepsilon = \eta = 10^{-3}$ and $\varepsilon = \eta = 10^{-2}$) leads instability. Higher accuracies lead to ever smaller vicinities of the origin that the state converges into. This clearly demonstrates that computational uncertainty must be taken into account in practical stabilization.

7. CONCLUSION

This work surveyed, in a brief form, some of the key modern nonsmooth stabilization tools and techniques. These include general practical stabilization, sliding-mode control, nonsmooth backstepping. Examples were provided. In addition, this work briefly discussed some computational aspects of practical stabilization one should be concerned about in applications.

REFERENCES

Abbasi, W., ur Rehman, F., and Shah, I. (2017). Backstepping based nonlinear adaptive control for the extended nonholonomic double integrator. *Kybernetika*, 53(4), 578–594.

Aeyels, D. (1985). Stabilization of a class of nonlinear systems by a smooth feedback control. *Systems & Control Letters*, 5(5), 289–294.

Braun, P., Grüne, L., and Kellett, C. (2017). Feedback design using nonsmooth control Lyapunov functions: A numerical case study for the nonholonomic integrator.

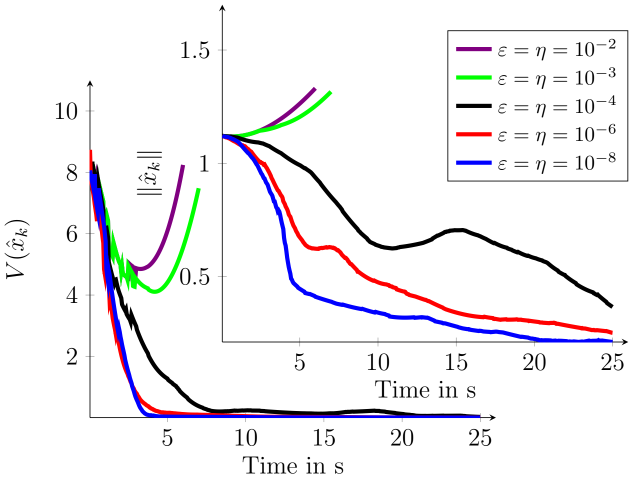

Fig. 3. Practical stabilization of the three-wheel robot with dynamic actuators under different computational accuracy.

In *Proceedings of the 56th IEEE Conference on Decision and Control*.

Braun, P., Grüne, L., and Kellett, C. (2018). Complete instability of differential inclusions using Lyapunov methods. In *2018 IEEE Conference on Decision and Control (CDC)*, 718–724. IEEE.

Brockett, R. (1983). Asymptotic stability and feedback stabilization. *Differential geometric control theory*, 27(1), 181–191.

Camilli, F., Grüne, L., and Wirth, F. (2008). Control Lyapunov functions and Zubov's method. *SIAM Journal on Control and Optimization*, 47(1), 301–326.

Cannarsa, P. and Sinestrari, C. (2004). *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58. Springer Science & Business Media.

Clarke, F. (2011). Lyapunov functions and discontinuous stabilizing feedback. *Annual Reviews in Control*, 35(1), 13–33.

Clarke, F., Ledyaev, Y., Sontag, E., and Subbotin, A. (1997). Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10), 1394–1407.

Clarke, F., Ledyaev, Y., Stern, R., and Wolenski, P. (1995). Qualitative properties of trajectories of control systems: a survey. *Journal of dynamical and control systems*, 1(1), 1–48.

Clarke, F., Ledyaev, Y., Stern, R., and Wolenski, P. (2008). *Nonsmooth Analysis and Control Theory*, volume 178. Springer Science & Business Media.

Clarke, F. and Vinter, R. (2009). Stability analysis of sliding-mode feedback control. *Control and Cybernetics*, 4(38), 1169–1192.

Coron, J.M. (1992). Global asymptotic stabilization for controllable systems without drift. *Mathematics of Control, Signals and Systems*, 5(3), 295–312.

Coron, J.M. (1995). On the stabilization in finite time of locally controllable systems by means of continuous time-varying feedback law. *SIAM Journal on Control and Optimization*, 33(3), 804–833.

Coron, J.M. and d'Andrea Novel, B. (1991). Smooth stabilizing time-varying control laws for a class of nonlinear systems. Application to mobile robots. In *Nonlinear Control Systems Design 1992. Selected Papers from the 2nd IFAC Symposium*, 413–18.

Coron, J.M. and Pomet, J.B. (1993). A remark on the design of time-varying stabilizing feedback laws for controllable systems without drift. In *Nonlinear Control Systems Design 1992*, 397–401. Elsevier.

Coron, J.M. and Rosier, L. (1994). A relation between continuous time-varying and discontinuous feedback stabilization. *Journal of Mathematical Systems, Estimation, and Control*, 4, 67–84.

Cortes, J. (2008). Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28(3), 36–73.

Crandall, M. and Lions, P.L. (1983). Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American mathematical society*, 277(1), 1–42.

Filippov, A. (1988). *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Springer Science & Business Media.

Fridland, L. and Levant, A. (1999). *Higher Order Sliding Modes*. Sliding mode in Automatic Control, Ecole Central de Lille.

Fridman, L. and Levant, A. (1996). Higher order sliding modes as a natural phenomenon in control theory. In *Robust Control via variable structure and Lyapunov techniques*, 107–133. Springer.

Kawski, M. (1989). Stabilization of nonlinear systems in the plane. *Systems & Control Letters*, 12(2), 169–175.

Kellett, C., Shim, H., and Teel, A. (2004). Further results on robustness of (possibly discontinuous) sample and hold feedback. *IEEE Transactions on Automatic Control*, 49(7), 1081–1089.

Kellett, C. and Teel, A. (2000). Uniform asymptotic controllability to a set implies locally Lipschitz control-Lyapunov function. In *Proceedings of the 39th IEEE Conference on Decision and Control*, volume 4, 3994–3999.

Khaneja, N. and Brockett, R. (1999). Dynamic feedback stabilization of nonholonomic systems (i). In *IEEE Conference on Decision and Control*, volume 2, 1640–1645. IEEE.

Kimura, S., Nakamura, H., and Yamashita, Y. (2015). Asymptotic stabilization of two-wheeled mobile robot via locally semiconcave generalized homogeneous control Lyapunov function. *SICE Journal of Control, Measurement, and System Integration*, 8(2), 122–130.

Matsumoto, R., Nakamura, H., Satoh, Y., and Kimura, S. (2015). Position control of two-wheeled mobile robot via semiconcave function backstepping. In *2015 IEEE Conference on Control Applications (CCA)*, 882–887. IEEE.

Morin, P., Pomet, J., and Samson, C. (1999). Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of lie brackets in closed loop. *SIAM Journal on Control and Optimization*, 38(1), 22–49.

Nakamura, H., Tsuzuki, T., Fukui, Y., and Nakamura, N. (2013). Asymptotic stabilization with locally semiconcave control Lyapunov functions on general manifolds. *Systems & Control Letters*, 62(10), 902–909.

Osinenko, P., Beckenbach, L., and Streif, S. (2018a). Practical sample-and-hold stabilization of nonlinear systems under approximate optimizers. *IEEE Control Systems Letters (L-CSS)*, 2(4), 569–574.

Osinenko, P., Devadze, G., and Streif, S. (2018b). Practical stability analysis of sliding-mode control with explicit computation of sampling time. *Asian Journal of Control*.

Pascoal, A. and Aguiar, A. (2002). Practical stabilization of the extended nonholonomic double integrator. *Proc. 10th Mediterranean Conferenceon Control and Automation*.

Perruquetti, W. and Barbot, J.P. (2002). *Sliding Mode Control in Engineering*. CRC Press.

Pomet, J.B. (1992). Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems & Control Letters*, 18(2), 147–158.

Ryan, E. (1994). On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback. *SIAM Journal on Control and Optimization*, 32(6), 1597–1604.

Samson, C. (1991). Velocity and torque feedback control of a nonholonomic cart. In *Advanced robot control*, 125–151. Springer.

Sankaranarayanan, V. and Mahindrakar, A. (2009). Switched control of a nonholonomic mobile robot. *Communications in Nonlinear Science and Numerical Simulation*, 14(5), 2319–2327.

Slotine, J.J.E. and Li, W.A. (1991). *Applied Nonlinear Control*. Prentice Hall.

Subbotin, A.I. (2013). *Generalized Solutions of First-Order PDEs: the Dynamical Optimization Perspective*. Springer Science & Business Media.

Young, K., Utkin, V.I., and Ozguner, U. (1996). A control engineer's guide to sliding mode control. In *Proceedings of the IEEE International Workshop on Variable Structure Systems VSS'96*, 1–14.

Zabczyk, J. (2009). *Mathematical control theory: an introduction*. Springer Science & Business Media.