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Distributed Formation Control of Mobile Robots Using Discrete-Time Distributed Population Dynamics ⋆

Juan Martinez-Piazuelo * **Gilberto Diaz-Garcia** *
Nicanor Quijano * **Luis Felipe Giraldo** *

* *Electronics Engineering Department, Universidad de los Andes, Bogota, Colombia, (e-mail: jp.martinez10@uniandes.edu.co, gj.diaz10@uniandes.edu.co, nquijano@uniandes.edu.co, lf.giraldo404@uniandes.edu.co).*

Abstract: This paper studies the distributed formation control of multiple differential-drive robots. To solve such problem, we propose a novel class of distributed population dynamics, formulated in discrete-time, and we obtain sufficient conditions to guarantee asymptotic stability in practical implementations where computations are necessarily discrete. Moreover, we apply the proposed dynamics to a real multi-robot platform where robots achieve geometric formations under partial information and limited communication capabilities. Our proposed method achieves comparable and even better performance than other distributed methods, and displays some invariance properties that make it attractive for several other engineering applications.

Keywords: Distributed control, mobile robots, discrete-time systems, Lyapunov stability.

1. INTRODUCTION

The automatic control of multi-robot systems is an important research topic in the field of distributed robotics. When compared to a single robot, a coordinated team of robots can provide several advantages in surveillance, transportation, exploration and even rescue applications. However, when it comes to the design of automatic controllers, the decentralized nature of multi-robot systems imposes some significant challenges for control engineers. Factors such as partial information and restricted communication forbid the application of centralized methods to multi-robot control, and require the design of novel control techniques that can cope with the informational constraints of distributed systems.

In this work, we focus on the distributed formation control of multiple mobile robots under a leader-follower scheme. More precisely, a leader robot moves through a pre-defined trajectory and some follower robots have to follow the leader while maintaining a desired geometric formation. Moreover, the problem is hardened by the fact that not all followers have information about the leader. Typically, this problem has been solved using the consensus algorithm of Olfati-Saber et al. (2007), however, some recent researches have solved this problem using distributed nonlinear model predictive control (Xiao and Chen (2019)); designing distributed estimators for the leader's state (Miao et al. (2018)); and applying game-theoretical continuous-time distributed population dynamics (Barreiro-Gomez et al. (2016)). As we show in this paper, the formation control problem can be formulated as a distributed optimization problem whose solution leads to the achievement of a desired collective behavior among the robots. Following the approach of Barreiro-Gomez et al. (2016), in this work we apply population dynamics to solve the distributed formation control problem. In contrast with the previous work, however, we propose a novel class of dynamics, formulated in discrete-time, and we provide sufficient conditions to guarantee the asymptotic stability of our method in practical implementations where computations are necessarily discrete. Furthermore, we apply our proposed method to a real multi-robot platform composed by six e-puck v2 robots, and we compare it against the classical consensus algorithm of Olfati-Saber et al. (2007), and against a game-theoretical distributed optimization method proposed by Li and Marden (2013). Our approach not only outperforms the other game-theoretical method, but also achieves a performance comparable to the consensus algorithm. Moreover, our method displays some invariance properties that make it suitable for other types of distributed control applications including distributed resource allocation (Quijano et al. (2017)). In summary, this paper proposes a novel class of discrete-time distributed population dynamics as a game-theoretical method for decentralized control, and depicts the application of the method to a real multi-robot platform.

The rest of this paper is organized as follows. In Section 2 we present the problem statement and the general structure of the proposed control architecture. In Section 3 we explain our proposed method based on discrete-time distributed population dynamics, and we develop the corresponding stability analysis. Afterwards, in Section 4 we show the application of our theoretical developements to a real multi-robot platform. Finally, Section 5 concludes the paper and presents some future directions.

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2. PROBLEM STATEMENT AND CONTROL ARCHITECTURE

In this work we study the distributed formation control of multiple differential-drive robots that move over $\mathbb{R}^2$ under a leader-follower scheme. For such, we consider a set of n robots (one leader and $n-1$ followers) and define the following notations: $\mathcal{R} = \{1, 2, \cdots, n\}$ is the entire set of robots, $\ell = 1$ is the index of the leader robot, and $\mathcal{F} = \{2, 3, \cdots, n\}$ is the set of follower robots. As robots can move over $\mathbb{R}^2$, the location of the i-th robot at time k is given by the tuple $(x_i[k], y_i[k])$ where $x_i[k], y_i[k] \in \mathbb{R}$ for all $i \in \mathcal{R}$ and all k. In this work, we assume that the leader robot follows a pre-defined trajectory within a rectangle of $\mathbb{R}^2$. Without loss of generality, such rectangle is defined as

$$\mathcal{X} = \left\{(x, y) \in \mathbb{R}^2_+ \,:\, x < \beta^x,\, y < \beta^y\right\}, \qquad (1)$$

where x and y denote the coordinate dimensions of $\mathbb{R}^2$; $\mathbb{R}^2_+$ denotes the strictly positive quadrant of $\mathbb{R}^2$; and β^x and β^y are positive scalars that define the width and height of the rectangular region $\mathcal{X}$, respectively. On the other hand, the goal of the follower robots is to follow the leader while maintaining a given geometric formation within $\mathcal{X}$. In this work, we assume that robots can communicate with each other through bidirectional communication channels. In general, the interaction between robots can be modeled as an undirected time-varying graph $\mathcal{G}[k] = (\mathcal{R}, \mathcal{E}[k], \mathbf{A}[k])$, where the set of robots $\mathcal{R}$ is the set of nodes; $\mathcal{E}[k]$ is the set of communication links between the robots at time k; and $\mathbf{A}[k] = [a_{ij}[k]]$ is an $n \times n$ adjancency matrix whose elements are $a_{ij}[k] = a_{ji}[k] = 1$ if the i-th robot communicates with the j-th robot at time k, and $a_{ij}[k] = a_{ji}[k] = 0$ otherwise. Moreover, we define $\mathcal{N}_i[k] = \{j \in \mathcal{R} \setminus \{i\} \,:\, a_{ij}[k] = 1\}$ as the set of robots that communicate with the i-th robot at time k. The communication between robots is used to obtain the x and y coordinates of other robots, and, in consequence, we assume that at time k the robot $i \in \mathcal{R}$ sends the coordinates $(x_i[k], y_i[k])$ to all robots $j \in \mathcal{N}_i[k]$, and receives the coordinates $(x_j[k], y_j[k])$ of each robot $j \in \mathcal{N}_i[k]$. Whether two robots can communicate might depend on factors like the spatial distance between the robots (as in Barreiro-Gomez et al. (2016)), or the topology of an available cloud-based communication network (as in Liu et al. (2014)). To keep our framework general, in this work we develop our theoretical analyses for arbitrary connected and undirected time-varying graph topologies.

Assumption 1: The undirected communication graph $\mathcal{G}[k]$ is connected for all k.

To achieve a geometric formation, each follower robot is given a reference displacement vector that defines its required position relative to the leader. Such reference displacement vector is given by $\boldsymbol{\delta}_i = \left[\,\delta_i^x,\, \delta_i^y\,\right]^\top$, for all $i \in \mathcal{F}$, where δ_i^x and δ_i^y are the reference displacements for the i-th follower with respect to the leader's x and y coordinates, respectively. For instance, Fig. 1 shows one example formation for a set of $n = 6$ robots and depicts some reference displacements. In this work, we assume that the reference displacement vectors of all the follower robots satisfy the following assumption.

Assumption 2: The required formation lies within $\mathcal{X}$ for all k.

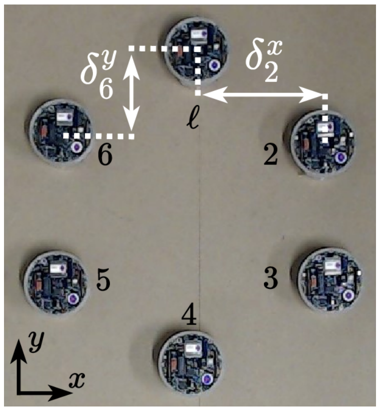

Fig. 1. Hexagonal formation for a set of $n = 6$ robots.

It is worth noting that depending on the graph topology, not all followers might have communication with the leader. Thus, the presented formation problem is a distributed control task under partial information. To solve such kind of problem, we can break the objective into two tasks. The first one is to determine, distributedly over $\mathcal{G}[k]$, the set-point coordinates in $\mathcal{X}$ that each follower has to reach so that certain formation is achieved. This task can be formulated as a set of time-varying optimization problems that each follower robot has to solve. More precisely, such optimization problems are given by

$$\min_{r_i^d[k]} \frac{\left(c_\ell^d[k] + \delta_i^d[k] - r_i^d[k]\right)^2}{2}, \; \forall d \in \mathcal{D},\, \forall i \in \mathcal{F},\, \forall k, \qquad (2)$$

where the set $\mathcal{D} = \{x, y\}$ contains the dimensions of the space where the robots move; the term $c_\ell^d[k]$ represents the value of the d-th dimension of the leader's location at time k, i.e., $c_\ell^x[k] = x_\ell[k]$ and $c_\ell^y[k] = y_\ell[k]$; the term $\delta_i^d[k]$ is the reference displacement over the d-th dimension for the i-th follower at time k; and $r_i^d[k]$ is the d-th component of the set-point coordinate for the i-th follower at time k, i.e., at time k the set-point coordinate for the i-th follower is $(r_i^x[k],\, r_i^y[k])$. Notice that (2) defines $|\mathcal{D}||\mathcal{F}| = 2(n-1)$ optimization problems. More precisely, it defines two independent optimization problems for each follower robot: one for x and one for y. Given that not all followers might have communication with the leader at time k, such optimization problems have to be solved with a distributed algorithm that satisfies the informational constraints imposed by $\mathcal{G}[k]$. In this work, we focus on the design of a distributed algorithm that guarantees that as k goes to infinity, the term $r_i^d[k]$ converges to $c_\ell^d[k] + \delta_i^d[k]$ for all $i \in \mathcal{F}$ and all $d \in \mathcal{D}$, i.e., a convergence in the asymptotic sense. For such matter, we use distributed population dynamics. The second task, on the other hand, is for each robot to actually navigate to its corresponding set-point coordinates. Such set-point coordinates are given by the pre-defined trajectory for the leader robot, and by $(r_i^x[k], r_i^y[k])$ for all $i \in \mathcal{F}$. In this work we focus mainly on the first task, and, in consequence, we use classical PID controllers for the navigation of each robot and we do not consider obstacle avoidance. Nevertheless, under the proposed approach, the local PID controllers can be replaced by any other navigation strategy without having to modify the distributed optimization of the first task. Thus, obstacle avoidance could be added using other

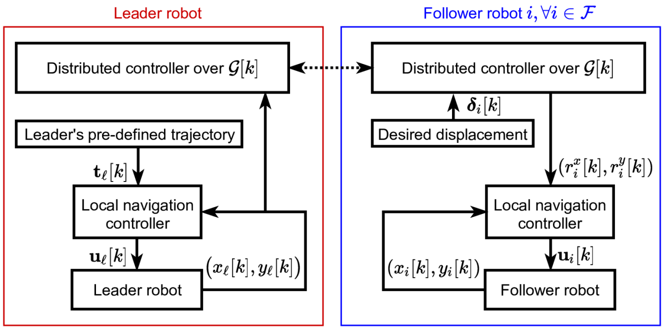

Fig. 2. Control architecture for the distributed formation problem.

navigation controllers such as the one proposed by Snape et al. (2011).

To summarize, Fig. 2 presents the control architecture described above. Here, $\mathbf{u}_j[k] \in \mathbb{R}^2$ is a vector that contains the voltages to be applied to the motors of the j-th robot at time k, and $\mathbf{t}_\ell[k] \in \mathcal{X}$ is a vector that contains the reference x and y coordinates for the leader robot at time k. Note that in all cases the navigation task is done using only the local information available to each robot. Regarding the leader, for instance, the local navigation controller uses the reference coordinates given by the pre-defined trajectory, and the current coordinates of the leader robot. Similarly, each follower robot navigates using its own location and the reference coordinates computed distributedly over $\mathcal{G}[k]$. Note that in the case of the leader the distributed controller over $\mathcal{G}[k]$ is not used to provide a reference for the leader's navigation. Instead, it is used to exchange information about the leader's location with the follower robots that can communicate with the leader (the dotted line is used to highlight that in general the leader does not have communication with all followers).

3. DISTRIBUTED CONTROL OVER THE NETWORK USING POPULATION DYNAMICS

In this section we present our proposed method to solve the distributed optimization problems of (2), which is based on population dynamics (Sandholm (2010)). However, we propose a novel discrete-time-class of distributed population dynamics. In the following sections we present our approach and we develop the corresponding stability analysis of our proposed method.

3.1 Distributed Optimization Using Population Dynamics

Consider a set of populations $\mathcal{D}$ that represent the dimensions of $\mathbb{R}^2$, i.e., $\mathcal{D} = \{x, y\}$, and let each population be composed by a large and finite number of agents that interact strategically with their population peers. Moreover, let the agents of each population be rational decision makers that, at any time k, select among n different strategies representing the set of robots, i.e., at any time k the set of available strategies is $\mathcal{R} = \{\ell, 2, \cdots, n\}$. To solve the optimization problems of (2) using population dynamics, we interpret the optimization variable $r_i^d[k]$ as the portion of agents of population $d \in \mathcal{D}$ that select the robot $i \in \mathbb{R}$ at time k. Hence, at any time k, the vector $\mathbf{r}^d[k] = \left[\, r_\ell^d[k],\, r_2^d[k],\, \cdots,\, r_n^d[k] \,\right]^\top$ describes the state of population $d \in \mathcal{D}$. Thus, $\mathbf{r}^d[k]$ contains the values of all the optimization variables of (2), as well as the value of an additional variable $r_\ell^d[k]$. Such additional variable does not appear in any of the objective functions of (2) and is not used for the leader's navigation. Instead, the variable $r_\ell^d[k]$ plays the role of an auxiliary variable that allows us to deal with certain invariance properties of the population dynamics (see Remark 1 in Section 3.2). Clearly, depending on the mechanism that population agents use to select robots at time k, different trajectories of the population state $\mathbf{r}^d[\cdot]$ emerge on the population $d \in \mathcal{D}$. Therefore, to solve the optimization problems of (2) using this framework, we have to design the strategic interaction of the population agents so that the dynamic evolution of $\mathbf{r}^d[\cdot]$ converges to the minimizer of (2) for all $d \in \mathcal{D}$. The Smith dynamics are one of the fundamental population dynamics and Barreiro-Gomez et al. (2017) have proposed a continuous-time distributed version of such dynamics. In this work, we build upon such distributed dynamics, but we formulate them with two fundamental differences: (i) we use a discrete-time formulation to obtain stability guarantees for practical implementations where computations are necessarily discrete; and (ii) we saturate the dynamics to obtain less conservative bounds in our theoretical analyses. More precisely, our proposed discrete-time distributed Smith dynamics are given by

$$r_i^d[k+1] = r_i^d[k] + \epsilon^d \sum_{j \in \mathcal{N}_i[k]} (f_i^d - f_j^d)\theta_{ij}^d \phi_{ij}^d, \qquad (3)$$

for all $i \in \mathcal{R}$ and all $d \in \mathcal{D}$, where the scalar $\epsilon^d > 0$ is the step size of the update and is assumed equal for all $i \in \mathcal{R}$; and where

$$\begin{aligned} f_i^d &= \delta_i^d[k] - r_i^d[k], \quad \forall i \in \mathcal{F} \\ f_\ell^d &= -c_\ell^d[k] \\ \theta_{ij}^d &= \begin{cases} \beta^d, & \text{if } f_i^d = f_j^d \\ \min\left(r_j^d[k],\, \beta^d\right), & \text{if } f_i^d > f_j^d \\ \min\left(r_i^d[k],\, \beta^d\right), & \text{if } f_i^d < f_j^d \end{cases} \\ \phi_{ij}^d &= \begin{cases} 1, & \text{if } |f_i^d - f_j^d| \le \beta^d \\ \beta^d / |f_i^d - f_j^d|, & \text{if } |f_i^d - f_j^d| > \beta^d. \end{cases} \end{aligned} \qquad (4)$$

Here, the saturations with β^d in θ_{ij}^d and ϕ_{ij}^d are included to obtain less conservative conditions for stability (see Remark 3 in Section 3.2). In addition, note that the term ϕ_{ij}^d is always well-defined, and, in fact, $0 \le \phi_{ij}^d \le 1$ for all $i, j \in \mathcal{R}$. On the other hand, the scalar function f_i^d represents the payoff provided by the i-th robot to the d-th population. Such scalar function is denoted as a fitness function and, as shown in (4), depends on local information available to the i-th robot (yet, we omit its arguments to simplify the notation). In consequence, notice that (3) defines a set of n equations where the i-th equation depends only on local information available to the i-th robot. Thus, to obtain a distributed computation of (3), it suffices that each robot $i \in \mathcal{R}$ computes its corresponding i-th equation. Furthermore, note that a particular equilibrium of the dynamics of (3) occurs when $f_i^d = f_j^d$ for all $i, j \in \mathcal{R}$ (by equilibrium we mean that $r_i^d[k+1] = r_i^d[k]$, for all $i \in \mathcal{R}$). Such equilibrium is independent of $r_\ell^d[k]$ and occurs when

$$r_i^d[k] = c_\ell^d[k] + \delta_i^d[k], \quad \forall i \in \mathcal{F}. \qquad (5)$$

Therefore, if we guarantee that the dynamics in (3) always converge to (5), then we can use such dynamics to solve (2) in a distributed fashion. The study of such convergence is the topic of the next section.

3.2 Stability Analysis

In this section we provide sufficient conditions to guarantee the asymptotic stability of our proposed discrete-time dynamics (3) to the minimizer of (2). However, before doing so, we need to introduce some additional results. First of all, note that (3) can be written as

$$\mathbf{r}^d[k+1] = \mathbf{r}^d[k] + \epsilon^d \mathbf{L}^d[k]\mathbf{f}^d[k], \quad \forall d \in \mathcal{D}, \tag{6}$$

where $\mathbf{f}^d[k] = [f_i^d]$ is an $n \times 1$ vector that collects all the fitness functions of the d-th population; and $\mathbf{L}^d[k] = [l_{ij}^d]$ is an $n \times n$ matrix whose elements depend on $\mathbf{r}^d[k]$ and $\mathcal{G}[k]$, and are given by

$$\begin{aligned} l_{ii}^d &= \sum_{j \in \mathcal{R}\setminus\{i\}} a_{ij}[k]\theta_{ij}^d \phi_{ij}^d, \quad \forall i \in \mathcal{R}, \forall d \in \mathcal{D} \\ l_{ij}^d &= -a_{ij}[k]\theta_{ij}^d \phi_{ij}^d, \quad \forall i,j \in \mathcal{R}, \forall d \in \mathcal{D}. \end{aligned} \tag{7}$$

Here, the arguments of l_{ii}^d and l_{ij}^d have been omitted to ease notation. Notice that the matrix $\mathbf{L}^d[k]$ is the Laplacian of a dynamical graph $\mathcal{G}^d[k]$ that depends on $\mathbf{r}^d[k]$ and the original graph $\mathcal{G}[k]$, and whose adjacency matrix has the elements $a_{ii}^d[k] = 0$ and $a_{ij}^d[k] = -l_{ij}^d$ for all $i,j \in \mathcal{R}$, with $i \neq j$, and for all $d \in \mathcal{D}$. Furthermore, observe that the matrix form given by (6) implies the following assumption.

Assumption 3: If $a_{ij}[k] = 1$, then the population portions $r_i^d[k]$ and $r_j^d[k]$ are updated at time k, for all $d \in \mathcal{D}$.

These observations lead to the following results.

Lemma 1. Consider the discrete-time distributed Smith dynamics given by (3), and consider the set

$$\mathcal{M}^d = \left\{ \mathbf{r}^d[k] \in \mathbb{R}^n \, : \, \sum_{i \in \mathcal{R}} r_i^d[k] = m^d \right\}, \tag{8}$$

where the scalar $m^d > 0$ is the total population mass. If Assumption 3 holds, then the set $\mathcal{M}^d$ is forward invariant under the dynamics (3). That is, $\mathbf{r}^d[k] \in \mathcal{M}^d$ implies that $\mathbf{r}^d[k+1] \in \mathcal{M}^d$ for all k.

Proof. For all $d \in \mathcal{D}$ and all k it holds that

$$\begin{aligned} \sum_{i \in \mathcal{R}} \sum_{j \in \mathcal{N}_i[k]} (f_i^d - f_j^d)\theta_{ij}^d \phi_{ij}^d &= \sum_{i \in \mathcal{R}} \sum_{j \in \mathcal{R}} a_{ij}[k](f_i^d - f_j^d)\theta_{ij}^d \phi_{ij}^d \\ &= 0. \end{aligned}$$

This follows from the facts that $a_{ij}[k] = a_{ji}[k]$, $\theta_{ij}^d = \theta_{ji}^d$, and $\phi_{ij}^d = \phi_{ji}^d$, for all $i,j \in \mathcal{R}$, all $d \in \mathcal{D}$, and all k. Therefore, we have that $\sum_{i \in \mathcal{R}} r_i^d[k+1] = \sum_{i \in \mathcal{R}} r_i^d[k]$ for all k, i.e., $\sum_{i \in \mathcal{R}} r_i^d[k] = \sum_{i \in \mathcal{R}} r_i^d[0] = m^d$ for all k. ■

Lemma 2. Consider the discrete-time distributed Smith dynamics given by (6), and consider the set

$$\Delta^d = \mathbb{R}_+^n \cap \mathcal{M}^d, \tag{9}$$

where $\mathbb{R}_+^n$ is the strictly positive orthant of $\mathbb{R}^n$. If Assumption 3 holds and $\mathbf{r}^d[k] \in \Delta^d$ for all k, then the following properties hold:

(i) The matrix $\mathbf{L}^d[k]$ is positive semi-definite for all k.

(ii) The dynamical graph $\mathcal{G}^d[k]$ has the same communication links as the original graph $\mathcal{G}[k]$ for all k.

Proof. To prove (i) we observe that $\mathbf{L}^d[k]$ is real, symmetric, and diagonally dominant (see (7)). In addition, if $\mathbf{r}^d[k] \in \Delta^d$ for all k, then the diagonal elements of $\mathbf{L}^d[k]$ are non-negative for all k. In consequence, $\mathbf{L}^d[k] \succeq 0$ for all k. To prove (ii) we notice that the fitness functions defined in (4) are finite over all Δ^d. In consequence, $\phi_{ij}^d > 0$ for any $\mathbf{r}^d[k] \in \Delta^d$ and for all $i,j \in \mathcal{R}$. Similarly, if $\mathbf{r}^d[k] \in \Delta^d$, we have that $\theta_{ij}^d > 0$ for all $i,j \in \mathcal{R}$. In consequence, $a_{ij}^d[k] > 0$ if and only if $a_{ij}[k] = 1$. Thus, $\mathcal{G}[k]$ and $\mathcal{G}^d[k]$ have the same set of communication links for all k. ■

Remark 1: Lemma 2 requires the forward invariance of the set Δ^d given in (9). Note that such invariance implies that $r_i^d[k] \in \mathbb{R}_+$ and $\sum_{i \in \mathcal{F}} r_i^d[k] = m^d - r_\ell^d[k]$, for all $i \in \mathcal{R}$, all $d \in \mathcal{D}$, and all k. Hence, $\sum_{i \in \mathcal{F}} r_i^d[k] < m^d$, for all $d \in \mathcal{D}$ and all k. Thus, to guarantee that the reference coordinate $(r_i^x[k], r_i^y[k])$, for all $i \in \mathcal{F}$, can achieve any value within $\mathcal{X}$, the population masses m^x and m^y have to be big enough. Given that $m^d = \sum_{i \in \mathcal{R}} r_i^d[0]$ for all $d \in \mathcal{D}$, in order to guarantee that any formation within $\mathcal{X}$ is attainable, it suffices to satisfy the following condition:

$$\begin{aligned} &\sum_{i \in \mathcal{R}} r_i^d[0] \geq n\beta^d, \quad \forall d \in \mathcal{D}, \\ &r_i^d[0] > 0, \quad \forall i \in \mathcal{R}, \forall d \in \mathcal{D}, \end{aligned} \tag{10}$$

where β^d is the spatial length of the d-th dimension of the rectangle $\mathcal{X}$ defined in (1). Note that if the initial position of the robots is within $\mathcal{X}$, then condition (10) can be satisfied under the informational constraints of $\mathcal{G}[k]$ by setting $r_\ell^d[0] \geq n\beta^d$ for all $d \in \mathcal{D}$, and $r_i^x[0] = x_i[0]$ and $r_i^y[0] = y_i[0]$ for all $i \in \mathcal{F}$.

With the aid of these preliminary results, we now provide sufficient conditions that guarantee the asymptotic stability of the minimizers of (2) under the proposed method. The next theorem illustrates our result.

Theorem 1. Consider the discrete-time distributed Smith dynamics given in (3) and (6). Let $\tilde{n}$ denote the maximum number of robots that any robot can communicate with (simultaneously) at any time k. Moreover, supose that Assumptions 1, 2, and 3 hold, and let the following conditions be satisfied:

(i) The initial population state satisfies condition (10), i.e., $\mathbf{r}^d[0] \in \mathbb{R}_+^n$ and $\sum_{i \in \mathcal{R}} r_i^d[0] \geq n\beta^d$, for all $d \in \mathcal{D}$.

(ii) For all $d \in \mathcal{D}$, it holds that $0 < \epsilon^d < 1/(\tilde{n}\beta^d)$.

Then the minimizers of (2) are asymptotically stable under the proposed discrete-time distributed Smith dynamics.

Proof. First, to induce the results of Lemma 2, we prove that under the given conditions it holds that $\mathbf{r}^d[k] \in \Delta^d$ for all k, where Δ^d is given by (9) with $m^d \geq n\beta^d$. From Lemma 1 we can conclude that $\mathbf{r}^d[k] \in \mathcal{M}^d$ for all k. Thus, we have to prove only the positiveness of $\mathbf{r}^d[k]$ for all k. To do so, given that by assumption $\mathbf{r}^d[0] \in \mathbb{R}_+^n$, we have to prevent any sign changes in $r_i^d[k]$ for all $i \in \mathcal{R}$ and all k. From (3)-(4) we note that the only terms of the summation in (3) that favor the decrement of $r_i^d[k]$ are those where $f_i^d - f_j^d < 0$. Thus, we should consider only the critical (impossible) case where $f_i^d < f_j^d$ for all $i,j \in \mathcal{R}$. Under such scenario we have that (3) can be written as

$$r_i^d[k+1] = r_i^d[k] - \epsilon^d \sum_{j\in\mathcal{N}_i[k]} \left|f_i^d - f_j^d\right| \min(r_i^d[k], \beta^d)\phi_{ij}^d,$$

for all $i \in \mathcal{R}$ and all $d \in \mathcal{D}$. Moreover, note that $r_i^d[k] \geq \min(r_i^d[k], \beta^d)$ for all k. Hence, to obtain a sufficient condition that prevents sign changes and guarantees positiveness, without loss of generality we can consider the case where θ_{ij}^d does not have a $\min(\cdot)$ saturator, i.e., the case where $\theta_{ij}^d = r_i^d[k]$ for any $\mathbf{r}^d[k] \in \Delta^d$ where $f_i^d < f_j^d$. Furthermore, we can also focus only on the worst (impossible) case where $|f_i - f_j| = \beta^d$ for all $i, j \in \mathcal{R}$ (such saturation is given by the term ϕ_{ij}^d). Joining all these critical cases we get that (3) can be written as

$$\begin{aligned} r_i^d[k+1] &= r_i^d[k]\left(1 - \epsilon^d \sum_{j\in\mathcal{N}_i[k]} \beta^d\right), \quad \forall i \in \mathcal{R}, \forall d \in \mathcal{D} \\ &= r_i^d[k]\big(1 - \epsilon^d \tilde{n}\beta^d\big), \quad \forall i \in \mathcal{R}, \forall d \in \mathcal{D}. \end{aligned}$$

Therefore, to prevent sign changes of $r_i^d[k]$ we require that $0 < \epsilon^d < 1/(\tilde{n}\beta^d)$, which is guaranteed by condition (ii). Observe that the strict bounds in this condition not only prevent sign changes, but also prevent that $r_i^d[k]$ becomes zero for any k. Thus, by conditions (i) and (ii) it holds that $\mathbf{r}^d[k] \in \Delta^d$ for all k, and Lemma 2 applies. From now on, we consider the original dynamics given by (3) and not only the worst cases, i.e., we consider the saturator in θ_{ij}^d.

Now that we have proved that Lemma 2 applies, we can proceed to prove the asymptotic stability of the proposed dynamics. For that, notice that the dynamics of (3) are invariant under the addition of a scalar to the fitness functions. This follows from the fact that $(f_i^d - f_j^d) = (f_i^d + \alpha - f_j^d - \alpha)$ for any $\alpha \in \mathbb{R}$. Thus, the dynamics under the original fitness vector $\mathbf{f}^d[k]$ are exactly the same as if the fitness vector $\tilde{\mathbf{f}}^d[k] = \mathbf{f}^d[k] + c_\ell^d[k]\mathbf{1}_n$ were used instead (here $\mathbf{1}_n$ denotes a column vector with n ones), i.e., we have that $\tilde{f}_\ell^d = 0$ and $\tilde{f}_i^d = c_\ell^d[k] + \delta_i^d[k] - r_i^d[k]$ for all $i \in \mathcal{F}$. In consequence, without loss of generality, we analyze the stability of $\mathbf{r}^d[k+1] = \mathbf{r}^d[k] + \epsilon^d \mathbf{L}^d[k]\tilde{\mathbf{f}}^d[k]$ to obtain convergence results that are also valid for (6). The advantage of using $\tilde{\mathbf{f}}^d[k]$, instead of $\mathbf{f}^d[k]$, is that $\tilde{\mathbf{f}}^d[k]$ can be written as $\tilde{\mathbf{f}}^d[k] = -\mathbf{B}\mathbf{e}^d[k]$, where $\mathbf{B}$ is an $n \times n$ diagonal matrix with the (ℓ,ℓ)-th element equal to zero and the other diagonal elements equal to 1; and $\mathbf{e}^d[k] = \mathbf{r}^d[k] - \mathbf{r}_*^d[k]$, where $\mathbf{r}_*^d[k]$ is the population state for which $\tilde{f}_i^d = \tilde{f}_j^d = 0$ for all $i, j \in \mathcal{R}$, i.e., the minimizer of (2). Moreover, from Assumption 2 we can conclude that such population state $\mathbf{r}_*^d[k]$ is always attainable with a population mass $m^d \geq n\beta^d$ (see condition (i)), i.e., $\mathbf{r}_*^d[k]$ is always within Δ^d. Under this formulation, we can use the function $V(\mathbf{r}^d[k]) = (\mathbf{e}^d[k])^\top \mathbf{B}\mathbf{e}^d[k]$ as a valid Lyapunov function candidate (i.e., $V(\mathbf{r}^d[k]) > 0$, for all $\mathbf{r}^d[k] \neq \mathbf{r}_*^d[k]$ and $V(\mathbf{r}^d[k]) = 0 \iff \mathbf{r}^d[k] = \mathbf{r}_*^d[k]$). It is worth mentioning that although $\mathbf{B}$ is a positive semi-definite matrix, the proposed Lyapunov function is positive definite over Δ^d. To see why, note that $\mathbf{B}$ has only one zero diagonal element and such element is associated to $r_\ell^d[k]$. Nevertheless, by Lemma 1 we have that $r_\ell^d[k] = m^d - \sum_{i\in\mathcal{F}} r_i^d[k]$ for all k. Thus, there are only $n-1$ independent population portions, and, in consequence, the line where $V(\mathbf{r}^d[k]) = 0$ contracts to a single point in $\mathbb{R}^n$ for every k. Continuing with the proof, for $\mathbf{r}_*^d[k]$ to be asymptotically stable we require that $V(\mathbf{r}^d[k+1]) - V(\mathbf{r}^d[k]) < 0$ for all k. If we denote $\tilde{V} = V(\mathbf{r}^d[k+1]) - V(\mathbf{r}^d[k])$ we get:

$$\tilde{V} = \left(\mathbf{e}^d + \epsilon^d \mathbf{L}^d \tilde{\mathbf{f}}^d\right)^\top \mathbf{B}\left(\mathbf{e}^d + \epsilon^d \mathbf{L}^d \tilde{\mathbf{f}}^d\right) - (\mathbf{e}^d)^\top \mathbf{B}\mathbf{e}^d,$$

where we have removed the time index in $\mathbf{e}^d[k]$, $\mathbf{L}^d[k]$, and $\tilde{\mathbf{f}}^d[k]$ as all the terms are now at time k. Solving this expression we get that $\tilde{V}$ equals:

$$\begin{aligned} &= \epsilon^d (\mathbf{e}^d)^\top \mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d + \epsilon^d (\mathbf{L}^d\tilde{\mathbf{f}}^d)^\top \mathbf{B}\mathbf{e}^d + (\epsilon^d)^2 (\mathbf{L}^d\tilde{\mathbf{f}}^d)^\top \mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d \\ &= \epsilon^d (\mathbf{B}\mathbf{e}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d + \epsilon^d (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{e}^d + (\epsilon^d)^2 (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d \\ &= -\epsilon^d (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d - \epsilon^d (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d + (\epsilon^d)^2 (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d \\ &= -2\epsilon^d (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d + (\epsilon^d)^2 (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d, \end{aligned}$$

where we have used the facts that $\mathbf{B} = (\mathbf{B})^\top$, $\mathbf{L}^d = (\mathbf{L}^d)^\top$, and $\tilde{\mathbf{f}}^d = -\mathbf{B}\mathbf{e}^d$. Notice that due to the form of $\mathbf{B}$ it holds that $(\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d \leq (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{L}^d\tilde{\mathbf{f}}^d$. Thus, setting $\mathbf{L}^d = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ (by spectral decomposition) we have that

$$\begin{aligned} (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\mathbf{B}\mathbf{L}^d\tilde{\mathbf{f}}^d &\leq (\tilde{\mathbf{f}}^d)^\top \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}\tilde{\mathbf{f}}^d \\ &\leq (\tilde{\mathbf{f}}^d)^\top \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}\mathbf{\Lambda}^{1/2}\mathbf{P}^{-1}\tilde{\mathbf{f}}^d \\ &\leq \lambda_{max}^{\mathbf{\Lambda}} (\tilde{\mathbf{f}}^d)^\top \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{P}^{-1}\tilde{\mathbf{f}}^d \\ &\leq \lambda_{max}^{\mathbf{L}^d} (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d. \end{aligned}$$

Here $\lambda_{max}^{\mathbf{\Lambda}} = \lambda_{max}^{\mathbf{L}^d}$ is the maximum eigenvalue of $\mathbf{L}^d$, and we have used the fact that a quadratic form $\mathbf{x}^\top\mathbf{Z}\mathbf{x}$ with symmetric $\mathbf{Z}$ is always upper-bounded by $\lambda_{max}^{\mathbf{Z}}\mathbf{x}^\top\mathbf{x}$. From the assumption that $\mathcal{G}[k]$ is connected, and due to Lemma 2, we can conclude that $\mathcal{G}^d[k]$ is also connected. This means that $\mathbf{L}^d$ has only one eigenvalue equal to zero, and, from Lemma 2, we have that the remaining eigenvalues are strictly positive. Thus, $\lambda_{max}^{\mathbf{L}^d} > 0$. Moreover, from Gershgorin Circle Theorem we have that $\lambda_{max}^{\mathbf{L}^d} \leq 2\max_{i\in\mathcal{R}} l_{ii}^d$, and from (7) we can conclude that $\lambda_{max}^{\mathbf{L}^d} \leq 2\tilde{n}\beta^d$. Hence,

$$\begin{aligned} \tilde{V} &\leq -2\epsilon^d (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d + 2\tilde{n}\beta^d (\epsilon^d)^2 (\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d \\ &\leq 2\epsilon^d (\tilde{n}\beta^d\epsilon^d - 1)(\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d. \end{aligned}$$

Notice that $(\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d$ is the quadratic form of the Laplacian of $\mathcal{G}^d[k]$. Given that $\mathcal{G}^d[k]$ is undirected and connected (by the symmetry of $\mathbf{L}^d$ and Lemma 2), we can conclude that $(\tilde{\mathbf{f}}^d)^\top \mathbf{L}^d\tilde{\mathbf{f}}^d$ is always positive and is zero only when $\tilde{\mathbf{f}}^d \in \mathrm{span}(\mathbf{1}_n)$, i.e., when $f_i^d = f_j^d$ for all $i, j \in \mathcal{R}$. Furthermore, if ϵ^d satisfies condition (ii), the term $(\tilde{n}\beta^d\epsilon^d - 1)$ is always strictly negative. In consequence, $\tilde{V}$ is always non-positive and is zero only when $\mathbf{r}^d[k] = \mathbf{r}_*^d[k]$. Thus, $\mathbf{r}_*^d[k]$ is asymptotically stable. ■

Remark 2: Observe that Theorem 1 is valid for any connected and undirected graph topology. Moreover, in the proof of Theorem 1 the Laplacian of $\mathcal{G}^d[k]$ only appears at time k. By induction, Theorem 1 applies unchanged for time-varying graphs that remain connected and undirected for all k.

Remark 3: Note that without the saturations in θ_{ij}^d and ϕ_{ij}^d, the term $\lambda_{max}^{\mathbf{L}^d} \leq 2\max_{i\in\mathcal{R}} l_{ii}^d$ would be upper-bounded by $2\tilde{n}n\beta^d$ and stability would not be guaranteed by condition (ii) (it can be shown that a similar issue occurs with the bound required for positiveness). However, given that the formations are assumed to lie within $\mathcal{X}$, the saturations in (4) allow us to consider only the region of interest and obtain less conservative bounds for ϵ^d.

Remark 4: Notice that the conjunction of Assumptions 1 and 3 implies that all robots update their corresponding variables at all times k. Thus, Theorem 1 assumes that there is a persistent synchronicity on the robots' operation, which might be hard to satisfy for some robotic applications. Although we leave the formal study of asynchronous population dynamics for a future work, in Section 4 we provide some experiments where the graph $\mathcal{G}[k]$ is not connected for all k, and so the synchronicity of all robots is not required for all k (the synchronicity is required only for each connected sub-graph at each time k). Such experiments illustrate the fact that the aforementioned persistent synchronicity of all robots is only a sufficient condition for asymptotic stability.

4. EXPERIMENTAL RESULTS

In this section we illustrate the theoretical developements of this paper on a real multi-robot platform comprised by a set of six e-puck v2 robots. Due to space limitations, we only show the results of a particular experiment under a time-invariant path graph (i.e., $\tilde{n} = 2$), which is the connected graph with the most distributed topology. However, videos of several other experiments are available at `youtu.be/t-SOGtblh_A` and some of them consider time-varying graphs that are not connected for all k. The results of our particular experiment are depicted in Fig. 3 where DT-DSD denotes our discrete-time distributed Smith dynamics; Consensus refers to the consensus algorithm of Olfati-Saber et al. (2007); and SBPG denotes the state-based potential games approach proposed by Li and Marden (2013). As key performance index (KPI) we have taken the sum of the objectives functions of (2) over all $i \in \mathcal{F}$ and all $d \in \mathcal{D}$, and we have normalized it up to 1. In all cases, robots start in the same initial position and the required formation is an hexagon. The bounds for the DT-DSD are taken according to Theorem 1 with $\beta^x = 125$ and $\beta^y = 90$ (the dimensions of our testbed); the bound for the Consensus is taken according to Olfati-Saber et al. (2007); and the bound of SBPG is set at the biggest value (within 0.001 precision) that converged in this particular experiment. As shown in Fig. 3, all the methods converge to the desired solution. In particular, to achieve a KPI lower than 0.02 our method required 37 iterations, Consensus required 28, and SBPG required 451 (iterations were taken every 0.1 seconds). Thus, we can conclude that our method outperforms the SBPG and obtains a performance comparable to the consensus algorithm. The advantage of our method in contrast with Consensus, is that it preserves the forward-invariance of Δ^d, and, in consequence, is suitable not only to achieve robotic formations as in this paper, but also for resource allocation problems that are relevant for many other distributed optimization and control applications (see Quijano et al. (2017)).

5. CONCLUDING REMARKS AND FUTURE WORK

In this paper we have proposed the discrete-time distributed Smith dynamics, and we have provided sufficient conditions for stability in practical implementations where computations are necessarily discrete. Moreover, we have illustrated the application of the proposed method on a real robotic platform with six e-puck v2 robots under a leader-follower scheme. Future work should focus on the application of the of the developed theory to other types of problems, as well as the extension of the stability analysis to asynchronous implementations, and the characterization of the convergence rate of our proposed method.

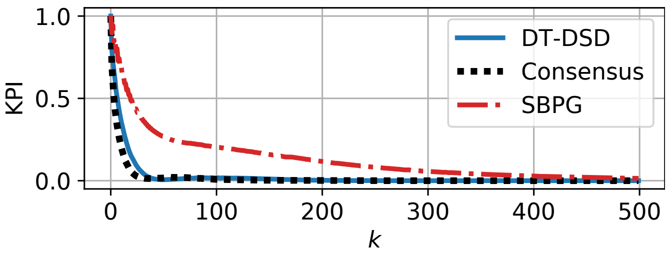

Fig. 3. Experimental results on real robots.

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