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Performance Assessment and Design of Quadratic Alarm Filters ⋆

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Abstract: Alarm filtering is a structurally simple, easy to implement, and effective method to improve industrial alarm systems. Owing to these advantages, alarm filters are widely used in industrial applications. Linear and quadratic are the main types of alarm filters. Although a linear filter can detect mean changes, it can not be used to detect variation changes. However, a quadratic filter can be used to detect both types of changes. Although this remarkable feature of quadratic filters has been addressed in the literature, no explicit performance analysis is performed yet. So, deriving an analytical solution for quadratic filters is of paramount importance. To this aim, we propose an analytical method for performance assessment and design of quadratic filters. On the other side, in industrial applications, many process variables are acquired. So one challenge is to identify the process variable that provides the best alarm performance after filtering. We will derive an analytical solution to this problem. Furthermore, we will prove that this optimal solution is a function of the statistical feature of historical data and alarm filter structure.

Keywords: Alarm systems, fault detection, performance monitoring, quadratic filters

1. INTRODUCTION

In process industries, alarm management systems are of prominent importance to maintain safe operation and satisfy a certain level of performance. An alarm system can be viewed as a classifier that notifies operators if the process is working abnormally. Ideally, we expect no false and missed alarms. However, in reality, numerous false alarms are distracting operators that degrade overall proficiency. The main concerns for analysis and design of industrial alarm systems are addressed in Wang et al. [2015], Kourti [2002] and Nimmo [1999]. The problem of unnecessary alarms is not only limited to industries; a study conducted by Lawless [1994] surprisingly shows that over 94% of alarm warnings in intensive care units are clinically worthless. Hagenouw [2007] reported that false alarms impede clinical care in anesthesia environments. Arrue et al. [2000] stated the problem of false alarms in forest-fire detection. Abe et al. [2009] conducted some experiments to explore the effects of missed alarms on a driver's trust in the alarm system. Inspired by this great demand to improve alarm systems, many researchers have proposed effective methods such as dead-bands (see Naghoosi et al. [2011]), delay timers (see Zang et al. [2015]), latches (see Kondaveeti et al. [2011]) and various filters (see Gustafsson and Palmqvist [1997]).

In a basic alarm system, a process variable is compared with a constant alarm threshold, and an alarm is raised if the process variable exceeds the threshold. The alarm system can be improved by adding a filter, and now the filtered data can be compared by a new alarm threshold. The main types of alarm filters are linear and quadratic filters, which have been introduced in Cheng et al. [2011]. Quadratic filters have a significant advantage over linear filters. In a quadratic filter, both mean and variance changes can be detected, but a linear filter can only identify the mean change. In Cheng et al. [2011], the authors introduced an algorithm to find the optimal filter weights. However, we need an analytical solution to facilitate the design procedure and provide intuition about the impact of filter weights on the performance of the alarm system. On the other side, in industrial applications, numerous process variables are measured by deployed sensors in the plants. So the problem is to determine the process variable that is the best representative in case of fault occurrence. Geng et al. [2005] addressed this problem by ranking alarm variables based on a fuzzy clustering algorithm. So we also need an analytical solution to identify the optimal process variable for filtering.

In this paper, we exploit an alarm index, proposed by Roohi et al. [2019], to evaluate the performance of quadratic filters. We provide an analytical expression for alarm performance of two classes of quadratic filters. Furthermore, we propose a new score for appraising process variables to indicate the optimal choice. Via an example, we illustrate that this optimal choice may be changed by modifying the filter formulation.

2. PROBLEM FORMULATION

Let $x[k], k \in \{0, 1, 2, \cdots\}$, indicate a process variable that is measured in a plant. Suppose that samples of this process variable are independent, identically distributed and follow

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$$\mathcal{X} \sim \begin{cases} \mathcal{N}(\mu_{x,\mathrm{n}}, \sigma^2_{x,\mathrm{n}}), & k < T_{ab}, \\ \mathcal{N}(\mu_{x,\mathrm{ab}}, \sigma^2_{x,\mathrm{ab}}), & k \geq T_{ab}, \end{cases} \tag{1}$$

where T_{ab} corresponds to the time of abnormality occurrence, $\sigma^2_{x,\mathrm{ab}}$ (resp. $\mu_{x,\mathrm{ab}}$) and $\sigma^2_{x,\mathrm{n}}$ (resp. $\mu_{x,\mathrm{n}}$) are corresponding to the variance (resp. mean) of x in abnormal and normal operation modes, respectively. Now let $y[k], k \in \{0, 1, 2, \cdots\}$, denote the output samples of a quadratic alarm filter. The formulation of a general quadratic filter of order N (see Cheng et al. [2011]) is given by

$$y[k] = \boldsymbol{x} Q \boldsymbol{x}^T, \tag{2}$$

where Q is a symmetric matrix and

$$\boldsymbol{x} \triangleq \big[x[k] \cdots x[k-N+1] \; 1\big].$$

The alarm system decides weather to raise an alarm or not based on what follows:

$$\begin{cases} \text{alarm}, & y[k] > y_{th}, \\ \text{no alarm}, & \text{otherwise}, \end{cases}$$

where y_{th} indicates the alarm threshold which is designed by the operator using the historical information of the plant. According the results of Roohi et al. [2019], we define the alarm performance index as

$$\mathcal{A}(y) \triangleq \frac{\sigma^2_{y,\mathrm{ab}} + \sigma^2_{y,\mathrm{n}}}{(\mu_{y,\mathrm{ab}} - \mu_{y,\mathrm{n}})^2}, \tag{3}$$

where $\sigma^2_{y,\mathrm{ab}}$ (resp. $\mu_{y,\mathrm{ab}}$) and $\sigma^2_{y,\mathrm{n}}$ (resp. $\mu_{y,\mathrm{n}}$) are corresponding to the variance (resp. mean) of y in abnormal and normal operation modes, respectively. A smaller index represents a better distinguishability of normal and abnormal operation modes. Intuitively, when the mean difference of abnormal and normal modes is large, and variance of each mode is small, it is easier to separate these two modes by using a constant threshold. The following lemma holds for the introduced alarm index.

Lemma 1. Suppose that $\mathcal{A}(y)$, $\mathcal{A}(y_c)$ and $\mathcal{A}(y_m)$ are associated with $y[k]$, $y[k] + c$ and $my[k]$ where $c \in \mathbb{R}$ and $m \in \mathbb{R} - \{0\}$ are known and constant. It follows that

$$\mathcal{A}(y) = \mathcal{A}(y_c) = \mathcal{A}(y_m). \tag{4}$$

Proof. The proof is straightforward from the definition of the alarm index. □

Intuitively, if a constant value is multiplied by, or added to, a process variable, the trip point can be modified accordingly to compensate it. According to Lemma 1, without loss of generality, we can fix the upper-left and lower-right elements of Q to 1 and 0, respectively. Now we impose some constraints on the structure of filters and study two special cases.

2.1 Case I: diagonal Q

Suppose that the matrix Q has the following structure

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, Q_1 is a diagonal matrix and defined as

$$Q_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & q_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & q_{N-1} \end{bmatrix}, \tag{5}$$

where q_i's are non-negative weights. In this case, the filter can be reformulated as

$$y[k] = x^2[k] + \sum_{i=1}^{N-1} q_i x^2[k-i].$$

For this problem, our goal is to evaluate $\mathcal{A}(y)$, given the statistical information of x.

2.2 Case II: a more general case

Inspired by Cheng et al. [2011] we assume the following structure for Q:

$$Q_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & \alpha_0 \\ 0 & q_1 & \ddots & \vdots & \alpha_1 q_1 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & q_{N-1} & \alpha_{N-1} q_{N-1} \\ \alpha_0 & \alpha_1 q_1 & \dots & \alpha_{N-1} q_{N-1} & 0 \end{bmatrix}. \tag{6}$$

Considering this expression, one may rewrite the filter equation as

$$y'[k] = (x[k] + \alpha_0)^2 + \sum_{i=1}^{N-1} q_i (x[k-i] + \alpha_i)^2 + c,$$

where $c = -\Big(\alpha_0^2 + \sum_{i=1}^{N-1} \alpha_i^2 q_i\Big)$. It is worth noting that c does not affect the alarm performance of filter (see Lemma 1) and can be discarded in further analysis. Now the problem is to find an explicit expression for $\mathcal{A}(y')$.

3. PERFORMANCE ASSESSMENT OF CASE I

For this problem, the filter can be rewritten as $\boldsymbol{x}_1 Q_1 \boldsymbol{x}_1^T$, where

$$\boldsymbol{x}_1 \triangleq \big[x[k] \cdots x[k-N+1]\big]. \tag{7}$$

To evaluate the alarm performance index, we first need the following lemma, which is introduced by Provost and Mathai [1992].

Lemma 2. Consider the quadratic form $P_1(\boldsymbol{x}_1) = \boldsymbol{x}_1 Q_1 \boldsymbol{x}_1^T$, where Q_1 is a symmetric matrix and $\mathcal{X}_1 \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where Σ is a positive definite matrix. The r^{th} moment of $P_1(\boldsymbol{x}_1)$ for $r \in \{1, 2\}$ is expressed as

$$E[P_1(\boldsymbol{x}_1)]^r = \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} g^{(r-1-r_1)} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} g^{(r_1-1-r_2)}, \tag{8}$$

where

$$g^{(j)} = 2^j j! \big(\mathrm{tr}(Q_1\Sigma)^{j+1} + (j+1)\boldsymbol{\mu}(Q_1\Sigma)^j Q \boldsymbol{\mu}^T\big),$$

for $j \in \{0, 1, 2, \cdots\}$.

By using this lemma, the mean and variance of $P_1(\boldsymbol{x}_1)$ is determined as

$$\begin{aligned} E[P_1(\boldsymbol{x}_1)] &= \binom{0}{0} g^{(0)} \\ &= \mathrm{tr}(Q_1\Sigma) + \boldsymbol{\mu} Q_1 \boldsymbol{\mu}^T, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathrm{Var}[P_1(\boldsymbol{x}_1)] &= E[P_1(\boldsymbol{x}_1)]^2 - \big(E[P_1(\boldsymbol{x}_1)]\big)^2 \\ &= \left(\binom{1}{0} g^{(1)} + \binom{1}{1}\big(g^{(0)}\big)^2\right) - \binom{0}{0}\big(g^{(0)}\big)^2 \\ &= 2\mathrm{tr}(Q_1\Sigma)^2 + 4\boldsymbol{\mu} Q_1 \Sigma Q_1 \boldsymbol{\mu}^T. \end{aligned} \tag{10}$$

According to the equation in (2), it follows that

$$\Sigma_{\boldsymbol{x}_1,\mathrm{n}} = \sigma_{x_n}^2 I_N,$$

$$\Sigma_{\boldsymbol{x}_1,\mathrm{ab}} = \sigma_{x,\mathrm{ab}}^2 I_N,$$

where $\Sigma_{\boldsymbol{x}_1,\mathrm{n}}$ and $\Sigma_{\boldsymbol{x}_1,\mathrm{ab}}$ are corresponding to the normal and abnormal operation modes, respectively, and I_N is an identity matrix of size N. Considering the equation in (5), we have

$$\begin{aligned} \mathrm{tr}(Q_1\Sigma_{\boldsymbol{x}_1,\mathrm{n}}) &= \sigma_{x,\mathrm{n}}^2\Big(1+\sum_{i=1}^{N-1} q_i\Big), \\ \mathrm{tr}(Q_1\Sigma_{\boldsymbol{x}_1,\mathrm{ab}}) &= \sigma_{x,\mathrm{ab}}^2\Big(1+\sum_{i=1}^{N-1} q_i\Big), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathrm{tr}(Q_1\Sigma_{\boldsymbol{x}_1,\mathrm{n}})^2 &= \sigma_{x,\mathrm{n}}^4\Big(1+\sum_{i=1}^{N-1} q_i^2\Big), \\ \mathrm{tr}(Q_1\Sigma_{\boldsymbol{x}_1,\mathrm{ab}})^2 &= \sigma_{x,\mathrm{ab}}^4\Big(1+\sum_{i=1}^{N-1} q_i^2\Big). \end{aligned} \tag{12}$$

Furthermore, according to the equation in (2) we have

$$\begin{aligned} \boldsymbol{\mu}_{\boldsymbol{x}_1,\mathrm{n}} &= \mu_{x,\mathrm{n}}\,[1\ 1\ \cdots\ 1]\,, \\ \boldsymbol{\mu}_{\boldsymbol{x}_1,\mathrm{ab}} &= \mu_{x,\mathrm{ab}}\,[1\ 1\ \cdots\ 1]\,, \end{aligned} \tag{13}$$

where $\boldsymbol{\mu}_{\boldsymbol{x}_1,\mathrm{n}}$ and $\boldsymbol{\mu}_{\boldsymbol{x}_1,\mathrm{ab}}$ are corresponding to the normal and abnormal operation modes, respectively. By performing some algebraic manipulations on the equations in (9), (11) and (13), the mean of filtered data is determined as

$$\mu_y = \begin{cases} (\sigma_{x,\mathrm{n}}^2+\mu_{x,\mathrm{n}}^2)\Big(1+\sum_{i=1}^{N-1} q_i\Big), & N \le k < T_{ab}, \\ (\sigma_{x,\mathrm{ab}}^2+\mu_{x,\mathrm{ab}}^2)\Big(1+\sum_{i=1}^{N-1} q_i\Big), & k \ge T_{ab}+N. \end{cases} \tag{14}$$

Moreover, by manipulating the equations in (10), (11) and (13), we have

$$\sigma_y^2 = \begin{cases} (2\sigma_{x,\mathrm{n}}^4+4\mu_{x,\mathrm{n}}^2\sigma_{x,\mathrm{n}}^2)\Big(1+\sum_{i=1}^{N-1} q_i^2\Big), & N \le k < T_{ab}, \\ (2\sigma_{x,\mathrm{ab}}^4+4\mu_{x,\mathrm{ab}}^2\sigma_{x,\mathrm{ab}}^2)\Big(1+\sum_{i=1}^{N-1} q_i^2\Big), & k \ge T_{ab}+N. \end{cases} \tag{15}$$

Now by substituting (14) and (15) into the definition of alarm index (see the equation in (3)) we have

$$\mathcal{A}(y) = \frac{\big(2(\sigma_{x,\mathrm{n}}^4+\sigma_{x,\mathrm{ab}}^4)+4(\mu_{x,\mathrm{n}}^2\sigma_{x,\mathrm{n}}^2+\mu_{x,\mathrm{ab}}^2\sigma_{x,\mathrm{ab}}^2)\big)\Big(1+\sum_{i=1}^{N-1} q_i^2\Big)}{\big(\sigma_{x,\mathrm{n}}^2+\mu_{x,\mathrm{n}}^2-(\sigma_{x,\mathrm{ab}}^2+\mu_{x,\mathrm{ab}}^2)\big)^2\Big(1+\sum_{i=1}^{N-1} q_i\Big)^2}. \tag{16}$$

Hence, the impact of filter weights on the alarm performance index is given by

$$\mathcal{A}(y) \propto \frac{1+\sum_{i=1}^{N-1} q_i^2}{\Big(1+\sum_{i=1}^{N-1} q_i\Big)^2}. \tag{17}$$

Now by setting $\left[\frac{\partial \mathcal{A}}{\partial q_1}\ \frac{\partial \mathcal{A}}{\partial q_2}\ \cdots \frac{\partial \mathcal{A}}{\partial q_{N-1}}\right] = 0$, the optimal alarm weights are determined as $q_i = q,\ \forall i \in \{1, 2, \cdots, N-1\}$, where q can be any positive real number.

Remark 1. Although the best performance (in the view of (3)) can be achieved by setting all q_i's to one, there are some cases that operators decide to change the weights due to some circumstances. An example is when operators assign higher weights to the newer samples of a process variable to reduce detection delay. For this condition, (17) can be exploited as a straightforward measure for the accuracy of alarm systems. Adding other constraints gives rise to a new optimization problem.

The effect of statistical parameters of raw data x on the alarm performance of filtered data is obtained as

$$\mathcal{A}_y(x) \propto \frac{\big(2(\sigma_{x,\mathrm{n}}^4+\sigma_{x,\mathrm{ab}}^4)+4(\mu_{x,\mathrm{n}}^2\sigma_{x,\mathrm{n}}^2+\mu_{x,\mathrm{ab}}^2\sigma_{x,\mathrm{ab}}^2)\big)}{\big(\sigma_{x,\mathrm{n}}^2+\mu_{x,\mathrm{n}}^2-(\sigma_{x,\mathrm{ab}}^2+\mu_{x,\mathrm{ab}}^2)\big)^2}. \tag{18}$$

This expression can be utilized as a measure to help operators for exploring historical data and determining the optimal process variable for filtering. In this paper, we call $\mathcal{A}_y(x)$, the alarm score. A smaller alarm score corresponds to a better alarm performance after filtering.

4. PERFORMANCE ASSESSMENT OF CASE II

The quadratic filter associated with Case II can be reformulated as

$$P_2(\boldsymbol{x}) = \boldsymbol{x}\begin{bmatrix} Q_1 & \boldsymbol{\alpha}^T \\ \boldsymbol{\alpha} & 0 \end{bmatrix}\boldsymbol{x}^T,$$

where $\boldsymbol{x} = [\boldsymbol{x}_1\ 1]$, $\boldsymbol{x}_1$ is introduced by the equation in (7), and $\boldsymbol{\alpha} = [\alpha_0\ \alpha_1 q_1\ \ldots\ \alpha_{N-1}q_{N-1}]$. Furthermore, $\boldsymbol{x}_1\boldsymbol{\alpha}^T = \boldsymbol{\alpha}\boldsymbol{x}_1^T$, thus

$$P(\boldsymbol{x}) = \boldsymbol{x}_1 Q_1 \boldsymbol{x}_1^T + 2\boldsymbol{\alpha}\boldsymbol{x}_1^T. \tag{19}$$

By performing some modification on a lemma presented by Provost and Mathai [1992], we introduce the following lemma.

Lemma 3. Considering the same assumptions that are made in Lemma 2, the r^{th} moment of $P_2(\boldsymbol{x})$ for $r \in \{1, 2\}$ is determined by replacing $g^{(j)}$ with $g_*^{(j)}$ in (8), where

$$g_*^{(j)} = \begin{cases} \frac{1}{2}j!\sum_{i=1}^{N-1}(2\lambda_i)^{j+1} + \frac{(j+1)!}{2}\sum_{i=1}^{N-1} b_i^{*2}(2\lambda_i)^{j-1}, & j \ge 1, \\ \frac{1}{2}\sum_{i=1}^{N-1}(2\lambda_i) + 2\boldsymbol{\alpha}\boldsymbol{\mu} + \boldsymbol{\mu}Q_1\boldsymbol{\mu}^T, & j = 1. \end{cases}$$

Here,

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_{N-1} \end{bmatrix} = Q_1\Sigma,$$

and

$$\boldsymbol{b}^* \triangleq (b_1^*, b_2^*, \cdots, b_{N-1}^*) = 2\big(\Sigma^{\frac{1}{2}}\boldsymbol{\alpha} + \Sigma^{\frac{1}{2}}\boldsymbol{\mu}Q_1\big).$$

According to this lemma, the mean and variance of $P_2(x)$ is determined as

$$E[P_2(\boldsymbol{x})] = \mathrm{tr}(Q_1\Sigma) + 2\boldsymbol{\alpha}\boldsymbol{\mu} + \boldsymbol{\mu}Q_1\boldsymbol{\mu}^T, \tag{20}$$

and

$$\begin{aligned}\mathrm{Var}[P_2(\boldsymbol{x})] &= E[P_2(\boldsymbol{x})]^2 - \big(E[P_2(\boldsymbol{x})]\big)^2 \\ &= \left(\binom{1}{0} g_*^{(1)} + \binom{1}{1}\big(g_*^{(0)}\big)^2\right) - \binom{0}{0}\left(g_*^{(0)}\right)^2 \\ &= 2\mathrm{tr}(\Sigma Q_1)^2 + \sum_{i=1}^{N-1} b_i^{*2}. \end{aligned} \tag{21}$$

Now let $\boldsymbol{b}_n^*$ be corresponding to the normal operation mode. Then

$$\boldsymbol{b}_n^* = 2\sigma_{x,\mathrm{n}}\boldsymbol{\alpha} + 2\sigma_{x,\mathrm{n}}\boldsymbol{\mu}\begin{bmatrix}1 & 0 & \cdots & 0 \\ 0 & q_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & q_{N-1}\end{bmatrix}.$$

Hence

$$\begin{cases} b_{i,\mathrm{n}}^* = 2\sigma_{x,\mathrm{n}}(\alpha_i + \mu_{x,\mathrm{n}}), & i = 0, \\ b_{i,\mathrm{n}}^* = 2q_i\sigma_{x,\mathrm{n}}(\alpha_i + \mu_{x,\mathrm{n}}), & i \geq 1, \end{cases} \tag{22}$$

where $b_{i,\mathrm{n}}^*$'s are elements of $\boldsymbol{b}_n^*$. The same result holds for the abnormal operation modes ($b_{i,\mathrm{ab}}^*$). By substituting (11) and (13) into the equation in (20) and performing some algebraic manipulations, the mean of filtered data is obtained by

$$\mu_{y'} = \begin{cases} \mu_{y',\mathrm{n}} = (\sigma_{x,\mathrm{n}}^2 + \mu_{x,\mathrm{n}}^2)\Big(1 + \sum_{i=1}^{N-1} q_i\Big) + \\ \qquad 2\mu_{x,\mathrm{n}}\Big(\alpha_0 + \sum_{i=1}^{N-1}\alpha_i q_i\Big), & N \leq k < T_{ab}, \\ \mu_{y',\mathrm{ab}} = (\sigma_{x,\mathrm{ab}}^2 + \mu_{x,\mathrm{ab}}^2)\Big(1 + \sum_{i=1}^{N-1} q_i\Big) + \\ \qquad 2\mu_{x,\mathrm{ab}}\Big(\alpha_0 + \sum_{i=1}^{N-1}\alpha_i q_i\Big), & k \geq T_{ab} + N. \end{cases} \tag{23}$$

The variance of filtered data is determined by substituting (12) and (22) into equation (21):

$$\sigma_{y'}^2 = \begin{cases} \sigma_{y',\mathrm{n}}^2 = 4\sigma_{x,\mathrm{n}}^2\Big((\alpha_0 + \mu_{x,\mathrm{n}})^2 + \sum_{i=1}^{N-1} q_i^2(\alpha_i + \mu_{x,\mathrm{n}})^2\Big) + \\ \qquad 2\sigma_{x,\mathrm{n}}^4\Big(1 + \sum_{i=1}^{N-1} q_i^2\Big), & N \leq k < T_{ab}, \\ \sigma_{y',\mathrm{ab}}^2 = 4\sigma_{x,\mathrm{ab}}^2\Big((\alpha_0 + \mu_{x,\mathrm{ab}})^2 + \sum_{i=1}^{N-1} q_i^2(\alpha_i + \mu_{x,\mathrm{ab}})^2\Big) + \\ \qquad 2\sigma_{x,\mathrm{ab}}^4\Big(1 + \sum_{i=1}^{N-1} q_i^2\Big), & k \geq T_{ab} + N. \end{cases} \tag{24}$$

So the alarm index corresponding to the second scenario is given by

$$\mathcal{A}(y') = \frac{\sigma_{y',\mathrm{ab}}^2 + \sigma_{y',\mathrm{n}}^2}{\big(\mu_{y',\mathrm{ab}}^2 - \mu_{y',\mathrm{n}}^2\big)^2}, \tag{25}$$

where $\sigma_{y',\mathrm{ab}}$, $\sigma_{y',\mathrm{n}}$, $\mu_{y',\mathrm{ab}}$, and $\mu_{y',\mathrm{n}}$ are given by the equations in (23) and (24). Now consider a special case where $\alpha_i = \alpha$, $\forall i \in \{1, 2, \cdots, N\}$, and let $\tilde{y}'$ indicates the filter output, which is represented by

$$\tilde{y}'[k] = (x[k] + \alpha)^2 + \sum_{i=1}^{N-1} q_i(x[k-i] + \alpha)^2. \tag{26}$$

Under this assumption, we can obtain the mean and variance of $\tilde{y}'$ as

$$\mu_{\tilde{y}'} = \begin{cases} \big(\sigma_{x,\mathrm{n}}^2 + \mu_{x,\mathrm{n}}^2 + 2\alpha\mu_{x,\mathrm{n}}\big)\Big(1 + \sum_{i=1}^{N-1} q_i\Big), & N \leq k < T_{ab}, \\ \big(\sigma_{x,\mathrm{ab}}^2 + \mu_{x,\mathrm{ab}}^2 + 2\alpha\mu_{x,\mathrm{ab}}\big)\Big(1 + \sum_{i=1}^{N-1} q_i\Big), & k \geq T_{ab} + N. \end{cases}$$

and

$$\sigma_{\tilde{y}'}^2 = \begin{cases} \big(4\sigma_{x,\mathrm{n}}^2(\alpha + \mu_{x,\mathrm{n}})^2 + 2\sigma_{x,\mathrm{n}}^4\big)\Big(1 + \sum_{i=1}^{N-1} q_i^2\Big), & N \leq k < T_{ab}, \\ \big(4\sigma_{x,\mathrm{ab}}^2(\alpha + \mu_{x,\mathrm{ab}})^2 + 2\sigma_{x,\mathrm{ab}}^4\big)\Big(1 + \sum_{i=1}^{N-1} q_i^2\Big), & k \geq T_{ab} + N. \end{cases}$$

Now the relation of q_i's and the alarm performance index is determined as

$$\mathcal{A}(\tilde{y}') \propto \frac{1 + \sum_{i=1}^{N-1} q_i^2}{\Big(1 + \sum_{i=1}^{N-1} q_i\Big)^2}.$$

This expression is similar to the result that we derived for Case I, so the optimal q_i's can be obtained similar to the one in Case I. By performing some calculations, the optimal value for α is determined as

$$\alpha_{\mathrm{opt}} = \frac{\Pi_1\Pi_2 - \Pi_3\Pi_4}{\Pi_1\Pi_4 - \Pi_5\Pi_2}, \quad \Pi_1\Pi_4 \neq \Pi_5\Pi_2, \tag{27}$$

where

$$\begin{aligned} \Pi_1 &= 2(\sigma_{x,\mathrm{ab}}^2\mu_{x,\mathrm{ab}} + \sigma_{x,\mathrm{n}}^2\mu_{x,\mathrm{n}}), \\ \Pi_2 &= (\sigma_{x,\mathrm{ab}}^2 + \mu_{x,\mathrm{ab}}^2) - (\sigma_{x,\mathrm{n}}^2 + \mu_{x,\mathrm{n}}^2), \\ \Pi_3 &= (\sigma_{x,\mathrm{ab}}^4 + \sigma_{x,\mathrm{n}}^4) + 2(\sigma_{x,\mathrm{ab}}^2\mu_{x,\mathrm{ab}}^2 + \sigma_{x,\mathrm{n}}^2\mu_{x,\mathrm{n}}^2), \\ \Pi_4 &= 2(\mu_{x,\mathrm{ab}} - \mu_{x,\mathrm{n}}), \\ \Pi_5 &= 2(\sigma_{x,\mathrm{ab}}^2 + \sigma_{x,\mathrm{n}}^2). \end{aligned}$$

Hence, the effect of statistical parameters of process variable x on the alarm performance of filter data $\tilde{y}'$ is expressed as

$$\mathcal{A}_{\tilde{y}'}(x) \propto \frac{\Pi_5\alpha_{\mathrm{opt}}^2 + \Pi_1\alpha_{\mathrm{opt}} + \Pi_3}{(\Pi_4\alpha_{\mathrm{opt}} + \Pi_2)^2}. \tag{28}$$

5. NUMERICAL EXAMPLE

In this section, we study an example to demonstrate the effectiveness of the proposed method and verify the theoretical analysis. Consider a plant with three process variables that are sampled at discrete times. The process variables are indicated by $x_1[k]$, $x_2[k]$ and $x_3[k]$, $k \in \{1, 2, \cdots, T\}$ and are available for the alarm system to detect abnormal operation of the plant. Now assume that a fault occurred in the plant at sample $k = T_{ab}$. Mean and variance of the process variables can be estimated as

$$\mu_{x_j,\mathrm{n}} = \frac{1}{T_{ab}} \sum_{i=0}^{T_{ab}-1} x_j[k],$$

$$\mu_{x_j,\mathrm{ab}} = \frac{1}{T - T_{ab} + 1} \sum_{i=T_{ab}}^{T} x_j[k],$$

and

$$\sigma^2_{x_j,\mathrm{n}} = \frac{1}{T_{ab}} \sum_{i=0}^{T_{ab}-1} (x_j[k] - \mu_{x_j,\mathrm{n}})^2,$$

$$\sigma^2_{x_j,\mathrm{ab}} = \frac{1}{T - T_{ab} + 1} \sum_{i=T_{ab}}^{T} (x_j[k] - \mu_{x_j,\mathrm{ab}})^2,$$

where $j \in \{1, 2, 3\}$. We assume that after estimation of mean and variance, the following distributions are obtained:

$$\mathcal{X}_1 \sim \begin{cases} \mathcal{N}(0.2, 0.4^2), & k < T_{ab}, \\ \mathcal{N}(1, 1^2), & k \geq T_{ab}, \end{cases}$$

$$\mathcal{X}_2 \sim \begin{cases} \mathcal{N}(0.4, 0.3^2), & k < T_{ab}, \\ \mathcal{N}(1, 0.4^2), & k \geq T_{ab}, \end{cases}$$

$$\mathcal{X}_3 \sim \begin{cases} \mathcal{N}(0.1, 0.4^2), & k < T_{ab}, \\ \mathcal{N}(2.1, 1.4^2), & k \geq T_{ab}. \end{cases}$$

Now we use the equations in (18) and (28) to obtain the appropriate process variable for fault detection. The result is presented in Table 1.

Table 1. Alarm score of process variables

	x_1	x_2	x_3
$\mathcal{A}_y(x)$	1.87	**0.92**	1.10
$\mathcal{A}_{\tilde{y}'}(x)$	1.63	0.69	**0.46**

In this table, a smaller alarm score represents a better alarm performance after filtering. This result indicates that for the filter structures of Case I and Case II, we should select x_2 and x_3, respectively. Furthermore, we can infer that x_1 is not the right choice for either filter structure. Now let y_1, y_2 and y_3 denote the filtered data corresponding to the process variables x_1, x_2 and x_3, respectively. This result can also be concluded from Fig. 1 which is obtained by conducting a Monte Carlo simulation. Fig. 2 shows the alarm index of filtered data for Case I. Details of the studied scenarios are presented in Table 2. From Fig. 2, we can see that the best performance (in terms of the equation in (3)) can be archived by setting all q_i's to one. However, considering new constraints (see Remark 1), another scenario may be a good candidate. In all scenarios, the analytical result captures well the Monte Carlo simulation.

For Case II, the obtained α_{opt} for each process variable is presented in Table 3.
The simulation result of Fig. 3 verifies this analysis. Finally, Fig. 4 and Fig. 5 show time trends of x_2 and y_2, respectively. By comparing the histograms of x_2 and y_2, we conclude that the filter reduced the overlapped area of normal and abnormal operation modes. This implies that the separation of these two modes can be achieved with higher accuracy after filtering.

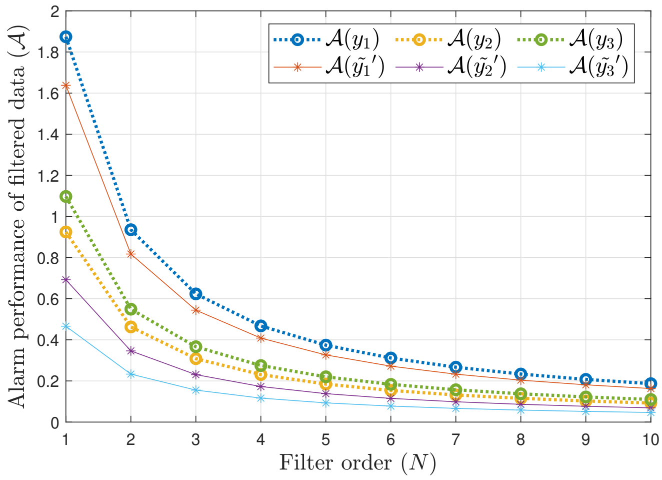

Fig. 1. Simulation result for various filter orders, where $q_i = 1,\ \forall i \in \{1, 2, \cdots, N-1\}$, and $\alpha = \alpha_{\mathrm{opt}}$.

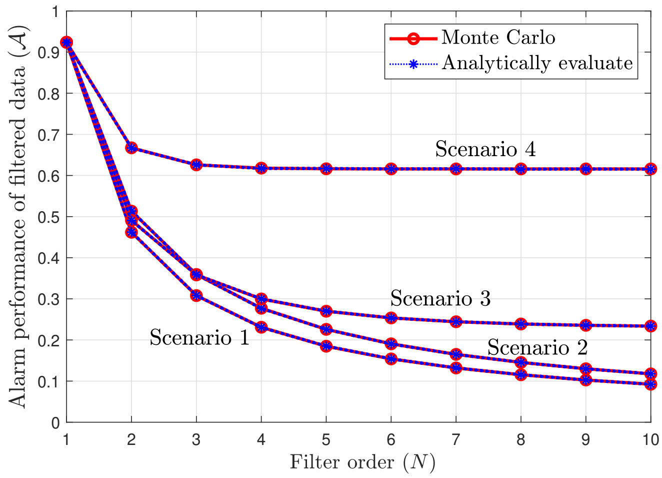

Fig. 2. Simulation result of $\mathcal{A}(y)$ where q_i's are selected according to Table 2.

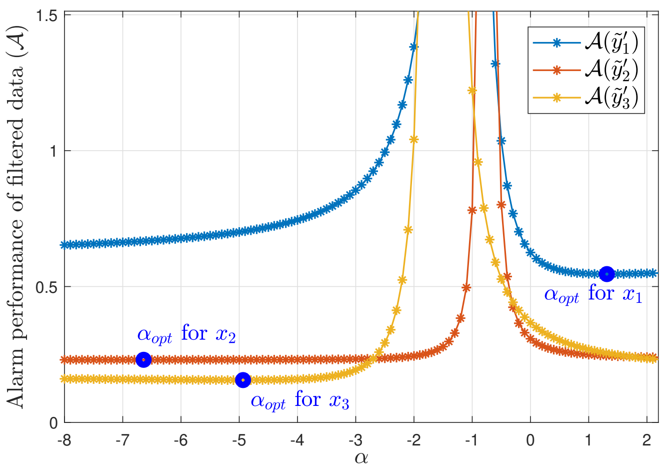

Fig. 3. Analytically evaluated optimal α (using the equation in (27)) and simulation result for various choices of α with $N = 3$.

Table 2. Simulation scenarios

Scenario	Filter weights
1	All set to 1
2	Set according to an arithmetic sequence with initial term 1 and common difference $1/N$
3	Set according to a geometric sequence with initial term 1 and common ratio 0.6
4	Set according to a geometric sequence with initial term 1 and common ratio 0.2

Table 3. Optimal α for process variables

	x_1	x_2	x_3
α_{opt}	1.31	-6.64	-4.93

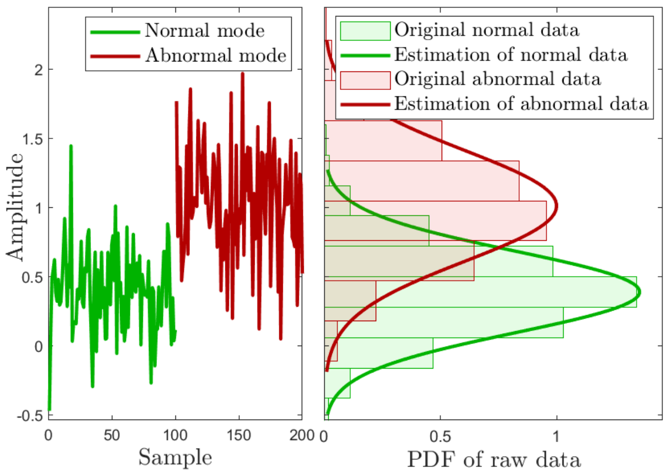

Fig. 4. Time trend and histogram of x_2.

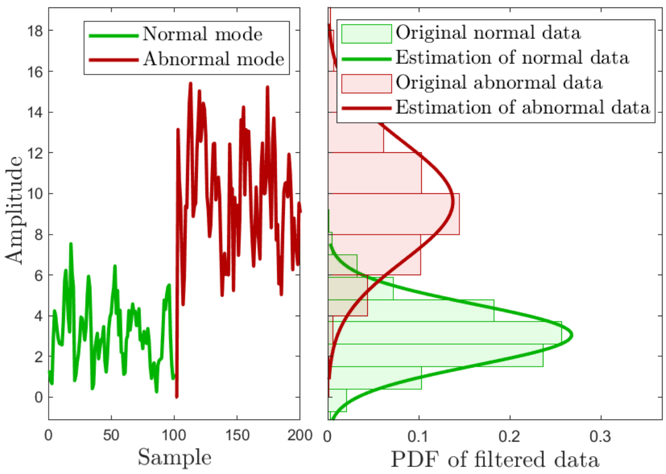

Fig. 5. Time trend and histogram of y_2 (filtered version of x_2 according to the scenario 1 with $N = 3$).

6. CONCLUSION

This paper addressed the problem of optimal quadratic filter design for industrial alarm management systems. We derived an explicit solution for the alarm performance of quadratic filters. We introduced a new score, which can be utilized to help plant operators to determine an appropriate process variable for alarm purposes. We also demonstrated that for different filter structures, this optimal choice might be different. The analysis of this paper can be combined with other alarm performance indices (e.g., alarm detection delay) to satisfy the requirements of various applications. It can also be served as a stepping stone to assess and design other forms of nonlinear alarm filters.

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