Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Dual lattices for non-strictly proper systems ⋆

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Abstract: This paper investigates the dual lattice structures of self-bounded and self-hidden subspaces of linear time-invariant systems arising in the solution of disturbance decoupling, regulator and unknown-input observation problems. The case that we are addressing in this paper is the one where the algebraic feedthrough matrices are allowed to be nonzero. We show that, in this general case, the additional constraints that need to be taken into account for the solution of the aforementioned control/estimation problems are no longer simple subspace inclusions as in the strictly proper case. As a consequence, mathematical apparatus underpinning the structure of the dual lattices of self bounded and self hidden subspaces in this more general framework becomes more challenging and richer.

Keywords: Dual lattices, self-boundedness, self-hiddenness, geometric control, disturbance decoupling, unknown-input observation.

1. INTRODUCTION

The two cornerstone problems that are traditionally addressed with geometric techniques are the disturbance decoupling problem by state feedback and the unknown-input observation problem Trentelman *et al.* (2001). Another fundamental control problem, which essentially combines the two above-mentioned problems, is the disturbance decoupling by dynamic output feedback. When asymptotic stability of the closed-loop is not required, the classical solution of these problems involves supremal output-nulling and infimal input-containing subspaces. When the additional stability constraint is introduced, the simplest adaptation of the classic solution involves the supremal stabilizability and the infimal detectability subspaces in place of the output-nulling and input-containing subspaces, respectively Trentelman *et al.* (2001), Stoorvogel and van der Woude (1991). An alternative solution, which is methodologically richer and computationally more efficient, relies on the concepts of self-boundedness and self-hiddenness. This solution avoids the computation of eigenspaces, which can lead to numerical issues, Basile and Marro (1986b).

A geometric theory of self-bounded and self-hidden subspaces requires characterizing lattices of structural invariants of the system. The so-called dual lattices show the interplay of output-nulling and self-bounded subspaces with their duals: they show in particular that these subspaces are not independent objects, but they are related by means of well-defined mappings, Basile and Marro (1986a). However, this analysis becomes crucial when dealing with the problem of disturbance decoupling by dynamic output feedback and the regulator problem, because in these contexts the solution involves the simultaneous use of self-bounded and self-hidden subspaces which in turn hinges on the universal bounds of these lattice structures.

In particular, a stream of literature flourished in the past twenty years showing that a controller that solves the problem by maximizing the freedom in the assignability of the closed-loop eigenvalues builds onto a pair of subspaces, one of which is self-bounded, and the other is self-hidden, see Del-Muro-Cuéllar (1997), Del-Muro-Cuéllar and Malabre (1997), Malabre *et al.* (1997). This pair is not unique: one can choose a self-hidden subspace from a certain lattice of subspaces and associate the correct self-bounded subspace, or, dually, one can select a self-bounded subspace from a certain lattice of subspaces and associate the correct self-hidden subspace. The analysis of the interplay between the two lattices provides a clear understanding of this mechanism.

Consequently, in this paper, we study the mathematics underlying this mechanism for systems that are not necessarily strictly proper by extending the dual lattice theory introduced in Basile and Marro (1986a). In the case of non-strictly proper systems, the structure of the lattices is considerably more complex. The disturbance decoupling problem by dynamic output feedback using the concepts of self boundedness and self hiddenness has only been addressed very recently in Padula and Ntogramatzidis (2019). The main challenge in addressing this general case is the fact that, while for purely dynamical systems the constraints on the lattices are expressed in terms of simple subspace inclusions in the state space, when we have nonzero feedthrough the constraints are given by inclusions in extended spaces (in the state + input space or in the state + output space).

We begin our investigation by first considering simple lattices of self-bounded and self-hidden subspaces of a biproper LTI system described by a quadruple (A,B,C,D). We prove the existence of a map between the set of output-nulling subspaces and the set of input-containing subspaces, which induces a bijection between self-bounded and self-hidden subspaces. Building on this result, we extend the theory of dual lattices to the interaction of different quadruples, as is standard in the context of the regulator problem and in the disturbance decoupling problem by dynamic output feedback. In this general framework we have 8 lattices whose interplay is analyzed in this paper under the conditions which arise in the solution of the aforementioned problems. Unlike the classical (strictly proper)

⋆ Partially supported by the Australian Research Council under the grants FT170100120 and DP190102478.

case, where these solvability conditions are simple subspace inclusions, when we have feedthrough matrices we need to generalize the conditions in extended vector spaces.

2. GEOMETRIC BACKGROUND

Let $\mathbb{T}$ denote either $\mathbb{N}$ or $\mathbb{R}_+$ in the discrete and continuous time, respectively. Consider a quadruple $\Sigma = (A, B, C, D)$ associated with the non-strictly proper LTI system

$$\begin{cases} \mathcal{D}\boldsymbol{x}(t) = A\,\boldsymbol{x}(t) + B\,\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\,\boldsymbol{x}(t) + D\,\boldsymbol{u}(t), \end{cases} \tag{1}$$

where the operator $\mathcal{D}$ denotes either the time derivative in the continuous time or the unit time shift in the discrete time and, for all $t \in \mathbb{T}$, $\boldsymbol{x}(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $\boldsymbol{u}(t) \in \mathcal{U} = \mathbb{R}^m$ is the input and $\boldsymbol{y}(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output. We denote by $\mathbb{C}_g$ a (continuous or discrete) self-conjugate stability domain. We denote by $\mathcal{R}$ the reachable subspace from the origin. We recall that $\mathcal{R}$ is also the smallest A-invariant subspace containing $\operatorname{im} B$, i.e., $\mathcal{R} = \langle A \,|\, \operatorname{im} B\rangle$. Dually, we denote by $\boldsymbol{Q}$ the unobservable subspace, i.e., $\boldsymbol{Q} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$, which is also the largest A-invariant subspace contained in $\ker C$, i.e., $\boldsymbol{Q} = \langle \ker C \,|\, A\rangle$.

A subspace $\mathcal{V}$ is said to be an output-nulling subspace if, for any initial state $\boldsymbol{x}_0 \in \mathcal{V}$, there exists a control function $\boldsymbol{u}$ such that the state trajectory generated by the system remains in $\mathcal{V}$ and the output remains identically at zero; equivalently, $\mathcal{V}$ is output-nulling if the subspace inclusion $\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}$ holds. Such a control function can again be expressed as the static state feedback $\boldsymbol{u}(t) = F\,\boldsymbol{x}(t)$. The condition of output-nullingness can be equivalently expressed by saying that there exists a feedback matrix F such that $\begin{bmatrix} A+BF \\ C+DF \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \oplus \{0\}$. In this case, we say that F is a *friend* of $\mathcal{V}$. We denote by $\mathfrak{F}_\Sigma(\mathcal{V})$ the set of friends of $\mathcal{V}$. The set of output-nulling subspaces $\mathcal{V}(\Sigma)$ of Σ is closed under addition. Thus, we can define the largest output-nulling subspace $\mathcal{V}^\star \stackrel{\text{def}}{=} \max \mathcal{V}(\Sigma) = \sum_{\mathcal{V} \in \mathcal{V}(\Sigma)} \mathcal{V}$ (also denoted as $\mathcal{V}^\star_\Sigma$ when we need to specify the system), which can be interpreted as the set of all initial states for which a control function exists for which the output can be maintained at zero. The sequence

$$\begin{cases} \mathcal{V}_0 = \mathcal{X} \\ \mathcal{V}_{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left((\mathcal{V}_i \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right), \quad i \in \mathbb{N} \end{cases} \tag{2}$$

is monotonically non-increasing and converges to $\mathcal{V}^\star$ in at most $n-1$ steps, i.e., $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_h = \mathcal{V}_{h+1} = \ldots$ implies $\mathcal{V}^\star = \mathcal{V}_h$, with $h \le n-1$.

Given an output-nulling subspace $\mathcal{V}$, we define the reachability subspace $\mathcal{R}_\mathcal{V}$ on $\mathcal{V}$ as the set of points that can be reached from the origin by means of control functions that maintain the state on $\mathcal{V}$ and the output at zero. Given a friend F of $\mathcal{V}$, there holds $\mathcal{R}_\mathcal{V} = \langle A + BF \,|\, \mathcal{V} \cap B \ker D\rangle$. The eigenvalues of $A + BF$, for $F \in \mathfrak{F}_\Sigma(\mathcal{V})$, can be divided into two multi-sets: the eigenvalues of the mapping $A + BF \,|\, \mathcal{V}$ and the eigenvalues of $A + BF \,|\, \frac{\mathcal{X}}{\mathcal{V}}$. In turn, the eigenvalues of $A + BF \,|\, \mathcal{V}$ can be divided into two multi-sets: the eigenvalues of $A + BF \,|\, \mathcal{R}_\mathcal{V}$ are freely assignable with a suitable choice of $F \in \mathfrak{F}_\Sigma(\mathcal{V})$, whereas those of $A + BF \,|\, \frac{\mathcal{V}}{\mathcal{R}_\mathcal{V}}$ are fixed, i.e., they are independent from $F \in \mathfrak{F}_\Sigma(\mathcal{V})$. Likewise, the eigenvalues of $A + BF \,|\, \frac{\mathcal{V}+\mathcal{R}}{\mathcal{V}}$ are assignable with a suitable choice of $F \in \mathfrak{F}_\Sigma(\mathcal{V})$, whereas those of $A + BF \,|\, \frac{\mathcal{X}}{\mathcal{V}+\mathcal{R}}$ are fixed for all $F \in \mathfrak{F}_\Sigma(\mathcal{V})$. We say that $\mathcal{V}$ is:

- *internally stabilizable* if there exists $F \in \mathfrak{F}_\Sigma(\mathcal{V})$ such that $\sigma(A + BF \,|\, \mathcal{V})$ is asymptotically stable, or, equivalently, if $\sigma(A + BF \,|\, \frac{\mathcal{V}}{\mathcal{R}_\mathcal{V}})$ is asymptotically stable;
- *externally stabilizable* if there exists $F \in \mathfrak{F}_\Sigma(\mathcal{V})$ such that $\sigma(A + BF \,|\, \frac{\mathcal{X}}{\mathcal{V}})$ is asymptotically stable, or, equivalently, if $\sigma(A + BF \,|\, \frac{\mathcal{X}}{\mathcal{V}+\mathcal{R}})$ is asymptotically stable.

An output-nulling subspace which is internally stabilizable is also referred to as a *stabilizability output-nulling subspace*. The set of stabilizability output-nulling subspaces is closed under addition, and thus it admits a maximum, that we denote by $\mathcal{V}^\star_g$: this subspace is the set of all initial states for which an input exists that maintains the output at zero and the state trajectory converges to the origin.

An output-nulling subspace $\mathcal{V}$ for which a friend F exists such that the spectrum of $A + BF \,|\, \mathcal{V}$ is arbitrary is called a reachability output-nulling subspace. The set of reachability output-nulling subspaces is closed under addition, and thus it admits a maximum, that we denote by $\mathcal{R}^\star$ or $\mathcal{R}^\star_\Sigma$: there holds $\mathcal{R}^\star \subseteq \mathcal{V}^\star_g \subseteq \mathcal{V}^\star$. The subspace $\mathcal{R}^\star$ is also the output-nulling reachability subspace on $\mathcal{V}^\star$, i.e., $\mathcal{R}^\star = \mathcal{R}_{\mathcal{V}^\star}$. This subspace can be interpreted as the set of all initial states that are reachable from the origin by control inputs that maintain the output at zero. The eigenstructure of $A + BF \,|\, \frac{\mathcal{V}^\star}{\mathcal{R}^\star}$ is the *invariant zero structure* of the system, and it is denoted by $\mathcal{Z}_\Sigma$.

An output-nulling subspace $\mathcal{V}$ is self-bounded if, for any initial state $\boldsymbol{x}_0 \in \mathcal{V}$, any control that gives an identically zero output is such that the entire state trajectory is forced to evolve on $\mathcal{V}$. In terms of inclusions, $\mathcal{V}$ is self-bounded if one of the following equivalent conditions holds:

$$(1)\ \ \mathcal{V} \supseteq \mathcal{V}^\star \cap B \ker D; \qquad (2)\ \ \mathcal{V} \supseteq \mathcal{R}^\star.$$

Thus, $\mathcal{R}^\star$ and $\mathcal{V}^\star$ are self-bounded. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are self-bounded and $\mathcal{V}_1 \subseteq \mathcal{V}_2$, then every friend of $\mathcal{V}_2$ is also a friend of $\mathcal{V}_1$, i.e., $\mathfrak{F}_\Sigma(\mathcal{V}_2) \subseteq \mathfrak{F}_\Sigma(\mathcal{V}_1)$. Since $\mathcal{R}^\star \subseteq \mathcal{V}^\star$, every friend of $\mathcal{V}^\star$ is also a friend of $\mathcal{R}^\star$. The intersection of self-bounded subspaces is self-bounded. We define Φ_Σ to be the set of self-bounded subspaces; then Φ_Σ admits both a maximum, which is $\mathcal{V}^\star$, and a minimum, which is $\mathcal{R}^\star$.

The dual of $\Sigma = (A, B, C, D)$ is $\Sigma^\top = (A^\top, C^\top, B^\top, D^\top)$. A subspace $\mathcal{S}$ is input-containing if $[\,A\ \ B\,]\big((\mathcal{S} \oplus \mathcal{U}) \cap \ker[\,C\ \ D\,]\big) \subseteq \mathcal{S}$. A subspace $\mathcal{L}$ is input-containing for Σ if and only if $\mathcal{L}^\perp$ is output-nulling for $\Sigma^\top$. The input-containingingness condition can be equivalently expressed by saying that there exists an output-injection matrix G such that $\begin{bmatrix} A+GC \\ B+GD \end{bmatrix} (\mathcal{S} \oplus \mathcal{U}) \subseteq \mathcal{S}$. In this case, we say that G is a *friend* of $\mathcal{S}$. We denote by $\mathfrak{G}_\Sigma(\mathcal{S})$ the set of friends of $\mathcal{S}$. The set of input-containing subspaces $\mathcal{S}(\Sigma)$ of Σ is closed under intersection. Thus, we can define the smallest input-containing subspace $\mathcal{S}^\star \stackrel{\text{def}}{=} \min \mathcal{S}(\Sigma) = \bigcap_{\mathcal{S} \in \mathcal{S}(\Sigma)} \mathcal{S}$ (also denoted as $\mathcal{S}^\star_\Sigma$). The sequence

$$\begin{cases} \mathcal{S}_0 = \{0\} \\ \mathcal{S}_{i+1} = [\,A\ \ B\,]\big((\mathcal{S}_i \oplus \mathcal{U}) \cap \ker[\,C\ \ D\,]\big), \quad i \in \mathbb{N} \end{cases} \tag{3}$$

is monotonically non-decreasing and converges to $\mathcal{S}^\star$ in at most $n-1$ steps, i.e., $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_h = \mathcal{S}_{h+1} = \ldots$ implies $\mathcal{S}^\star = \mathcal{S}_h$, with $h \le n-1$. There holds also $\mathcal{S}^\star_\Sigma = (\mathcal{V}^\star_{\Sigma^\top})^\perp$. Given an input-containing subspace $\mathcal{S}$ and a corresponding friend G, we define the detectability subspace associated to it as $\mathcal{Q}_{\mathcal{S}} \stackrel{\text{def}}{=} \langle \mathcal{S} + C^{-1}\,\mathrm{im}\,D \,|\, A + G\,C\rangle$, and is the orthogonal complement of the reachability subspace on $\mathcal{S}^{\perp}$. The eigenvalues of $A{+}G\,C$, for $G \in \mathfrak{G}_{\Sigma}(\mathcal{S})$, can be divided into the eigenvalues of the mapping $A + G\,C\,|\,\mathcal{S}$ and those of $A + G\,C\,|\,\frac{\mathcal{X}}{\mathcal{S}}$. In turn, the eigenvalues of $A + G\,C\,|\,\mathcal{S}$ can be divided into two multi-sets: the eigenvalues of $A + G\,C\,|\,(\mathcal{S} \cap \mathcal{Q})$ are fixed, while those of $A + G\,C\,|\,\frac{\mathcal{S}}{\mathcal{S}\cap\mathcal{Q}}$ are assignable with a suitable choice of $G \in \mathfrak{G}_{\Sigma}(\mathcal{S})$. The eigenvalues of $A{+}G\,C\,|\,\frac{\mathcal{Q}_{\mathcal{S}}}{\mathcal{S}}$ are fixed, while those of $A{+}G\,C\,|\,\frac{\mathcal{X}}{\mathcal{Q}_{\mathcal{S}}}$ are assignable with a suitable $G \in \mathfrak{G}_{\Sigma}(\mathcal{S})$. We say that $\mathcal{S}$ is

- *internally detectable* if there exists $G \in \mathfrak{G}_{\Sigma}(\mathcal{S})$ such that $\sigma(A + G\,C\,|\,\mathcal{S})$ is asymptotically stable, or, equivalently, if $\sigma(A + G\,C\,|\,\mathcal{S} \cap \mathcal{Q})$ is asymptotically stable;
- *externally detectable* if there exists $G \in \mathfrak{G}_{\Sigma}(\mathcal{S})$ such that $\sigma(A + G\,C\,|\,\frac{\mathcal{X}}{\mathcal{S}})$ is asymptotically stable, or, equivalently, if $\sigma(A + G\,C\,|\,\frac{\mathcal{Q}_{\mathcal{S}}}{\mathcal{S}})$ is asymptotically stable.

An input-containing subspace that is externally detectable is also referred to as a *detectability input-containing subspace*. The set of detectability input-containing subspaces has a minimum denoted by $\mathcal{S}^{\star}_{g}$. An input-containing subspace $\mathcal{S}$ for which a friend G exists such that the spectrum of $A + G\,C\,|\,\frac{\mathcal{X}}{\mathcal{S}}$ is arbitrary is called an unobservability input-containing subspace. The set of unobservability input-containing subspaces is closed under intersection, and thus it admits a minimum $\mathcal{Q}^{\star}$ (or $\mathcal{Q}^{\star}_{\Sigma}$): there holds $\mathcal{S}^{\star} \subseteq \mathcal{S}^{\star}_{g} \subseteq \mathcal{Q}^{\star}$. There holds also $\mathcal{Q}^{\star} = \mathcal{Q}_{\mathcal{S}^{\star}}$. The eigenstructure $A + G\,C\,|\,\frac{\mathcal{Q}^{\star}}{\mathcal{S}^{\star}}$ coincides with the invariant zero structure of the system, so that $\mathcal{Z}_{\Sigma} = \sigma\left(A + B\,F \,\middle|\, \frac{\mathcal{V}^{\star}}{\mathcal{R}^{\star}}\right) = \sigma\left(A + G\,C \,\middle|\, \frac{\mathcal{Q}^{\star}}{\mathcal{S}^{\star}}\right)$. Finally, we recall that $\mathcal{Q}^{\star}_{\Sigma} = (\mathcal{R}^{\star}_{\Sigma^{\top}})^{\perp}$.

An input-containing subspace $\mathcal{S}$ is self-hidden if one of the equivalent conditions

$$(1)\;\; \mathcal{S} \subseteq \mathcal{S}^{\star} + C^{-1}\,\mathrm{im}\,D; \qquad (2)\;\; \mathcal{S} \subseteq \mathcal{Q}^{\star}.$$

holds. Thus, $\mathcal{Q}^{\star}$ and $\mathcal{S}^{\star}$ are self-hidden subspaces. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are self-hidden subspaces and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then every friend of $\mathcal{S}_1$ is also a friend of $\mathcal{S}_2$, i.e., $\mathfrak{G}_{\Sigma}(\mathcal{S}_1) \subseteq \mathfrak{G}_{\Sigma}(\mathcal{S}_2)$. In particular, every friend of $\mathcal{S}^{\star}$ is also a friend of $\mathcal{Q}^{\star}$. The sum of self-hidden subspaces is self-hidden. We define Ψ_{Σ} to be the set of self-hidden subspaces; then Ψ_{Σ} admits both a maximum, which is $\mathcal{Q}^{\star}$, and a minimum, which is $\mathcal{S}^{\star}$. Finally, we recall the identities $\mathcal{R}^{\star} = \mathcal{V}^{\star} \cap \mathcal{S}^{\star}$ and $\mathcal{Q}^{\star} = \mathcal{V}^{\star} + \mathcal{S}^{\star}$.

3. INTERMEDIATE RESULTS

Before we introduce the main dual lattice structures for LTI systems with possibly nonzero feedthrough matrices, we give an overview of the fundamental results which are used in the solution of disturbance decoupling problems by state feedback and of unknown-input observation problems. The proofs of these results can be found in (Ntogramatzidis, 2008, Lemma 3). First, we consider the inclusion $\mathrm{im}\,L \subseteq \mathcal{V}^{\star}$, which is the solvability condition of the disturbance decoupling problem by static state feedback for a system ruled by $\mathcal{D}\,\boldsymbol{x}(t) = A\,\boldsymbol{x}(t) + B\,\boldsymbol{u}(t) + L\,\boldsymbol{w}(t)$ and $\boldsymbol{y}(t) = C\,\boldsymbol{x}(t)$. We consider the augmented system $\Sigma_L = (A, [\,B \;\; L\,], C, [\,D \;\; 0\,])$, and we denote the corresponding supremal output-nulling and reachability subspaces by $\mathcal{V}^{\star}_{L}$ and $\mathcal{R}^{\star}_{L}$, respectively.

Theorem 3.1. Let $\mathrm{im}\,L \subseteq \mathcal{V}^{\star}$. The following results hold:

i) $\mathcal{V}^{\star} = \mathcal{V}^{\star}_{L}$;
ii) $\Phi_{\Sigma_L} \subseteq \Phi_{\Sigma}$;
iii) For all $\mathcal{V} \in \Phi_{\Sigma_L}$, there holds $\mathrm{im}\,L \subseteq \mathcal{V}$.

Theorem 3.2. $\mathrm{im}\,L \subseteq \mathcal{V}^{\star}$ if and only if $\mathrm{im}\,L \subseteq \mathcal{R}^{\star}_{L}$.

Theorem 3.3. If $\mathrm{im}\,L \subseteq \mathcal{V}^{\star}$, the subspace $\mathcal{R}^{\star}_{L}$ is the smallest of all the self-bounded subspaces $\mathcal{V}$ satisfying $\mathrm{im}\,L \subseteq \mathcal{V}$.

The following three results are a generalization of the last three: they are concerned with a geometric condition in the form $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{V}^{\star} \oplus \{0\}) + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ which arises in the solution of the decoupling problem of a measurable disturbance $\boldsymbol{w}$ with control law in the form $\boldsymbol{u}(t) = F\,\boldsymbol{x}(t) + S\,\boldsymbol{w}(t)$ for a system described by $\mathcal{D}\,\boldsymbol{x}(t) = A\,\boldsymbol{x}(t){+}B\,\boldsymbol{u}(t){+}L_1\,\boldsymbol{w}(t)$ and $\boldsymbol{y}(t) = C\,\boldsymbol{x}(t){+}D\,\boldsymbol{u}(t) + L_2\,\boldsymbol{w}(t)$. We consider the augmented system $\Sigma_d = (A, [\,B \;\; L_1\,], C, [\,D \;\; L_2\,])$, and we denote the corresponding supremal output-nulling and reachability subspaces by $\mathcal{V}^{\star}_{d}$ and $\mathcal{R}^{\star}_{d}$, respectively.

Theorem 3.4. Let $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{V}^{\star} \oplus \{0\}) + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$. Then:

i) $\mathcal{V}^{\star} = \mathcal{V}^{\star}_{d}$;
ii) $\Phi_{\Sigma_d} \subseteq \Phi_{\Sigma}$;
iii) $\forall\, \mathcal{V} \in \Phi_{\Sigma_d}$, $\;\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq \mathcal{V} \oplus \{0\} + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$.

Theorem 3.5. $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{V}^{\star} \oplus \{0\}) + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ if and only if $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{R}^{\star}_{d} \oplus \{0\}) + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$.

Theorem 3.6. If $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{V}^{\star}{\oplus}\{0\}){+}\mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$, the subspace $\mathcal{R}^{\star}_{d}$ is the smallest of all the self-bounded subspaces $\mathcal{V}$ satisfying $\mathrm{im}\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \subseteq (\mathcal{V} \oplus \{0\}) + \mathrm{im}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$.

We now dualize all the previous results. The first three involve an inclusion in the form $\mathcal{S}^{\star} \subseteq \ker M$, for some matrix M.

We consider the augmented system $\Sigma_M = (A, B, \left[\begin{smallmatrix} C \\ M \end{smallmatrix}\right], \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right])$, and we denote the corresponding infimal input-containing and unobservability subspaces by $\mathcal{S}^{\star}_{M}$ and $\mathcal{Q}^{\star}_{M}$, respectively.

Theorem 3.7. Let $\mathcal{S}^{\star} \subseteq \ker M$. The following results hold:

i) $\mathcal{S}^{\star} = \mathcal{S}^{\star}_{M}$;
ii) $\Psi_{\Sigma_M} \subseteq \Psi_{\Sigma}$;
iii) For all $\mathcal{S} \in \Psi_{\Sigma_M}$, there holds $\mathcal{S} \subseteq \ker M$.

Theorem 3.8. $\mathcal{S}^{\star} \subseteq \ker M$ if and only if $\mathcal{Q}^{\star}_{M} \subseteq \ker M$.

Theorem 3.9. If $\mathcal{S}^{\star} \subseteq \ker M$, the subspace $\mathcal{Q}^{\star}_{M}$ is the largest of all the self-hidden subspaces $\mathcal{S}$ satisfying $\mathcal{S} \subseteq \ker M$.

Finally, we consider the generalization $(\mathcal{S}^{\star} \oplus \mathcal{U}) \cap \ker[\,C \;\; D\,] \subseteq \ker[\,M_1 \;\; M_2\,]$ of the condition $\mathcal{S}^{\star} \subseteq \ker M$, which arises in the solution of unknown-input observation problems. Correspondingly, we consider the augmented system $\Sigma_o = \left(A, B, \left[\begin{smallmatrix} C \\ M_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} D \\ M_2 \end{smallmatrix}\right]\right)$, and we denote the corresponding infimal input-containing and unobservability subspaces by $\mathcal{S}^{\star}_{o}$ and $\mathcal{Q}^{\star}_{o}$, respectively.

Theorem 3.10. Let $(\mathcal{S}^{\star} \oplus \mathcal{U}) \cap \ker[C\,D] \subseteq \ker[M_1 \; M_2]$. Then:

i) $\mathcal{S}^{\star} = \mathcal{S}^{\star}_{o}$;
ii) $\Psi_{\Sigma_o} \subseteq \Psi_{\Sigma}$;
iii) $\forall\, \mathcal{S} \in \Psi_{\Sigma_o}$, $\;(\mathcal{S} \oplus \mathcal{U}) \cap \ker[\,C \;\; D\,] \subseteq \ker[\,M_1 \;\; M_2\,]$.

Theorem 3.11. We have $(\mathcal{S}^{\star} \oplus \mathcal{U}) \cap \ker[\,C \;\; D\,] \subseteq \ker[\,M_1 \;\; M_2\,]$ if and only if
$(\mathcal{Q}^{\star}_{o} \oplus \mathcal{U}) \cap \ker[\,C \;\; D\,] \subseteq \ker[\,M_1 \;\; M_2\,]$.

Theorem 3.12. If $(\mathcal{S}^\star \oplus \mathcal{U}) \cap \ker[\,C\;\;D\,] \subseteq \ker[\,M_1\;\;M_2\,]$, the subspace $\mathcal{Q}_o^\star$ is the largest of all self-hidden subspaces $\mathcal{S}$ satisfying $(\mathcal{S} \oplus \mathcal{U}) \cap \ker[C\;D] \subseteq \ker[M_1\;M_2]$.

4. DUAL LATTICE STRUCTURES

We begin by presenting a simple result, recalled without proof for the sake of brevity.

Lemma 4.1. Let $\mathcal{V}$ be an output-nulling subspace and let $\mathcal{S}$ be an input-containing subspace for $\Sigma = (A, B, C, D)$. Then, $\mathcal{S} \supseteq B \ker D$ and $\mathcal{V} \subseteq C^{-1} \operatorname{im} D$.

Theorem 4.1. Let $\mathcal{V}$ be an output-nulling subspace and let $\mathcal{S}$ be an input-containing subspace for $\Sigma = (A, B, C, D)$. Then:

- $\mathcal{V} \cap \mathcal{S}$ is an output-nulling subspace for $\Sigma = (A, B, C, D)$;
- $\mathcal{V}+\mathcal{S}$ is an input-containing subspace for $\Sigma = (A, B, C, D)$.

Theorem 4.2. Let $\mathcal{V}$ be a self-bounded subspace and let $\mathcal{S}$ be a self-hidden subspace for the quadruple (A, B, C, D). Then:

- $\mathcal{V} + \mathcal{S}^\star$ is a self-hidden subspace for (A, B, C, D);
- $\mathcal{S} \cap \mathcal{V}^\star$ is a self-bounded subspace for (A, B, C, D).

As a consequence of Theorem 4.2, the two functions

$$f : \Phi_\Sigma \longrightarrow \Psi_\Sigma, \quad \mathcal{V} \mapsto \mathcal{V} + \mathcal{S}^\star \qquad \text{and} \qquad g : \Psi_\Sigma \longrightarrow \Phi_\Sigma, \quad \mathcal{S} \mapsto \mathcal{S} \cap \mathcal{V}^\star$$

are well-defined. It is easy to see also that g is the inverse of f, i.e., $f \circ g$ is the identity function in Φ_Σ and $g \circ f$ is the identity function in Ψ_Σ. Indeed, if $\mathcal{V} \in \Phi_\Sigma$, using the modular distributive rule we find

$$\begin{aligned}(g \circ f)(\mathcal{V}) &= g(\mathcal{V} + \mathcal{S}^\star) = (\mathcal{V} + \mathcal{S}^\star) \cap \mathcal{V}^\star \\ &= \mathcal{V} + (\mathcal{S}^\star \cap \mathcal{V}^\star) = \mathcal{V} + \mathcal{R}^\star = \mathcal{V},\end{aligned}$$

where the last equality follows from the fact that, since $\mathcal{V}$ is self-bounded, it contains $\mathcal{R}^\star$. Similarly, one can verify that $(f \circ g)(\mathcal{S}) = \mathcal{S}$. Neither of these two maps are injective. However, if we restrict them to the lattices Φ_Σ and Ψ_Σ, as we have seen above, they become bijective. Indeed

$$\begin{aligned}&(\mathcal{V}^\star \cap \mathcal{S}^\star) + \mathcal{S}^\star = \mathcal{S}^\star \\ &(\mathcal{V}^\star \cap B \ker D) + \mathcal{S}^\star = (\mathcal{V}^\star + \mathcal{S}^\star) \cap \mathcal{S}^\star = \mathcal{S}^\star,\end{aligned}$$

and dually

$$\begin{aligned}&(\mathcal{S}^\star + C^{-1} \operatorname{im} D) \cap \mathcal{V}^\star = (\mathcal{V}^\star \cap \mathcal{S}^\star) + \mathcal{V}^\star = \mathcal{V}^\star \\ &(\mathcal{S}^\star + \mathcal{V}^\star) \cap \mathcal{V}^\star = \mathcal{V}^\star.\end{aligned}$$

5. SYSTEMS WITH GROUPED INPUTS AND OUTPUTS

Consider the following system

$$\begin{cases} \mathcal{D}\,\boldsymbol{x}(t) = A\,\boldsymbol{x}(t) + B_1\,\boldsymbol{u}_1(t) + B_2\,\boldsymbol{u}_2(t) \\ \boldsymbol{y}_1(t) = C_1\,\boldsymbol{x}(t) + D_{1,1}\,\boldsymbol{u}_1(t) + D_{1,2}\,\boldsymbol{u}_2(t) \\ \boldsymbol{y}_2(t) = C_2\,\boldsymbol{x}(t) + D_{2,1}\,\boldsymbol{u}_1(t) + D_{2,2}\,\boldsymbol{u}_2(t) \end{cases} \tag{4}$$

This representation, where the inputs $\boldsymbol{u}_1(t) \in \mathcal{U}_1$ and $\boldsymbol{u}_2(t) \in \mathcal{U}_2$ and outputs $\boldsymbol{y}_1(t) \in \mathcal{Y}_1$ and $\boldsymbol{y}_2(t) \in \mathcal{Y}_2$ are divided into subgroups, is useful in several control and estimation contexts, including **(i)** the disturbance decoupling by state feedback (where one input is manipulable and the other can be thought of as a disturbance), **(ii)** the unknown-input observation (where one output is used to reconstruct the other output by an observer that has no access to the system input) and **(iii)** the disturbance decoupling by dynamic output feedback and the regulator problem (where the above mentioned splitting of inputs and outputs appear simultaneously).

We consider the following sub-systems associated with (4):

$$\begin{aligned}\hat{\Sigma} &= (A, B_1, C_2, D_{2,1}) \\ \tilde{\Sigma} &= \Big(A, [\,B_1\;\;B_2\,], C_2, [\,D_{2,1}\;\;D_{2,2}\,]\Big) \\ \check{\Sigma} &= (A, B_2, C_1, D_{1,2}) \\ \bar{\Sigma} &= \left(A, B_2, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{1,2} \\ D_{2,2} \end{bmatrix}\right).\end{aligned}$$

The above quadruples play a fundamental role in the aforementioned problems. In these problems, the matrix $D_{1,1}$ is solely responsible for the well-posedness of the feedback interconnection, but it does not play any role in the subspaces associated with the solution of the problems.

We denote by $(\hat{\mathcal{V}}_i)_{i\in\mathbb{N}}$ and $(\hat{\mathcal{S}}_i)_{i\in\mathbb{N}}$ the two sequences (2) and (3) written for the sub-system $\hat{\Sigma}$, that converge in at most $n-1$ steps to $\hat{\mathcal{V}}^\star$ and $\hat{\mathcal{S}}^\star$, respectively. Likewise, we denote by $(\tilde{\mathcal{V}}_i)_{i\in\mathbb{N}}$ and $(\tilde{\mathcal{S}}_i)_{i\in\mathbb{N}}$ the two sequences (2) and (3) for $\tilde{\Sigma}$ that converge in at most $n-1$ steps to the subspaces $\tilde{\mathcal{V}}^\star$ and $\tilde{\mathcal{S}}^\star$.

Clearly, $\hat{\mathcal{V}}^\star \subseteq \tilde{\mathcal{V}}^\star$ and $\hat{\mathcal{S}}^\star \subseteq \tilde{\mathcal{S}}^\star$; indeed, $\hat{\mathcal{V}}_i \subseteq \tilde{\mathcal{V}}_i$ for all $i \in \mathbb{N}$. By induction, if $\tilde{\mathcal{V}}_i \supseteq \hat{\mathcal{V}}_i$ for a certain $i \in \mathbb{N}$, then

$$\begin{aligned}\tilde{\mathcal{V}}_{i+1} &= \begin{bmatrix} A \\ C_2 \end{bmatrix}^{-1} \left((\tilde{\mathcal{V}}_i \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B_1 & B_2 \\ D_{2,1} & D_{2,2} \end{bmatrix} \right) \\ &\supseteq \begin{bmatrix} A \\ C_2 \end{bmatrix}^{-1} \left((\tilde{\mathcal{V}}_i \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix} \right) \\ &\supseteq \begin{bmatrix} A \\ C_2 \end{bmatrix}^{-1} \left((\hat{\mathcal{V}}_i \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix} \right) = \hat{\mathcal{V}}_{i+1}.\end{aligned}$$

We have also $\hat{\mathcal{S}}_i \subseteq \tilde{\mathcal{S}}_i$ for all $i \in \mathbb{N}$. By induction, if $\tilde{\mathcal{S}}_i \supseteq \hat{\mathcal{S}}_i$ for a certain $i \in \mathbb{N}$, then

$$\begin{aligned}\tilde{\mathcal{S}}_{i+1} &= [\,A\;\;B_1\;\;B_2\,]\Big((\tilde{\mathcal{S}}_i \oplus \mathcal{U}_1 \oplus \mathcal{U}_2) \cap \ker[\,C_2\;\;D_{2,1}\;\;D_{2,2}\,]\Big) \\ &\supseteq [\,A\;\;B_1\;\;B_2\,]\Big((\hat{\mathcal{S}}_i \oplus \mathcal{U}_1 \oplus \mathcal{U}_2) \cap \ker[\,C_2\;\;D_{2,1}\;\;D_{2,2}\,]\Big) \\ &= \Big\{\boldsymbol{x} \,\Big|\, \exists \boldsymbol{\xi} \in \hat{\mathcal{S}}_i, \exists \boldsymbol{u}_1 \in \mathcal{U}_1, \exists \boldsymbol{u}_2 \in \mathcal{U}_2 : \\ &\qquad \boldsymbol{x} = A\,\boldsymbol{\xi} + B_1\,\boldsymbol{u}_1 + B_2\,\boldsymbol{u}_2 \text{ and} \\ &\qquad \boldsymbol{0} = C_2\,\boldsymbol{\xi} + D_{2,1}\,\boldsymbol{u}_1 + D_{2,2}\,\boldsymbol{u}_2\Big\} \\ &\supseteq \Big\{\boldsymbol{x} \,\Big|\, \exists \boldsymbol{\xi} \in \hat{\mathcal{S}}_i, \exists \boldsymbol{u}_1 \in \mathcal{U}_1 : \; \boldsymbol{x} = A\,\boldsymbol{\xi} + B_1\,\boldsymbol{u}_1 \\ &\qquad \text{and} \;\; \boldsymbol{0} = C_2\,\boldsymbol{\xi} + D_{2,1}\,\boldsymbol{u}_1\Big\} = \hat{\mathcal{S}}_{i+1}.\end{aligned}$$

However, when the inclusion

$$\operatorname{im} \begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix} \subseteq (\hat{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix} \tag{5}$$

holds, from the proof of the first statement of Theorem 3.4 we have $\hat{\mathcal{V}}_i = \tilde{\mathcal{V}}_i$ for all $i \in \mathbb{N}$, Ntogramatzidis (2008).

Even if we still have $\hat{\mathcal{S}}_i \subseteq \tilde{\mathcal{S}}_i$ for all $i \in \mathbb{N}$, the sum $\tilde{\mathcal{V}}_i + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}_i + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}_i + \hat{\mathcal{S}}_j$ holds for all $i, j \in \mathbb{N}$, as the following result shows.

Lemma 5.1. Let (5) hold. Then, $\hat{\mathcal{V}}_i + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}_i + \hat{\mathcal{S}}_j$ for all $i, j \in \mathbb{N}$.

Proof: We start proving that $\tilde{\mathcal{S}}_j \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$ for all $j \in \mathbb{N}$. We proceed by induction. The statement is trivially true for $j = 0$. Suppose that $\tilde{\mathcal{S}}_i \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_i$ for a certain $i \in \mathbb{N}$, and we prove that $\tilde{\mathcal{S}}_{i+1} \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_{i+1}$. Let $\boldsymbol{x} \in \tilde{\mathcal{S}}_{i+1}$. There exist $\boldsymbol{x}_1 \in \tilde{\mathcal{S}}_i$, $\boldsymbol{u}_1 \in \mathcal{U}_1$ and $\boldsymbol{u}_2 \in \mathcal{U}_2$ such that

$$\begin{aligned} \boldsymbol{x} &= A\,\boldsymbol{x}_1 + B_1\,\boldsymbol{u}_1 + B_2\,\boldsymbol{u}_2 \\ C_2\,\boldsymbol{x}_1 &+ D_{2,1}\,\boldsymbol{u}_1 + D_{2,2}\,\boldsymbol{u}_2 = \boldsymbol{0}. \end{aligned}$$

From (5), there exist two matrices M and N of suitable sizes such that $B_2 = V\,M + B_1\,N$ and $D_{2,2} = D_{2,1}\,N$, where V is a basis matrix of $\hat{\mathcal{V}}^\star$. We can rewrite the previous two identities as $\boldsymbol{x} = A\,\boldsymbol{x}_1 + B_1\,\boldsymbol{u}_1 + (V\,M + B_1\,N)\,\boldsymbol{u}_2$ and $C_2\,\boldsymbol{x}_1 + D_{2,1}\,\boldsymbol{u}_1 + (D_{2,1}\,N)\,\boldsymbol{u}_2 = \boldsymbol{0}$, or, equivalently,

$$\begin{aligned} \boldsymbol{x} &= A\,\boldsymbol{x}_1 + B_1\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2) + V\,M\,\boldsymbol{u}_2 \\ C_2\,\boldsymbol{x}_1 &+ D_{2,1}\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2) = \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{x}_1 \in \tilde{\mathcal{S}}_i \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_i$, from the inductive assumption, we can write $\boldsymbol{x}_1 = \boldsymbol{x}_v + \boldsymbol{x}_s$, where $\boldsymbol{x}_v \in \hat{\mathcal{V}}^\star$ and $\boldsymbol{x}_s \in \hat{\mathcal{S}}_i$, so that

$$\begin{aligned} \boldsymbol{x} &= A\,\boldsymbol{x}_v + A\,\boldsymbol{x}_s + B_1\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2) + V\,M\,\boldsymbol{u}_2 \\ C_2\,\boldsymbol{x}_v &+ C_2\,\boldsymbol{x}_s + D_{2,1}\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2) = \boldsymbol{0}. \end{aligned}$$

Let $F \in \mathfrak{F}_{\hat{\Sigma}}(\hat{\mathcal{V}}^\star)$. Adding and subtracting $B_1\,F\,\boldsymbol{x}_v$ in the right hand-side of the first equation and $D_{2,1}\,F\,\boldsymbol{x}_v$ in the right hand-side of the second equation gives

$$\begin{aligned} \boldsymbol{x} &= A\,\boldsymbol{x}_s + B_1\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2 - F\,\boldsymbol{x}_v) + (A + B_1\,F)\,\boldsymbol{x}_v + V\,M\,\boldsymbol{u}_2, \\ \boldsymbol{0} &= C_2\,\boldsymbol{x}_s + D_{2,1}\,(\boldsymbol{u}_1 + N\,\boldsymbol{u}_2 - F\,\boldsymbol{x}_v) + (C_2 + D_{2,1}\,F)\,\boldsymbol{x}_v. \end{aligned}$$

Clearly, $(A + B_1\,F)\,\boldsymbol{x}_v + V\,M\,\boldsymbol{u}_2 \in \hat{\mathcal{V}}^\star$ and $(C_2 + D_{2,1}\,F)\,\boldsymbol{x}_v = \boldsymbol{0}$. Defining $\boldsymbol{\omega} = \boldsymbol{u}_1 + N\,\boldsymbol{u}_2 - F\,\boldsymbol{x}_v$ and $\xi = A\,\boldsymbol{x}_s + B_1\,\omega$; since there holds also $C_2\,\boldsymbol{x}_s + D_{2,1}\,\omega = \boldsymbol{0}$ with $\boldsymbol{x}_s \in \hat{\mathcal{S}}_i$, it follows that $\xi \in \hat{\mathcal{S}}_{i+1}$. Thus, $\boldsymbol{x} \in \hat{\mathcal{S}}_{i+1} + \hat{\mathcal{V}}^\star$ as required.

We have proved that $\tilde{\mathcal{S}}_j \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$ for all $j \in \mathbb{N}$. Clearly, there holds also $\hat{\mathcal{V}}^\star + \tilde{\mathcal{S}}_j \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$ for all $j \in \mathbb{N}$. Since $\hat{\mathcal{V}}^\star = \tilde{\mathcal{V}}^\star$, we have $\tilde{\mathcal{V}}^\star + \tilde{\mathcal{S}}_j \subseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$ for all $j \in \mathbb{N}$. Since we showed that $\tilde{\mathcal{V}}^\star + \tilde{\mathcal{S}}_j \supseteq \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$, we obtain $\hat{\mathcal{V}}^\star + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}_j$ for all $j \in \mathbb{N}$. Finally, since $\hat{\mathcal{V}}_i \supseteq \hat{\mathcal{V}}^\star$ for all $i \in \mathbb{N}$, then also $\hat{\mathcal{V}}_i + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}_i + \hat{\mathcal{S}}_j$ for all $i, j \in \mathbb{N}$. ∎

As a simple consequence of Lemma 5.1, $\hat{\mathcal{V}}_{n-1} + \tilde{\mathcal{S}}_{n-1} = \hat{\mathcal{V}}_{n-1} + \hat{\mathcal{S}}_{n-1}$ can be written as

$$\hat{\mathcal{V}}^\star + \tilde{\mathcal{S}}^\star = \hat{\mathcal{V}}^\star + \hat{\mathcal{S}}^\star, \tag{6}$$

and $\hat{\mathcal{V}}_1 + \tilde{\mathcal{S}}_{n-1} = \hat{\mathcal{V}}_1 + \hat{\mathcal{S}}_{n-1}$, taking into account $\hat{\mathcal{V}}_1 = \left[\begin{smallmatrix} A \\ C_2 \end{smallmatrix}\right]^{-1} (\mathcal{X} \oplus \operatorname{im} D_{2,1}) = C_2^{-1} \operatorname{im} D_{2,1}$, becomes

$$C_2^{-1} \operatorname{im} D_{2,1} + \tilde{\mathcal{S}}^\star = C_2^{-1} \operatorname{im} D_{2,1} + \hat{\mathcal{S}}^\star. \tag{7}$$

We denote by $\mathcal{V}_m$ the largest reachability output-nulling subspace of the "disturbed system" with inputs $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and output $\boldsymbol{y}_2$, i.e. $\mathcal{V}_m \stackrel{\text{def}}{=} \tilde{\mathcal{R}}^\star = \tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star = \min \Phi_{\tilde{\Sigma}}$. If the condition $\operatorname{im}\left[\begin{smallmatrix} B_2 \\ D_{2,2} \end{smallmatrix}\right] \subseteq (\hat{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\left[\begin{smallmatrix} B_1 \\ D_{2,1} \end{smallmatrix}\right]$ is satisfied, in view of Theorem 3.4 we have $\hat{\mathcal{V}}^\star = \tilde{\mathcal{V}}^\star$, and we have $\mathcal{V}_m = \hat{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star$.

We now consider the two quadruples $\check{\Sigma} = (A, B_2, C_1, D_{1,2})$ and $\bar{\Sigma} = \left(A, B_2, \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} D_{1,2} \\ D_{2,2} \end{smallmatrix}\right]\right)$. We denote by $(\check{\mathcal{V}}_i)_{i\in\mathbb{N}}$ and $(\check{\mathcal{S}}_i)_{i\in\mathbb{N}}$ the sequences that converge in at most $n-1$ steps to $\check{\mathcal{V}}^\star$ and $\check{\mathcal{S}}^\star$, respectively. Similarly, we denote by $(\bar{\mathcal{V}}_i)_{i\in\mathbb{N}}$ and $(\bar{\mathcal{S}}_i)_{i\in\mathbb{N}}$ the two sequences that converge in at most $n-1$ steps to $\bar{\mathcal{V}}^\star$ and $\bar{\mathcal{S}}^\star$, respectively.

In general, it is clear that $\bar{\mathcal{V}}^\star \subseteq \check{\mathcal{V}}^\star$ and $\bar{\mathcal{S}}^\star \subseteq \check{\mathcal{S}}^\star$. However, when the inclusion $\ker[\, C_2 \;\; D_{2,2} \,] \supseteq (\check{\mathcal{S}}^\star \oplus \mathcal{U}_2) \cap \ker[\, C_1 \;\; D_{1,2} \,]$ holds, we have $\check{\mathcal{S}}_i = \bar{\mathcal{S}}_i$ for all $i \in \mathbb{N}$ from the dual of (Ntogramatzidis, 2008, Lemma 3). The following result can be proved by dualizing the proof of Lemma 5.1.

Lemma 5.2. Let $\ker[\, C_2 \;\; D_{2,2} \,] \supseteq (\check{\mathcal{S}}^\star \oplus \mathcal{U}_2) \cap \ker[\, C_1 \;\; D_{1,2} \,]$. For all $i, j \in \mathbb{N}$ there holds

$$\bar{\mathcal{V}}_i \cap \check{\mathcal{S}}_j = \check{\mathcal{V}}_i \cap \check{\mathcal{S}}_j. \tag{8}$$

Proof: Consider the statement of Lemma 5.1. We take the orthogonal complement in both sides of the condition $\operatorname{im}\left[\begin{smallmatrix} B_2 \\ D_{2,2} \end{smallmatrix}\right] \subseteq (\hat{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\left[\begin{smallmatrix} B_1 \\ D_{2,1} \end{smallmatrix}\right]$, and we obtain

$$\ker[\, B_2^\top \;\; D_{2,2}^\top \,] \supseteq (\mathcal{S}^\star_{\hat{\Sigma}^\top} \oplus \mathcal{Y}_2) \cap \ker[\, B_1^\top \;\; D_{2,1}^\top \,]. \tag{9}$$

Taking the orthogonal complement of both sides of the equation $\hat{\mathcal{V}}_i + \tilde{\mathcal{S}}_j = \hat{\mathcal{V}}_i + \hat{\mathcal{S}}_j$ gives

$$\hat{\mathcal{V}}_i^\perp \cap \tilde{\mathcal{S}}_j^\perp = \hat{\mathcal{V}}_i^\perp \cap \hat{\mathcal{S}}_j^\perp. \tag{10}$$

Taking the orthogonal complement of each subspace of the sequence $(\hat{\mathcal{V}}_i)_{i\in\mathbb{N}}$ gives

$$\begin{cases} \hat{\mathcal{V}}_0^\perp = \{0\} \\ \hat{\mathcal{V}}_{i+1}^\perp = [\, A^\top \;\; C_2^\top \,]\left((\hat{\mathcal{V}}_i^\perp \oplus \mathcal{Y}_2) \cap \ker[\, B_1^\top \;\; D_{2,1}^\top \,]\right) \end{cases}$$

Taking the orthogonal complement of each subspace of the sequence $(\hat{\mathcal{S}}_i)_{i\in\mathbb{N}}$ gives

$$\begin{cases} \hat{\mathcal{S}}_0^\perp = \mathcal{X} \\ \hat{\mathcal{S}}_{i+1}^\perp = \begin{bmatrix} A^\top \\ B_1^\top \end{bmatrix}^{-1} \left((\hat{\mathcal{S}}_i^\perp \oplus \{0\}) \cap \operatorname{im} \begin{bmatrix} C_2^\top \\ D_{2,1}^\top \end{bmatrix} \right). \end{cases}$$

Finally, if we take the orthogonal complement of each subspace of the sequence $(\tilde{\mathcal{S}}_i)_{i\in\mathbb{N}}$ we obtain

$$\begin{cases} \tilde{\mathcal{S}}_0^\perp = \mathcal{X} \\ \tilde{\mathcal{S}}_{i+1}^\perp = \begin{bmatrix} A^\top \\ B_1^\top \\ B_2^\top \end{bmatrix}^{-1} \left((\tilde{\mathcal{S}}_i^\perp \oplus \{0\}) \cap \operatorname{im} \begin{bmatrix} C_2^\top \\ D_{2,1}^\top \\ D_{2,2}^\top \end{bmatrix} \right). \end{cases}$$

With the substitutions $A^\top \to A$, $C_2^\top \to B_2$, $B_1^\top \to C$, $D_{2,1}^\top \to D_{1,2}$, $B_2^\top \to C_2$, $D_{2,2}^\top \to D_{2,2}$, condition (9) becomes $\ker[\, C_2 \;\; D_{2,2} \,] \supseteq (\check{\mathcal{S}}^\star \oplus \mathcal{U}_2) \cap \ker[\, C_1 \;\; D_{1,2} \,]$. With these substitutions, $(\hat{\mathcal{V}}_i^\perp)_{i\in\mathbb{N}}$ becomes $(\check{\mathcal{S}}_i)_{i\in\mathbb{N}}$, $(\hat{\mathcal{S}}_i^\perp)_{i\in\mathbb{N}}$ becomes $(\check{\mathcal{V}}_i)_{i\in\mathbb{N}}$, and $(\tilde{\mathcal{S}}_i^\perp)_{i\in\mathbb{N}}$ becomes $(\bar{\mathcal{V}}_i)_{i\in\mathbb{N}}$. Thus, (10) is exactly (8). ∎

From Lemma 5.2, we have in particular when $i = j = n-1$

$$\check{\mathcal{V}}^\star \cap \check{\mathcal{S}}^\star = \bar{\mathcal{V}}^\star \cap \check{\mathcal{S}}^\star. \qquad (11)$$

From $i = n-1$ and $j = 1$, taking into account that $\check{\mathcal{S}}_1 = [\,A\ \ B_2\,](\{0\}\oplus\mathcal{U}_2)\cap\ker[\,C_1\ \ D_{2,2}\,]) = [\,A\ \ B_2\,]((\{0\}\oplus\ker D_{1,2}) = B_2\ker D_{1,2}$, we have $\check{\mathcal{V}}^\star \cap B_2 \ker D_{1,2} = \bar{\mathcal{V}}^\star$.

Let $\mathcal{S}_M \overset{\text{def}}{=} \bar{\mathcal{Q}}^\star = \bar{\mathcal{V}}^\star + \bar{\mathcal{S}}^\star = \max \Psi_{\bar{\Sigma}}$. If $\ker[\,C_2\ \ D_{2,2}\,] \supseteq (\check{\mathcal{S}}^\star \oplus \mathcal{U}_2) \cap \ker[\,C_1\ \ D_{1,2}\,]$, in view of Theorem 3.10 we have $\bar{\mathcal{S}}^\star = \check{\mathcal{S}}^\star$. Hence, in this case $\mathcal{S}_M = \bar{\mathcal{V}}^\star + \check{\mathcal{S}}^\star$.

We now focus on the relationships between the remaining lattices. We start with a simple result, the proof is omitted for the sake of brevity.

Lemma 5.3. The following inclusions hold:

- $\bar{\mathcal{V}}^\star \subseteq \hat{\mathcal{V}}^\star \subseteq \tilde{\mathcal{V}}^\star$;
- $\bar{\mathcal{S}}^\star \subseteq \check{\mathcal{S}}^\star \subseteq \tilde{\mathcal{S}}^\star$;

The following result is an immediate consequence of Lemma 5.3.

Corollary 5.1. If $\check{\mathcal{S}}^\star \subseteq \hat{\mathcal{V}}^\star$, then $\bar{\mathcal{S}}^\star \subseteq \tilde{\mathcal{V}}^\star$.

Lemma 5.4. Let $\check{\mathcal{S}}^\star \subseteq \hat{\mathcal{V}}^\star$. Then:

- the subspace $\mathcal{V}_m + \mathcal{S}_M$ is self-bounded for $\tilde{\Sigma}$;
- the subspace $\mathcal{V}_m \cap \mathcal{S}_M$ is self-hidden for $\bar{\Sigma}$.

Proof: We find

$$\begin{aligned}\mathcal{V}_m + \mathcal{S}_M &= (\tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star) + (\bar{\mathcal{S}}^\star + \bar{\mathcal{V}}^\star)\\ &= \big((\tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star) + \bar{\mathcal{S}}^\star\big) + \bar{\mathcal{V}}^\star\\ &= \big((\tilde{\mathcal{V}}^\star + \bar{\mathcal{S}}^\star) \cap (\tilde{\mathcal{S}}^\star + \bar{\mathcal{S}}^\star)\big) + \bar{\mathcal{V}}^\star\\ &= (\tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star) + \bar{\mathcal{V}}^\star = \mathcal{V}_m + \bar{\mathcal{V}}^\star, \qquad (12)\end{aligned}$$

where we have used the modular distributive rule and Lemma 5.3. We now show that $\mathcal{V}_m + \mathcal{S}_M$ is output-nulling for $\tilde{\Sigma}$. The inclusion $\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} \bar{\mathcal{V}}^\star \subseteq (\bar{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_2 \\ D_{1,2} \\ D_{2,2} \end{bmatrix}$ implies $\begin{bmatrix} A \\ C_2 \end{bmatrix} \bar{\mathcal{V}}^\star \subseteq (\bar{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix}$, which in turn implies $\begin{bmatrix} A \\ C_2 \end{bmatrix} \bar{\mathcal{V}}^\star \subseteq (\bar{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 & B_2 \\ D_{2,1} & D_{2,2} \end{bmatrix}$. Adding this inclusion to $\begin{bmatrix} A \\ C_2 \end{bmatrix} \mathcal{V}_m \subseteq (\mathcal{V}_m \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 & B_2 \\ D_{2,1} & D_{2,2} \end{bmatrix}$ (recall that $\mathcal{V}_m \in \Phi_{\tilde{\Sigma}}$), it follows that

$$\begin{bmatrix} A \\ C_2 \end{bmatrix} (\mathcal{V}_m + \bar{\mathcal{V}}^\star) \subseteq ((\mathcal{V}_m + \bar{\mathcal{V}}^\star) \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 & B_2 \\ D_{2,1} & D_{2,2} \end{bmatrix}. \qquad (13)$$

Thus, $\mathcal{V}_m + \mathcal{S}_M$ is output-nulling for $\tilde{\Sigma}$. The fact that $\mathcal{V}_m + \bar{\mathcal{V}}^\star$ is self-bounded follows immediately from the inclusion

$$\mathcal{V}_m + \bar{\mathcal{V}}^\star \supseteq \mathcal{V}_m \supseteq \tilde{\mathcal{V}}^\star \cap [\,B_1\ \ B_2\,]\ \ker[\,D_{2,1}\ \ D_{2,2}\,].$$

The second statement follows by duality, by using the fact that

$$\begin{aligned}\mathcal{V}_m \cap \mathcal{S}_M &= (\bar{\mathcal{S}}^\star + \bar{\mathcal{V}}^\star) \cap (\tilde{\mathcal{V}}^\star \cap \tilde{\mathcal{S}}^\star)\\ &= ((\bar{\mathcal{S}}^\star + \bar{\mathcal{V}}^\star) \cap \tilde{\mathcal{V}}^\star) \cap \tilde{\mathcal{S}}^\star\\ &= [(\bar{\mathcal{S}}^\star \cap \tilde{\mathcal{V}}^\star) + (\bar{\mathcal{V}}^\star \cap \tilde{\mathcal{V}}^\star)] \cap \tilde{\mathcal{S}}^\star\\ &= (\bar{\mathcal{S}}^\star + \bar{\mathcal{V}}^\star) \cap \tilde{\mathcal{S}}^\star = \mathcal{S}_M \cap \tilde{\mathcal{S}}^\star.\end{aligned}$$

■

Corollary 5.2. Let $\check{\mathcal{S}}^\star \subseteq \hat{\mathcal{V}}^\star$. The following results hold:

- If $\operatorname{im}\begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix} \subseteq (\hat{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix}$, then $\mathcal{V}_m + \mathcal{S}_M$ is self-bounded for $\hat{\Sigma}$.
- If $\ker[\,C_2\ \ D_{2,2}\,] \supseteq (\check{\mathcal{S}}^\star \oplus \mathcal{U}_2) \cap \ker[\,C_1\ \ D_{1,2}\,]$, then $\mathcal{V}_m \cap \mathcal{S}_M$ is self-hidden for $\check{\Sigma}$.

Proof: Recall that $\operatorname{im}\begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix} \subseteq (\hat{\mathcal{V}}^\star \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix}$ implies $\operatorname{im}\begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix} \subseteq (\mathcal{V}_m \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix}$ from Theorem 3.4. Using this inclusion into (14) we obtain

$$\begin{aligned}\begin{bmatrix} A \\ C_2 \end{bmatrix} \big(\mathcal{V}_m + \bar{\mathcal{V}}^\star\big) &\subseteq \big((\mathcal{V}_m + \bar{\mathcal{V}}^\star) \oplus \{0\}\big) + \operatorname{im}\begin{bmatrix} B_1 & B_2 \\ D_{2,1} & D_{2,2} \end{bmatrix}\\ &\subseteq \big((\mathcal{V}_m + \bar{\mathcal{V}}^\star) \oplus \{0\}\big) + \operatorname{im}\begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix} + (\mathcal{V}_m \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B_2 \\ D_{2,2} \end{bmatrix}\\ &= \big((\mathcal{V}_m + \bar{\mathcal{V}}^\star) \oplus \{0\}\big) + \operatorname{im}\begin{bmatrix} B_1 \\ D_{2,1} \end{bmatrix}.\end{aligned}$$

We also need to prove that $\mathcal{V}_m + \mathcal{S}_M \supseteq \hat{\mathcal{V}}^\star \cap B_1 \ker D_{2,1}$: this follows from $\mathcal{V}_m + \mathcal{S}_M \supseteq \hat{\mathcal{V}}^\star \cap [\,B_1\ \ B_2\,]\ \ker[\,D_{2,1}\ \ D_{2,2}\,]$. The second claim can be proved by duality. ■

6. CONCLUSION

In this paper, we have first considered simple lattices of self-bounded and self-hidden subspaces of a biproper LTI system. Then, we analyzed the interaction of systems described by different quadruples in more complex lattice structures, in line with the standard framework that is usually considered in the regulator problem and in the disturbance decoupling problem by dynamic output feedback.

REFERENCES

G. Basile and G. Marro. Dual-lattice theorems in the geometric approach. *Journal of Optimization Theory and Applications*, 48(2):229–244, 1986.

G. Basile and G. Marro. Self-bounded controlled invariants versus stabilizability. *Journal of Optimization Theory and Applications*, 48(2):245–263, 1986.

B. Del-Muro-Cuéllar, *Commande des Systèmes Lineaires: Rejet de Perturbations et Pôles fixes.* Thèse de Doctorat. Ecole Centrale, Univ. de Nantes, France, 1997.

B. Del-Muro-Cuéllar, and M. Malabre, Fixed poles of disturbance rejection by dynamic measurement feedback: a geometric approach. *Automatica*, 37(2):231–238, 2001.

M. Malabre, J.C. Martinez-Garcia, and B. Del-Muro-Cuéllar. On the fixed poles for disturbance rejection. *Automatica*, 33(6):1209–1211, 1997.

L. Ntogramatzidis, "Self-bounded subspaces for non strictly proper systems and their application to the disturbance decoupling problem". *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 423–428, 2008.

F. Padula and L. Ntogramatzidis, "On the well-posedness in the solution of the disturbance decoupling by dynamic output feedback with self bounded and self hidden subspaces". *Automatica*, 106:315–326, 2019.

A.A. Stoorvogel and J.W. van der Woude. The disturbance decoupling problem with measurement feedback and stability for systems with direct feedthrough matrices. *Systems & Control Letters*, 17(3):217–226, 1991.

H. Trentelman, A. Stoorvogel, and M. Hautus, *Control theory for linear systems*, ser. Communications and Control Engineering. Great Britain: Springer, 2001.