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Robust Cadence Tracking for Switched FES-Cycling with an Unknown Time-Varying Input Delay Using a Time-Varying Estimate ⋆

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Abstract: For an individual affected by a lower limb movement disorder, motorized functional electrical stimulation (FES) induced cycling provides a means of functional restoration and therapeutic exercise. However, there exists a potentially destabilizing input delay between the application (and removal) of the stimulation and the production of muscle force. Exacerbating the problem, fatigue results in decreased force production and a time-varying input delay. Moreover, switching between FES and motor control can be destabilizing. This paper implements a time-varying estimate of the delay and develops a control method and switching conditions to account for the time-varying input delayed response of muscle. The controller is shown to yield semi-global uniformly ultimately bounded tracking for the uncertain switched nonlinear dynamic system with input delays.

Keywords: Functional Electrical Stimulation (FES), Rehabilitation Robot, Input Delay, Switched System, Human-Robot Interaction

1. INTRODUCTION

Throughout the world, there are many millions of people with neurological conditions (NCs) such as stroke, traumatic brain injury (TBI), Parkinson's Disease (PD), and spinal cord injury (SCI), among others Cousin et al. (2019). To combat the negative health effects of NCs, and to improve the overall quality of life of those who are affected, increased efforts have been made in the area of hybrid exoskeletons, which combine rehabilitation robots with functional electrical stimulation (FES) Cousin et al. (2019). One such use of hybrid exoskeletons is FES-induced cycling for individuals with lower limb movement disorders Cousin et al. (2019). However, closed-loop FES-cycling presents several challenges, predominantly fatigue and the fact that an input delay results from FES-induced muscle contractions Downey et al. (2017). Fatigue is undesirable primarily because it reduces the number of exercise repetitions, which lowers the rehabilitative effectiveness and exacerbates the input delay, possibly resulting in instability Downey et al. (2017). Under a constant stimulation intensity muscle force tends to decay due to fatigue Ding et al. (2002). Other challenges are that FES-cycling requires switching the control between multiple muscle groups and often a motor (to assist with fatigue reduction) Bellman et al. (2017), there is uncertainty in the parameters of the dynamic model and unknown disturbances Li et al. (2014), and there exists an unknown yet complex nonlinear mapping from the FES input to the generated muscle force Idsø et al. (2004).

When dealing with an FES-induced input delay, results such as Obuz et al. (2015, 2016); Karafyllis et al. (2015); Sharma et al. (2011) developed controllers for continuous exercises (e.g., leg extensions). Continuous exercises focus on the contraction delay between the application of the electrical stimulus and muscle contraction. Exercises that require limb coordination by switching between multiple muscle groups (e.g., cycling) must also consider residual forces that result from the delayed muscle response after the electrical stimulus is removed Allen et al. (2019b,a). Special consideration is required for these residual forces because they may be produced by antagonistic muscles, resulting in unfavorable biomechanics as well as an increased rate of fatigue, which is detrimental to rehabilitative outcomes. FES controllers have recently been developed to compensate for the delayed response of muscle. In Obuz et al. (2015) and Obuz et al. (2016), for a continuous leg extension exercise, an input delay that is both time-varying and unknown is examined. In Karafyllis et al. (2015), exact model knowledge of the lower limb dynamics and a constant but unknown delay are assumed to yield a global asymptotic tracking controller. In Sharma et al. (2011), a uniformly ultimately bounded result was achieved for a known delay and uncertain dynamics. More recently, a closed-loop FES controller was developed for switched system dynamics, first with an unknown constant input delay Allen et al. (2019b), and later with a time-varying unknown input delay Allen et al. (2019a).

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In recent years, general input delayed systems have been studied extensively Krstic (2009); Karafyllis and Krstic (2017); Chakraborty et al. (2016); Obuz et al. (2017); Mazenc et al. (2017); Wang et al. (2015); Enciu et al. (2018). In Chakraborty et al. (2016) the input delay is assumed to be known. Since the input delay cannot always be measured, results such as Obuz et al. (2017) have assumed the delay to be unknown. More recently, input delay compensation has been studied for switched systems Enciu et al. (2018); Wang et al. (2015); Mazenc et al. (2017). Non-time-varying input delays are considered for a linear system in Enciu et al. (2018) and for a class of nonlinear systems in Wang et al. (2015). Whereas, in Mazenc et al. (2017) time-varying delays are considered for a family of linear time-varying systems. The aforementioned results on general input delayed systems, however, do not compensate for important FES specific factors such as the need for complex state-dependent switching to yield effective agonist muscle contractions despite the contraction delay, while also reducing residual antagonistic forces that result from the delay after the stimulation has been removed.

In this paper, like in the author's previous works Allen et al. (2019b,a), a cadence tracking controller that is robust to a time-varying input delay is developed that incorporates a delay-dependent trigger condition, which appropriately schedules both the activation and deactivation of the FES and the motor. However, compared to the previous result, this paper incorporates a time-varying estimate of the delay along with the associated stability analysis. This paper also improves the switching conditions to ensure that residual forces are not produced by antagonist muscles.

2. DYNAMICS

2.1 Cycle-Rider System

In this paper, delayed functions are defined as

$$h_\tau \triangleq \begin{cases} h\left(t-\tau\left(t\right)\right) & t-\tau\left(t\right)\geq t_0 \\ 0 & t-\tau\left(t\right)<t_0 \end{cases},$$

where $t \in \mathbb{R}_{\geq 0}$ denotes the time and the initial time is denoted by $t_0 \in \mathbb{R}_{\geq 0}$. The time-varying electromechanical delay, i.e., the delay between the application/removal of the current and the onset/elimination of muscle force production is denoted by $\tau : \mathbb{R}_{\geq 0} \to \mathbb{S}$, where $\mathbb{S} \subset \mathbb{R}$ represents a set of all possible delay values Merad et al. (2016). The combined motorized cycle-rider system can be modeled as Bellman et al. (2017) [1]

$$\begin{aligned}\tau_M\left(q,\dot{q},\tau,t\right)+\tau_e\left(q,t\right)=M\left(q\right)\ddot{q}+V\left(q,\dot{q}\right)\dot{q} \\ +G\left(q\right)+P\left(q,\dot{q}\right)+b_c\dot{q}+d\left(t\right),\end{aligned} \tag{1}$$

where $q : \mathbb{R}_{\geq 0} \to \mathcal{Q}$, $\dot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $\ddot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ denote the measurable crank angle and velocity, and unmeasured acceleration, respectively. The set $\mathcal{Q} \subseteq \mathbb{R}$ denotes the set of all possible crank angles. The inertial effects, gravitational effects, centripetal-Coriolis effects, and passive viscoelastic tissue forces are denoted by $M : \mathcal{Q} \to \mathbb{R}_{>0}$, $G : \mathcal{Q} \to \mathbb{R}$, $V : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$, and $P : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$, respectively. The disturbances and viscous damping effects applied about the crank axis are denoted by $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $b_c \in \mathbb{R}_{>0}$, respectively. The torque contributions due to the FES induced muscle contractions and the motor are denoted as $\tau_M : \mathcal{Q} \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\tau_e : \mathcal{Q} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively defined as

$$\tau_M\left(q,\dot{q},\tau,t\right) \triangleq \sum_{m\in\mathcal{M}} B_m\left(q,\dot{q}\right)u_m\left(q_\tau,\dot{q}_\tau,\tau,t\right), \tag{2}$$

$$\tau_e\left(q,t\right) \triangleq B_e u_E\left(q,t\right), \tag{3}$$

where the unknown control effectiveness of the electrically stimulated muscle groups in (2) are denoted by $B_m : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$, where $m \in \mathcal{M} \triangleq \{RH, RQ, RG, LH, LQ, LG\}$ indicates the right (R) and left (L) hamstrings (H), quadriceps femoris (Q), and gluteal (G) muscle groups. The unknown motor control effectiveness is denoted by $B_e \in \mathbb{R}_{>0}$. The delayed FES input (i.e., pulse width) delivered to the rider's muscles, denoted by $u_m : \mathcal{Q} \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\forall m \in \mathcal{M}$, and the electric motor control current denoted by $u_E : \mathcal{Q} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, are defined as

$$u_m\left(q_\tau,\dot{q}_\tau,\tau,t\right) \triangleq k_m\sigma_{m,\tau}\left(q_\tau,\dot{q}_\tau\right)u_\tau, \tag{4}$$

$$u_E\left(q,t\right) \triangleq k_e\sigma_e\left(q\right)u_e\left(t\right), \tag{5}$$

where $k_m, k_e \in \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$ are selectable constants. The subsequently designed non-delayed FES and motor inputs are denoted by $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $u_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively. The delayed switching signals denoted by $\sigma_{m,\tau}\left(q_\tau,\dot{q}_\tau\right)$, $\forall m \in \mathcal{M}$ indicate which muscle groups receive the delayed FES input u_τ at the time $t - \tau\left(t\right)$. The state-dependent FES switching signal, denoted by $\sigma_m\left(q,\dot{q}\right)$ is designed to activate/deactivate the muscles at the appropriate time. The piecewise left-continuous switching signal for each muscle group is denoted as $\sigma_m : \mathcal{Q} \times \mathbb{R} \to \{0,\, 1\}$ and is designed as

$$\sigma_m\left(q,\dot{q}\right) \triangleq \begin{cases} 1, & q_\alpha \in \mathcal{Q}_m \\ 0, & \text{otherwise} \end{cases}, \tag{6}$$

$\forall m \in \mathcal{M}$, where the trigger condition $q_\alpha : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is defined as $q_\alpha \triangleq f\left(q,\dot{q}\right)$, where f is designed such that the rider's muscles are stimulated sufficiently prior to the crank entering the FES region and for stimulation to cease sufficiently prior to the crank leaving the FES region. The function f does not require explicit knowledge of the actual delay, but rather uses the fact that the delay can be upper bounded as determined from experimental results such as Merad et al. (2016). This allows q_α to act as a trigger condition that adjusts the activation/deactivation of the FES input based on the delay upper bound. Due to negligible motor delay, the motor switching signal is implemented at time t. Hence, in (5), $\sigma_e : \mathcal{Q} \times \mathbb{R} \to \{0,\, 1\}$ denotes a piecewise left-continuous switching signal for the motor and is defined as

$$\sigma_e\left(q,\dot{q}\right) \triangleq \begin{cases} 1, \; q \in \mathcal{Q}_e \\ 1, \; q \in \mathcal{Q}_{FES}, \; \sum\limits_{m\in\mathcal{M}} \sigma_m = 0 \\ 0, \; \text{otherwise} \end{cases}. \tag{7}$$

Definitions for the subsequent desired FES regions, denoted by $\mathcal{Q}_m \subset \mathcal{Q}$, are based on Bellman et al. (2017), which uses the fact that each muscle group is kinematically efficient in specific regions of the crank cycle. In this paper, the trigger conditions are designed to produce the FES-induced muscle contractions within each muscle's respective FES region. Based on Bellman et al. (2017), $\mathcal{Q}_m$ is defined for each muscle group as

[1] For notational brevity, all explicit dependence on time, t, within the terms $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ is suppressed.

$$\mathcal{Q}_m \triangleq \{q \in \mathcal{Q} \mid T_m(q) > \varepsilon_m\}, \tag{8}$$

$\forall m \in \mathcal{M}$, where $\varepsilon_m \in (0, \max(T_m)]$ is the lower threshold for each torque transfer ratio denoted by $T_m : \mathcal{Q} \to \mathbb{R}$, which limits the FES regions such that each muscle group only contributes to forward pedaling (i.e., positive crank motion). The union of all the muscle regions defined in (8) represents the entire FES region, denoted by $\mathcal{Q}_{FES}$, and defined as $\mathcal{Q}_{FES} \triangleq \underset{m\in\mathcal{M}}{\cup}\{\mathcal{Q}_m\}$. The kinematic deadzones are defined as $\mathcal{Q}_e \triangleq \mathcal{Q} \setminus \mathcal{Q}_{FES}$. Substituting (2)-(5) into (1) yields [2]

$$B_M^\tau u_\tau + B_E u_e = M\ddot{q} + V\dot{q} + G + P + b_c\dot{q} + d, \tag{9}$$

where $B_M^\tau \triangleq \sum_{m\in\mathcal{M}} B_m(q,\dot{q}) k_m \sigma_{m,\tau}(q_\tau, \dot{q}_\tau)$ and $B_E(q) \triangleq B_e k_e \sigma_e(q,\dot{q})$.

The parameters in (9) capture the torques that affect the dynamics of the combined cycle-rider system, but the exact value of these parameters are unknown for each rider and the cycle. However, the subsequently designed FES and motor controllers only require known bounds on the aforementioned parameters.

The switched system in (9) has the following properties Bellman et al. (2017). **Property: 1** $c_m \leq M \leq c_M$, where $c_m, c_M \in \mathbb{R}_{>0}$ are known constants. **Property: 2** $|V| \leq c_V|\dot{q}|$, where $c_V \in \mathbb{R}_{>0}$ is a known constant and $|\cdot|$ denotes the absolute value. **Property: 3** $|G| \leq c_G$, where $c_G \in \mathbb{R}_{>0}$ is a known constant. **Property: 4** $|P| \leq c_{P1} + c_{P2}|\dot{q}|$, where $c_{P1}, c_{P2} \in \mathbb{R}_{>0}$ are known constants. **Property: 5** $b_c\dot{q} \leq c_c|\dot{q}|$, where $c_c \in \mathbb{R}_{>0}$ is a known constant. **Property: 6** $|d| \leq c_d$, where $c_d \in \mathbb{R}_{>0}$ is a known constant. **Property: 7** $\frac{1}{2}\dot{M} = V$. **Property: 8** The muscle control effectiveness B_m is lower and upper bounded $\forall m \in \mathcal{M}$, and thus, when $\sum_{m\in\mathcal{M}} \sigma_{m,\tau} > 0$, $c_b \leq B_M^\tau \leq c_B$, where $c_b, c_B \in \mathbb{R}_{>0}$ are known constants. **Property: 9** The motor control effectiveness is bounded such that when $\sigma_e = 1$, $c_e \leq B_E \leq c_E$, where $c_e, c_E \in \mathbb{R}_{>0}$ are known constants. **Property: 10** The delay is upper and lower bounded such that $\underline{\tau} \leq \tau \leq \bar{\tau}$, where $\underline{\tau}, \bar{\tau} \in \mathbb{R}_{>0}$ are known constants.

3. CONTROL DEVELOPMENT

The control objective is for the bicycle crank to track a desired cadence $\dot{q}_d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ despite an unknown time-varying input delay and uncertainties in the dynamic model. A measurable cadence tracking error, denoted by $\dot{e} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as

$$\dot{e} \triangleq \dot{q}_d - \dot{q}, \tag{10}$$

where, the measurable auxiliary position tracking error, denoted by $e : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as

$$e \triangleq q_d - q. \tag{11}$$

To facilitate the subsequent analysis, a measurable auxiliary tracking error, denoted by $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as

$$r \triangleq \dot{e} + \alpha_1 e + \alpha_2 e_u, \tag{12}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0}$ are selectable constants. An auxiliary error signal is designed to incorporate a delay-free input term into the closed-loop error system. This signal is denoted by $e_u : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and is defined as

$$e_u \triangleq -\int_{t-\hat{\tau}}^{t} u(\theta)\, d\theta. \tag{13}$$

where $\hat{\tau} : \mathbb{R}_{\geq 0} \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a time-varying estimate of the delay. Taking the time derivative of (12), multiplying by M, using (9), (11), and (13), and adding and subtracting $B_M^\tau u_{\hat{\tau}} + e$ yields the open-loop error system

$$\begin{aligned} M\dot{r} = &-Vr - e + \chi + B_M^\tau (u_{\hat{\tau}} - u_\tau) - B_E u_e \\ &+ (M\alpha_2 - B_M^\tau) u_{\hat{\tau}} - M\alpha_2 u - M\alpha_2 \dot{\hat{\tau}} u_{\hat{\tau}}, \end{aligned} \tag{14}$$

where $\chi : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ denotes an auxiliary term defined as

$$\begin{aligned} \chi \triangleq\ & M\ddot{q}_d + V(\dot{q}_d + \alpha_1 e + \alpha_2 e_u) + G \\ &+ P + b_c\dot{q} + d + M\alpha_1\dot{e} + e. \end{aligned}$$

The auxiliary term χ can be bounded by using Properties 1-6 as

$$|\chi| \leq \Phi + \rho(\|z\|)\|z\|, \tag{15}$$

where $\Phi \in \mathbb{R}_{>0}$ is a known constant, $\rho(\cdot)$ is a positive, strictly increasing, and radially unbounded function, and $z \in \mathbb{R}^3$ is a vector of the error signals defined as

$$z \triangleq \left[\, e \;\; r \;\; e_u \,\right]^T. \tag{16}$$

The delay estimate is updated using a predictor of the form

$$\dot{\hat{\tau}} = \mathrm{proj}\left(g(t, q, \dot{q}, \hat{\tau})\right) \tag{17}$$

where $g : \mathbb{R}_{\geq 0} \times \mathcal{Q} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a piecewise left-continuous function that is used to update the estimate of the delay, and $\mathrm{proj}(\cdot)$ is a smooth projection operator (see Cai et al. (2006)), with bounds based on the known delay bounds of Property 10.

The FES control input, based on the subsequent stability analysis, and (14) and (15), is designed as

$$u = k_s r, \tag{18}$$

where $k_s \in \mathbb{R}_{>0}$ is a selectable constant. Likewise, the motor control input is designed as

$$u_e = k_1 \mathrm{sgn}(r) + (k_2 + k_3) r, \tag{19}$$

where $k_1, k_2, k_3 \in \mathbb{R}_{>0}$ are selectable constants, and $\mathrm{sgn}(\cdot)$ denotes the signum function. The closed-loop error system is obtained by substituting (18) and (19) into (14) to yield

$$\begin{aligned} M\dot{r} = &-Vr - B_E(k_1\mathrm{sgn}(r) + (k_2 + k_3) r) \\ &+ k_s B_M^\tau (r_{\hat{\tau}} - r_\tau) + (M\alpha_2 - B_M^\tau) k_s r_{\hat{\tau}} \\ &- M\alpha_2 k_s r - M\alpha_2 k_s \dot{\hat{\tau}} r_{\hat{\tau}} - e + \chi. \end{aligned} \tag{20}$$

Lyapunov-Krasovskii functionals $Q_1, Q_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ are defined based on the closed-loop error system in (20) and the subsequent stability analysis as

$$Q_1 \triangleq \frac{1}{2} k_s \left(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3\right) \int_{t-\hat{\tau}}^{t} r(\theta)^2\, d\theta, \tag{21}$$

$$Q_2 \triangleq \frac{\omega_2 k_s}{\bar{\tau}} \int_{t-\bar{\tau}}^{t} \int_{s}^{t} r(\theta)^2\, d\theta ds, \tag{22}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}_{>0}$ are selectable constants. Auxiliary bounding constants $\beta_1, \beta_2, \delta_1, \delta_2 \in \mathbb{R}_{>0}$

[2] For notational brevity, all functional dependencies are hereafter suppressed unless required for clarity of exposition.

are defined, to facilitate the subsequent stability analysis, as

$$\beta_1 \triangleq \min\left(\alpha_1 - \frac{\varepsilon_2\alpha_2^2}{2},\ \frac{1}{4}c_m\alpha_2 k_s,\ \frac{\omega_2}{3k_s\bar{\tau}^2} - \frac{1}{2\varepsilon_2} - \frac{k_s\omega_3}{2\varepsilon_3}(2+\varepsilon_4)\right), \tag{23}$$

$$\beta_2 \triangleq \min\left(\alpha_1 - \frac{\varepsilon_2\alpha_2^2}{2},\ c_e k_2 - k_s\left(\varepsilon_3\omega_3 + \omega_2 + \frac{1}{2}(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4)\right),\ \frac{\omega_2}{3k_s\bar{\tau}^2} - \frac{1}{2\varepsilon_2} - \frac{k_s\omega_3}{2\varepsilon_3}(2+\varepsilon_4)\right) \tag{24}$$

$$\delta_1 \triangleq \min\left(\frac{\beta_1}{2},\ \frac{\omega_2}{3\bar{\tau}\left(\frac{1}{2}\left(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3\right)\right)},\ \frac{1}{3\bar{\tau}}\right), \tag{25}$$

$$\delta_2 \triangleq \min\left(\frac{\beta_2}{2},\ \frac{\omega_2}{3\bar{\tau}\left(\frac{1}{2}\left(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3\right)\right)},\ \frac{1}{3\bar{\tau}}\right). \tag{26}$$

4. STABILITY ANALYSIS

In the following analysis, switching times are denoted by $\{t_n^i\}$, $i \in \{m, e\}$, $n \in \{0, 1, 2, ...\}$, which represent the time instances when B_M^τ becomes nonzero $(i = m)$, or the time instances when B_M^τ becomes zero $(i = e)$. Let $V_L : \mathbb{R}^5 \to \mathbb{R}_{>0}$ denote a continuously differentiable, positive definite, common Lyapunov function candidate defined as

$$V_L \triangleq \frac{1}{2}e^2 + \frac{1}{2}Mr^2 + \frac{1}{2}\omega_3 e_u^2 + Q_1 + Q_2. \tag{27}$$

By inspection, the common Lyapunov function candidate V_L satisfies the following inequalities:

$$\lambda_1 \|y\|^2 \le V_L \le \lambda_2 \|y\|^2, \tag{28}$$

where $y \in \mathbb{R}^5$ is defined as

$$y \triangleq \begin{bmatrix} z & \sqrt{Q_1} & \sqrt{Q_2} \end{bmatrix}^T, \tag{29}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ are known constants defined as

$$\lambda_1 \triangleq \frac{1}{2}\min(1, c_m, \omega_3), \quad \lambda_2 \triangleq \max\left(1, \frac{c_M}{2}, \frac{\omega_3}{2}\right).$$

Let the set of initial conditions be defined as

$$S_{\mathcal{D}} \triangleq \left\{ y \in \mathbb{R}^5 \mid \|y\| < \sqrt{\frac{\lambda_1}{\lambda_2}}\gamma \right\}, \tag{30}$$

where $\gamma \in \mathbb{R}_{>0}$ is a known constant defined as[3] $\gamma \triangleq \inf\left\{\rho^{-1}\left((\sqrt{\kappa}, \infty)\right)\right\}$, where $\kappa \triangleq c_m\alpha_2 k_s \min\left(\frac{1}{4}\beta_1, 2\beta_2\right)$.

Theorem 1. The closed-loop error system in (20) is uniformly ultimately bounded in the sense that

$$\|y(t)\|^2 \le \frac{\lambda_2}{\lambda_1}\|y(t_0)\|^2 \exp(-\lambda_3(t - t_0)) + \frac{v}{\lambda_1\lambda_3}\left(1 - \exp(-\lambda_3(t - t_0))\right), \tag{31}$$

where $v \triangleq \frac{(\bar{\tau} - \underline{\tau})\Upsilon^2}{k_s} + \frac{2\Phi^2}{c_m\alpha_2 k_s}$, $\Upsilon \in \mathbb{R}_{>0}$ is a known constant, $\lambda_3 \triangleq \lambda_2^{-1}\min(\delta_1, \delta_2)$, $\forall t \in [t_0, \infty)$ provided $y(t_0) \in S_{\mathcal{D}}$, and the following gain conditions are satisfied

[3] For a set A, the inverse image is defined as $\rho^{-1}(A) \triangleq \{a \mid \rho(a) \in A\}$.

$$\alpha_1 \ge \frac{\varepsilon_2\alpha_2^2}{2}, \quad \omega_2 \ge 3k_s\bar{\tau}^2\left(\frac{1}{2\varepsilon_2} + \frac{k_s\omega_3}{2\varepsilon_3}(2+\varepsilon_4)\right), \tag{32}$$

$$\max\left(|c_M\alpha_2 - c_b|, |c_m\alpha_2 - c_B|\right) \le \varepsilon_1\omega_1, \tag{33}$$

$$(\bar{\tau} - \underline{\tau}) \le \frac{1}{k_s^2 c_B^2}\left(2c_m\alpha_2 - 4\varepsilon_3\omega_3 - 4\omega_2 - 2(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4)\right), \tag{34}$$

$$\left|\dot{\tau}\right| \le \varepsilon_4 < 1, \quad \omega_4 = \frac{1}{1-\varepsilon_4}, \quad \sqrt{\lambda_1^{-1}\lambda_3^{-1}v} < \gamma, \tag{35}$$

$$k_1 \ge \frac{1}{c_e}\left(c_b k_s\bar{\tau}\Upsilon + \Phi\right), \quad k_3 \ge \frac{c_b k_s}{c_e}, \tag{36}$$

$$k_2 \ge \frac{k_s}{c_e}\left(\frac{1}{2}(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4) + \varepsilon_3\omega_3 + \omega_2\right). \tag{37}$$

Proof. When $B_M^\tau > 0$, the FES effect is present in the system because the rider's muscles are generating a force (i.e., $t \in [t_n^m, t_{n+1}^e)$). Additionally, since B_E and B_M^τ are discontinuous, a generalized solution to the time derivative of (27) exists almost everywhere (a.e.) within $t \in [t_0, \infty)$, and $\dot{V}_L(y) \overset{\text{a.e.}}{\in} \dot{\tilde{V}}_L(y)$, where $\dot{\tilde{V}}_L$ is the generalized time derivative of V_L. Let $y(t)$ for $t \in [t_0, \infty)$ be a Filippov solution to the differential inclusion $\dot{y} \in K[h](y)$ and let $h : \mathbb{R}^5 \to \mathbb{R}^5$ be defined as $h \triangleq \left[\dot{e}\ \dot{r}\ \dot{e}_u\ \dot{\sqrt{Q_1}}\ \dot{\sqrt{Q_2}}\right]^T$ (see Fischer et al. (2013)). After utilizing (11)-(13), (20), and applying Leibniz Rule, the generalized time derivative of (27) is

$$\begin{aligned}\dot{\tilde{V}}_L \subseteq\ & e(r - \alpha_1 e - \alpha_2 e_u) + \frac{1}{2}\dot{M}r^2 + r\big(-Vr - e + \chi \\ & + k_s B_M^\tau (r_{\hat{\tau}} - r_\tau) - B_E(k_1 K[\mathrm{sgn}(r)] + (k_2 + k_3) r) \\ & + (M\alpha_2 - B_M^\tau)k_s r_{\hat{\tau}} - M\alpha_2 k_s r - M\alpha_2 k_s \dot{\hat{\tau}} r_{\hat{\tau}}\big) \\ & + \frac{1}{2}k_s\left(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3\right) r^2 \\ & - \frac{1}{2}k_s\left(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3\right)\left(1 - \dot{\hat{\tau}}\right) r_{\hat{\tau}}^2 \\ & + \omega_3 e_u\left(-k_s r + k_s r_{\hat{\tau}} \cdot \left(1 - \dot{\hat{\tau}}\right)\right) \\ & + \frac{\omega_2 k_s}{\bar{\tau}}\left(\bar{\tau}r^2 - \int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta\right),\end{aligned} \tag{38}$$

where, $K[\mathrm{sgn}(\cdot)] = SGN(\cdot)$ such that $SGN(\cdot) = \{1\}$ if $(\cdot) > 0$, $[-1, 1]$ if $(\cdot) = 0$, and $\{-1\}$ if $(\cdot) < 0$.

When $B_M^\tau > 0$, $B_E = 0$ or $B_E > 0$, according to the defined switching laws in (6) and (7). The more restrictive of the two cases is when $B_E = 0$ (i.e., when only the delayed FES input is controlling the system). Therefore, the details of the case when $B_M^\tau > 0$ and $B_E > 0$ are not included in the subsequent proof because (38) with $B_E = 0$ can upper bound (38) with $B_E > 0$.

Setting $B_E = 0$, canceling common terms, choosing $\left|\dot{\hat{\tau}}\right| \le \varepsilon_4 < 1$, selecting ε_1 and ω_1 such that $\max\left(|c_M\alpha_2 - c_b|, |c_m\alpha_2 - c_B|\right) \le \varepsilon_1\omega_1$, and using Properties 1, 7, and 8 with (38) yields

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\alpha_1 e^2 + \alpha_2 |ee_u| + |r||\chi| + k_s c_B |r(r_{\hat{\tau}} - r_\tau)| \\ & + k_s(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)|rr_{\hat{\tau}}| - c_m\alpha_2 k_s r^2 \\ & + k_s\omega_3|e_u r| + k_s\omega_3\left(1-\dot{\hat{\tau}}\right)|e_u r_{\hat{\tau}}| \\ & + \frac{1}{2}k_s(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3) r^2 \\ & - \frac{1}{2}k_s(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3)\left(1-\dot{\hat{\tau}}\right) r_{\hat{\tau}}^2 \\ & + \frac{\omega_2 k_s}{\bar{\tau}}\left(\bar{\tau}r^2 - \int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta\right). \end{aligned} \tag{39}$$

Choosing $\omega_4 = \frac{1}{1-\varepsilon_4}$, using Young's Inequality and substituting (15) into (39), and completing the squares on $|r||\chi|$, yields

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\left(\alpha_1 - \frac{\varepsilon_2\alpha_2^2}{2}\right)e^2 + \left(\frac{1}{2\varepsilon_2} + \frac{k_s\omega_3}{2\varepsilon_3}(2+\varepsilon_4)\right)e_u^2 \\ & - k_s\left(\frac{1}{2}c_m\alpha_2 - \frac{1}{2}(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4)\right. \\ & \left. - \varepsilon_3\omega_3 - \omega_2\right) r^2 + k_s c_B |r(r_{\hat{\tau}} - r_\tau)| \\ & - \frac{\omega_2 k_s}{\bar{\tau}}\int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta - \frac{1}{4}c_m\alpha_2 k_s r^2 \\ & + \frac{2}{c_m\alpha_2 k_s}\left(\rho^2(\|z\|)\|z\|^2 + \Phi^2\right). \end{aligned} \tag{40}$$

Using the Cauchy-Schwarz inequality and (18), an upper bound for e_u^2 can be expressed as

$$e_u^2 \leq \bar{\tau}k_s^2 \int_{t-\bar{\tau}}^{t} r^2(\theta)\, d\theta. \tag{41}$$

Furthermore, Q_1 can be upper bounded as

$$Q_1 \leq \frac{1}{2}k_s(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3)\int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta, \tag{42}$$

and an upper bound for Q_2 can be obtained as

$$Q_2 \leq \omega_2 k_s \int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta. \tag{43}$$

Using the definition of β_1 in (23) and (41)-(43) the subsequent upper bound can be developed

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\beta_1\|z\|^2 - k_s\left(\frac{1}{2}c_m\alpha_2 - \varepsilon_3\omega_3 - \omega_2\right. \\ & \left. - \frac{1}{2}(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4)\right) r^2 \\ & + k_s c_B|r(r_{\hat{\tau}} - r_\tau)| - \frac{1}{3\bar{\tau}}Q_2 \\ & - \frac{\omega_2}{3\bar{\tau}\left(\frac{1}{2}(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3)\right)}Q_1 \\ & + \frac{2}{c_m\alpha_2 k_s}\left(\rho^2(\|z\|)\|z\|^2 + \Phi^2\right). \end{aligned} \tag{44}$$

Provided that $\|y(t)\| < \gamma,\ \forall t \in [t_0, \infty)$, where γ is defined in (30) and is a known constant, it can be proven that $\dot{r} < \Upsilon$, which will subsequently allow for the Mean Value Theorem to be used to further upper bound (44). Specifically, from (15) and (20), Properties 1-6, 8, and 9, and the fact that $\|y\| \geq \|z\|$, it can be shown that

$$\dot{r} < c_1 + c_2\gamma + c_3\gamma^2 \leq \Upsilon,$$

where $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ are known constants. Subsequently, by completing the squares, the fact that $\|y\| \geq \|z\|$, imposing the aforementioned gain conditions in (32)-(37), and using the Mean Value Theorem, (44) can be upper bounded as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\left(\frac{1}{2}\beta_1 - \frac{2}{c_m\alpha_2 k_s}\rho^2(\|y\|)\right)\|z\|^2 \\ & - \frac{1}{2}\beta_1\|z\|^2 - \frac{1}{3\bar{\tau}}Q_2 + \frac{(\bar{\tau} - \underline{\tau})\Upsilon^2}{k_s} + \frac{2\Phi^2}{c_m\alpha_2 k_s} \\ & - \frac{\omega_2}{3\bar{\tau}\left(\frac{1}{2}(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3)\right)}Q_1. \end{aligned} \tag{45}$$

Provided $y(t) \in \mathcal{D} \triangleq \left\{y \in \mathbb{R}^5 \mid \|y\| < \gamma\right\}, \forall t \in \left[t_n^m, t_{n+1}^e\right)$ and using (25) and the definition of v in (31), (45), can be upper bounded as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\delta_1\|y\|^2 + v. \tag{46}$$

From (28), the bound in (46) can be bounded even further as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\frac{\delta_1}{\lambda_2}V_L + v, \tag{47}$$

$\forall t \in \left[t_n^m, t_{n+1}^e\right)$.

When $B_M^\tau = 0$, the FES effect is absent from the system because the rider's muscles are not generating a force (i.e., $t \in \left[t_n^e, t_{n+1}^m\right)$). Therefore, to maintain control authority the switching laws in (6) and (7) were designed such that $B_E > 0$ whenever $B_M^\tau = 0$ (i.e., the system is controlled by the motor only). Setting $B_M^\tau = 0$ in (38), choosing ε_1 and ω_1 such that $c_M\alpha_2 - c_b \leq \varepsilon_1\omega_1$, choosing $\left|\dot{\hat{\tau}}\right| \leq \varepsilon_4 < 1$, canceling common terms, and using Properties 1 and 7-9 yields an upper bound for (38) as

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\alpha_1 e^2 + \alpha_2|ee_u| + |r||\chi| - c_e k_1|r| + c_b k_s|rr_{\hat{\tau}}| \\ & - c_e(k_2 + k_3)r^2 + k_s(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)|rr_{\hat{\tau}}| \\ & - c_m\alpha_2 k_s r^2 + k_s\omega_3|e_u r| + k_s\omega_3\left(1-\dot{\hat{\tau}}\right)|e_u r_{\hat{\tau}}| \\ & + \frac{1}{2}k_s(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3) r^2 \\ & - \frac{1}{2}k_s(\omega_4(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4) + \varepsilon_3\omega_3)\left(1-\dot{\hat{\tau}}\right) r_{\hat{\tau}}^2 \\ & + \frac{\omega_2 k_s}{\bar{\tau}}\left(\bar{\tau}r^2 - \int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta\right). \end{aligned} \tag{48}$$

After using the Mean Value Theorem, completing the squares on $|r||\chi|$, using Young's Inequality, substituting (15) into (48), and selecting the gain conditions according to (32)-(37), yields the following upper bound for (48)

$$\begin{aligned}\dot{V}_L \overset{\text{a.e.}}{\leq} & -\left(\alpha_1 - \frac{\varepsilon_2\alpha_2^2}{2}\right)e^2 + \frac{1}{4c_m\alpha_2 k_s}\rho^2(\|z\|)\|z\|^2 \\ & - \left(c_e k_2 - k_s\left(\frac{1}{2}(\varepsilon_1\omega_1 + c_M\alpha_2\varepsilon_4)(1+\omega_4)\right.\right. \\ & \left.\left. + \varepsilon_3\omega_3 + \omega_2\right)\right) r^2 - \frac{\omega_2 k_s}{\bar{\tau}}\int_{t-\bar{\tau}}^{t} r(\theta)^2 d\theta \\ & + \left(\frac{1}{2\varepsilon_2} + \frac{k_s\omega_3}{2\varepsilon_3}(2+\varepsilon_4)\right)e_u^2. \end{aligned} \tag{49}$$

After following a similar development to the case when $B_M^\tau > 0$, (49) can be upper bounded as

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\frac{\delta_2}{\lambda_2}V_L, \tag{50}$$

$\forall t \in \left[t_n^e, t_{n+1}^m\right)$. An upper bound for both (47) and (50) can be obtained by adding the constant v and substituting the decay rate $\lambda_3 \triangleq \lambda_2^{-1}\min(\delta_1, \delta_2)$ into (50) to yield

$$\dot{V}_L \overset{\text{a.e.}}{\leq} -\lambda_3 V_L + v. \tag{51}$$

The decay rate in (51) represents the most conservative decay rate across all regions (i.e., $\forall t \in [t_0, \infty)$). Furthermore, it can be verified that (27) is a common Lyapunov function across all regions of the crank cycle. The differential inequality in (51) can be solved to yield the bound

$$V_L(t) \leq V_L(t_0)\exp(-\lambda_3(t-t_0)) + \lambda_3^{-1} v (1 - \exp(-\lambda_3(t-t_0))), \tag{52}$$

provided that $\|y(t)\| < \gamma, \ \forall t \in [t_0, \infty)$. It could be shown that a sufficient condition for $\|y(t)\| < \gamma, \ \forall t \in [t_0, \infty)$ is that $y(t_0) \in S_{\mathcal{D}}$, provided the gain conditions in (32) to (37) are satisfied. Therefore, provided that $y(t_0) \in S_{\mathcal{D}}$ and the aforementioned gain conditions are met, (27) can be used with (52) to yield the exponential bound in (31). From (27) and (51), $e, r, e_u \in \mathcal{L}_\infty$. By (18) and (19), u, $u_e \in \mathcal{L}_\infty$ and the remaining signals are bounded.

5. CONCLUSION

In this paper, robust cadence tracking controllers and delay-dependent switching conditions are developed for a switched uncertain nonlinear dynamic system in the presence of time-varying input delays and bounded unknown additive disturbances. A Lyapunov based stability analysis is provided. To better estimate the delay a time-varying estimate is developed. The switching conditions are designed to appropriately schedule both the activation and deactivation of the FES and the motor, allowing the residual muscle forces to be managed. Ongoing efforts seek to better understand the FES delay to allow for an improved estimation of the delay. Additional work will include validation of the controllers effectiveness on participants with NCs and to perform an in-depth analysis.

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