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Copositive Conditions for LMI-Based Controller and Observer Design

Juan Carlos Arceo * **Jimmy Lauber** *

* *Université Polytechnique Hauts-de-France, LAMIH UMR CNRS 8201, F-59313 Valenciennes, France.*

Abstract: In this report we illustrate that nonlinear control and observer design can be described as a Positivstellensatz problem, we consider a quadratic candidate function to stabilize nonlinear systems that have been modeled via the nonlinear sector methodology as an exact convex representation. New shape-independent conditions for satisfying positiveness in double sums are established based on the concept of copositivity. The conditions obtained are compared with previous approaches found in the literature via a numerical example.

Keywords: Copositive, Exact Convex Representations, Linear Matrix Inequalities (LMIs), Nonlinear Sector, Fuzzy Systems, Relaxed Conditions, Takagi-Sugeno.

1. INTRODUCTION

The use of convex structures in nonlinear control design allows to reduce complexity in stability analysis, when convex structures are combined with the direct method of Lyapunov instead of having to check stability conditions for infinite points it is sufficient to check stability only for a finite number of linear subsystems. Thus, conditions for guaranteeing stability are frequently expressed in terms of linear matrix inequalities (LMIs) which are transformed into a convex optimization problem, LMIs have been widely used in control theory for many reasons, one of them is that they offer tractable solutions, it means that the algorithms that solve these convex optimization problems have polynomial-time complexity see Gahinet et al. (1994), and their solution can be computed via commercially available interior-point algorithms Boyd et al. (1994). Moreover, nonlinear synthesis based on LMIs allows to easily include performance parameters within the designs such as decay rate, disturbance rejection, input and output constraints, state constraints, uncertainties in the model, and more, by just stacking more constraints to the LMI problem to solve Tanaka and Wang (2004).

Different methods exist for rewriting nonlinear systems as an exact convex representation, some of these are explained in Taniguchi et al. (2001); Baranyi (2004); Sala and Ariño (2009), in this paper we focus in the ones obtained by applying the nonlinear sector methodology from Taniguchi et al. (2001), this method employs membership functions to interpolate between the extreme values for each nonlinearity.

There are several sources of conservativeness, some have been tackled in previous results; them may come from the choice of the candidate Lyapunov function (it can be quadratic, nonquadratic, periodic, polynomial or piece-wise) Guerra et al. (2009); Lendek et al. (2013); Parrilo (2000); Johansson et al. (1999), conditions that are shape-independence of the membership functions Lam and Lauber (2013), and nonuniqueness in exact convex representations Sala (2009), among others. One of these problems is how to properly take into account interactions between the linear subsystems, i.e., how are we sure that all the interactions among subsystems are stable? Asymptotically sufficient and necessary conditions have been obtained in Sala and Arino (2007), these are based on Polya's theorem for positive forms on the standard simplex. However, there is a trade-off between the number of constraints required to prove positiveness of a function and conservativeness of the condition, a less conservative condition requires increasing the number of inequality constraints and this number grows exponentially due to permutations, which is reflected on increasing computational time to find a solution Boyd et al. (1994).

Given this context, the aim of this report is to establish sufficient shape-independent quadratic stability conditions for nonlinear control and observer design by exploiting the concept of copositive matrix while keeping the problem as tractable.

The document is organized as follows: section 2 briefly explains the nonlinear sector methodology used to obtain an exact convex representation for a nonlinear system, some of the existing definitions, lemmas and notation are established, then, stability conditions for nonlinear controller and observer design are given based on a quadratic candidate Lyapunov function, in section 3 new theorems for nonlinear control and observer design are obtained based on sufficient conditions for copositivity, in section 4 a numerical example is used to illustrate the effectiveness of the approach by means of a comparison with previous lemmas and results, in section 5 a conclusion based on the obtained results is presented.

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2. PRELIMINARIES

In this section the procedure for obtaining an exact convex representation for a nonlinear system via the nonlinear sector methodology is briefly explained, then, stability conditions for nonlinear control and observer design based on a quadratic candidate Lyapunov function are transformed into a problem of Positivstellensatz. Some of the lemmas and definitions used in the literature to satisfy this constrained positiveness condition are given.

2.1 Nonlinear sector methodology

Consider a continuous time nonlinear system

$$\dot{x}(t) = f(x(t)) + g(x(t))\,u(t), \quad y(t) = z(x(t)), \tag{1}$$

these systems can be rewritten for all x within a compact of interest Ω by means of the nonlinear sector methodology from Taniguchi et al. (2001); thus, obtaining an exact convex representation for

$$\dot{x} = A(x)x + B(x)u, \quad y = C(x)x.$$

The method consists on rewriting each of the ρ non constant terms (denoted by z_i, with $i \in \{1, 2, \cdots, \rho\}$) in the matrices $A(x)$, $B(x)$ and $C(x)$ in (3) for all $x \in \Omega$, via the interpolation or membership functions defined as $h_i = w^1_{i_1} w^2_{i_2} \cdots w^r_{i_r}$, where $[i_1, i_2, \cdots, i_r]$ is a $\rho-$digit binary representation of $(i-1)$ with $i \in \{1, 2, \cdots, r\}$ and $r = 2^\rho$ is the number of membership or interpolation functions necessary to obtain this exact convex representation, the weight functions w^i_0 and w^i_1 that compose the membership functions are defined as

$$w^i_0 = \frac{z^1_i - z_i}{z^1_i - z^0_i}, \quad w^i_1 = \frac{z_i - z^0_i}{z^1_i - z^0_i},$$

where z^1_i and z^0_i stands for the maximum and minimum values of each non constant term z_i within Ω, respectively. Each non constant term is expressed as the convex sum of its constant extreme values $z_i = w^i_0\left(z^0_i\right) + w^i_1\left(z^1_i\right)$, the constant matrices can be obtained by evaluating

$$A_i = A(x)|_{h_i=1},\ B_i = B(x)|_{h_i=1},\ C_i = C(x)|_{h_i=1}, \tag{2}$$

it yields to the following *exact convex representation*

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(A_i x(t) + B_i u(t)\right), \quad y(t) = \sum_{i=1}^{r} h_i C_i x(t). \tag{3}$$

The membership functions (denoted by h_i) belong to the standard simplex (i.e., they hold the convex sum properties)

$$0 \le h_i \le 1, \quad \sum_{i=1}^{r} h_i = 1. \tag{4}$$

Developments thereafter assume that the mathematical model of the system has been rewritten as a convex structure, and also that the origin of the system $x = 0$ is contained in the modeling region Ω.

Definition 1. (Positive Johnson (1970)). Let a matrix $Q \in \mathbb{R}^{n\times n}$, be called positive definite if $x^T Q x > 0$ holds for $x \neq 0$ and $x \in \mathbb{R}^n$. A matrix Q is positive definite if and only if its symmetric part is positive definite $\frac{1}{2}\left(Q + Q^T\right) > 0$.

Definition 2. (Copositive Parrilo (2000)). Let a matrix Q be called strictly copositive if $h^T Q h > 0$ holds for $h \succeq 0$, with $Q \in \mathbb{R}^{r\times r}$ and $h \in \mathbb{R}^r$.

2.2 Stability conditions for nonlinear control design

First, consider a nonlinear system (1) written as an exact convex representation (3), assuming that the state vector x can be fully measured and applying a parallel distributed compensation control law (denoted as PDC) has the form

$$u = F(x)x = \sum_{j=1}^{r} h_j F_j x, \tag{5}$$

where h_j are the same membership functions as the original system, then, our closed-loop system is

$$\dot{x} = (A(x) + B(x)F(x))\,x = \sum_{i,j=1}^{r} h_i h_j\left(A_i + B_i F_j\right)x. \tag{6}$$

Consider a quadratic candidate Lyapunov function of the form $V(x) = x^T P x$, it is positive definite if $P > 0$, its time derivative is $\dot{V}(x) = x^T P\dot{x} + \dot{x}^T P x$, substituting (6) gives

$$\dot{V} = x^T\left[\sum_{i,j=1}^{r} h_i h_j\left(P\left(A_i + B_i F_j\right) + \left(F_j^T B_i^T + A_i^T\right)P\right)\right]x,$$

it will be negative $\forall x$ if the following holds

$$\sum_{i,j=1}^{r} h_i h_j\left(P\left(A_i + B_i F_j\right) + \left(F_j^T B_i^T + A_i^T\right)P\right) < 0,$$

since $h_i \ge 0$ and $h_j \ge 0$ in Ω they can be just removed

$$P\left(A_i + B_i F_j\right) + \left(F_j^T B_i^T + A_i^T\right)P < 0,$$

applying congruence with $X = P^{-1}$, yields to

$$-A_i X - B_i F_j X - X F_j^T B_i^T - X A_i^T > 0,$$

applying substitution with $M_j = F_j X$, then, $\dot{V}$ is negative definite if $\sum_{i,j=1}^{r} h_i h_j Q_{ij} > 0$, with $Q_{ij} = -A_i X - B_i M_j - M_j^T B_i^T - X A_i^T$.

2.3 Stability conditions for nonlinear observer design

The nonlinear observer for the system (3) is defined as

$$\hat{x} = \sum_{i,j=1}^{r} h_i h_j\left(A_i\hat{x} + B_i u + L_j\left(y - \hat{y}\right)\right), \ \hat{y} = \sum_{i=1}^{r} h_i C_i \hat{x}, \tag{7}$$

the observation error is $e = x - \hat{x}$ and its time derivative is

$$\dot{e} = \dot{x} - \dot{\hat{x}} = \sum_{i,j=1}^{r} h_i h_j\left(A_i - L_j C_i\right)e. \tag{8}$$

Consider a quadratic candidate Lyapunov function for the error $V = e^T P e$, is positive definite if $P > 0$, its time derivative is $\dot{V} = e^T P\dot{e} + \dot{e}^T P e$ and substituting (8) gives

$$\dot{V} = e^T\left[\sum_{i,j=1}^{r} h_i h_j\left(PA_i - PL_j C_i - C_i^T L_j^T P + A_i^T P\right)\right]e,$$

it is negative $\forall e$ if the following holds

$$\sum_{i,j=1}^{r} h_i h_j\left(-PA_i + PL_j C_i + C_i^T L_j^T P - A_i^T P\right) > 0,$$

since $h_i \ge 0$ and $h_j \ge 0$ in Ω they can be just removed

$$-PA_i + PL_j C_i + C_i^T L_j^T P - A_i^T P > 0,$$

substituting $N_j = PL_j$, then $\dot{V}$ is negative definite if $\sum_{i,j=1}^{r} h_i h_j Q_{ij} > 0$, with $Q_{ij} = -PA_i + N_j C_i + C_i^T N_j^T - A_i^T P$.

2.4 The problem of Positivstellensatz

Both cases controller and observer design have been transformed into a problem of Positivstellensatz, which is basically proving that a constrained polynomial is positive Chesi (2010). Notice that Q_{hh} is a quadratic function of h and increasing the number of membership functions involved will increase the degree of this polynomial function as in results based on Polya's theorem such as Fang et al. (2006); Sala and Ariño (2009); Kruszewski et al. (2009). The stability conditions obtained for both cases are of the form

$$Q_{hh} > 0, \tag{9}$$

with $Q_{hh} = \sum_{i,j=1}^{r} h_i h_j Q_{ij}$, where Q_{ij} are matrices with constant terms and some of them are decision variables (terms to be found). A trivial solution for (9) would be $Q_{ij} > 0\ \forall i, j$, but this is a conservative solution, it does not take into account the interaction between subsystems nor the positiveness in the h functions. There are many shape-independent relaxations available to satisfy the positiveness of (9) such as :

- *Lemma 1.* (Tanaka et al. (1998)). A sufficient condition for the expression (9) to be positive is

$$\begin{aligned} Q_{ii} &> 0, \\ Q_{ij} + Q_{ji} &\geq 0,\ \forall j < i, \end{aligned} \tag{10}$$

holds for $i, j \in \{1, 2, \cdots, r\}$.

- *Lemma 2.* (Tuan et al. (2001)). The equation (9) is positive for $h_i \geq 0$ if

$$\begin{bmatrix} \frac{1}{r-1} Q_{ii} & \frac{1}{2}\left(Q_{ij} + Q_{ji}\right) \\ \frac{1}{2}\left(Q_{ij} + Q_{ji}\right) & \frac{1}{r-1} Q_{jj} \end{bmatrix} > 0, \quad \forall j < i. \tag{11}$$

holds for $i, j \in \{1, 2, \cdots, r\}$.

- *Lemma 3.* (Kim and Lee (2000)). The expression (9) holds if we guarantee that

$$\begin{aligned} &Q_{ii} - Z_{ii} \geq 0, \\ &Q_{ij} + Q_{ji} - 2Z_{ij} \geq 0,\ \forall j < i, \\ &\begin{bmatrix} Z_{11} & Z_{21} & \cdots & Z_{r1} \\ Z_{21} & Z_{22} & \cdots & Z_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} > 0. \end{aligned} \tag{12}$$

- *Lemma 4.* (Xiaodong and Qingling (2003)). The inequality in (9) holds if there are matrices such that

$$\begin{aligned} &Q_{ii} - Z_{ii} \geq 0, \\ &Q_{ij} + Q_{ji} - Z_{ij} - Z_{ij}^T \geq 0,\ \forall j < i, \\ &\begin{bmatrix} Z_{11} & Z_{21}^T & \cdots & Z_{r1}^T \\ Z_{21} & Z_{22} & \cdots & Z_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} > 0. \end{aligned} \tag{13}$$

All the previous lemmas consider the h functions as positive scalars subject to an algebraic and inequality constraint (i.e., $h \geq 0$ and $\sum h = 1$); is easy to check that the condition $h \leq 1$ is *also guaranteed*, since you can only satisfy $\sum h = 1$ with positive scalars smaller or equal to 1.

Verifying that a matrix Q is not copositive is a well known NP-complete problem Murty and Kabadi (1987), it can not be verified by checking its eigenvalues, all the positive definite matrices are copositive, but the converse is false.

Lemma 5. (Parrilo (2000)). A sufficient condition for a matrix Q to be copositive is that it can be written as the sum of a positive semidefinite matrix P_d and a nonnegative matrix Λ with entries λ_{ij} :

$$Q = P_d + \Lambda \tag{14}$$

with $P_d \geq 0$ and $\lambda_{ij} \geq 0$; there are copositive matrices that can not be expressed in this form Quist et al. (1998).

3. MAIN RESULTS

In this section positiveness of the membership functions is considered into the Lyapunov analysis for nonlinear control and observer design, the new theorems obtained are based on the concept of copositive matrix.

Theorem 1. The origin of the system (3) is asymptotically stable under a control of the form (5) if there are proper size matrices $X = X^T > 0$, M_j, $\lambda_{ij} = \lambda_{ij}^T \geq 0$, for all $j < i$ such that :

$$\begin{bmatrix} Q_{11} - \lambda_{11} & Q_{21}^T - \lambda_{21}^T & \cdots & Q_{r1}^T - \lambda_{r1}^T \\ Q_{21} - \lambda_{21} & Q_{22} - \lambda_{22} & \cdots & Q_{r2}^T - \lambda_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r1} - \lambda_{r1} & Q_{r2} - \lambda_{r2} & \cdots & Q_{rr} - \lambda_{rr} \end{bmatrix} > 0. \tag{15}$$

is strictly positive definite for $Q_{ii} = -A_i X - B_i M_i - X A_i^T - M_i^T B_i^T$, $Q_{ij} = \frac{1}{2}\left(-A_i X - B_i M_j - X A_i^T - M_j^T B_i^T\right) + \frac{1}{2}\left(-A_j X - B_j M_i - X A_j^T - M_i^T B_j^T\right)$ with $j < i$ and $i, j \in \{1, 2, \cdots, r\}$. The gains for the controller are $F_j = M_j X^{-1}$.

Proof. The closed-loop system (6) can be written as

$$\dot{x}(t) = \sum_{i=1}^{r} h_i^2 G_{ii} x(t) + 2 \sum_{i<j\leq r} h_i h_j \left(\frac{G_{ij} + G_{ji}}{2}\right) x(t),$$

with $G_{ij} = A_i + B_i F_j$ and consider a quadratic candidate Lyapunov function $V = x^T P x$, with $P > 0$, its time-derivative is $\dot{V} = x^T P \dot{x} + \dot{x}^T P x = -x^T Q_{hh} x$, where Q_{hh} is equal to

$$Q_{hh} = \sum_{i,j=1}^{r} h_i h_j \left(\Delta_{ij}^T P + P \Delta_{ij}\right), \tag{16}$$

this can be rewritten as

$$\bar{h}^T \begin{bmatrix} P\Delta_{11} + \Delta_{11}^T P & P\Delta_{21} + \Delta_{21}^T P & \cdots & P\Delta_{r1} + \Delta_{r1}^T P \\ P\Delta_{21} + \Delta_{21}^T P & P\Delta_{22} + \Delta_{22}^T P & \cdots & P\Delta_{r2} + \Delta_{r2}^T P \\ \vdots & \vdots & \ddots & \vdots \\ P\Delta_{r1} + \Delta_{r1}^T P & P\Delta_{r2} + \Delta_{r2}^T P & \cdots & P\Delta_{rr} + \Delta_{rr}^T P \end{bmatrix} \bar{h}, \tag{17}$$

with $\bar{h} = [h_1 I\ h_2 I\ \cdots\ h_r I]^T$ and $\Delta_{ij} = \frac{1}{2}(G_{ij} + G_{ji})$. Now, consider a copositive matrix $\Gamma_{ij} \in \mathbb{R}^{rn \times rn}$, which means that the quadratic form

$$\Gamma_{hh} = \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix}^T \underbrace{\begin{bmatrix} \gamma_{11} & \gamma_{21}^T & \cdots & \gamma_{r1}^T \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{r1} & \gamma_{r2}^T & \cdots & \gamma_{rr} \end{bmatrix}}_{\Gamma_{ij}} \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix},$$

is positive for $h_i, h_j \geq 0$, with $\gamma_{ij} \in \mathbb{R}^{n \times n}$ for all $1 \leq j < i \leq r$, a sufficient condition for Γ_{ij} to be copositive is that it is composed by blocks of positive semidefinite matrices, therefore, $\gamma_{ij} \geq 0$. According to Lemma 5, if Γ_{ij} is a copositive matrix, then, (17) is also copositive if (18) holds.

$$\begin{bmatrix} P\Delta_{11}+\Delta_{11}^T P-\gamma_{11} & P\Delta_{21}+\Delta_{21}^T P-\gamma_{21} & \cdots & P\Delta_{r1}+\Delta_{r1}^T P-\gamma_{r1} \\ P\Delta_{21}+\Delta_{21}^T P-\gamma_{21} & P\Delta_{22}+\Delta_{22}^T P-\gamma_{22} & \cdots & P\Delta_{r2}+\Delta_{r2}^T P-\gamma_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ P\Delta_{r1}+\Delta_{r1}^T P-\gamma_{r1} & P\Delta_{r2}+\Delta_{r2}^T P-\gamma_{r2} & \cdots & P\Delta_{rr}+\Delta_{rr}^T P-\gamma_{rr} \end{bmatrix} > 0. \tag{18}$$

Then, pre- and post- multiplying it by a proper size diagonal matrix of X, with $X=P^{-1}$ yields to

$$\begin{bmatrix} \Delta_{11}X+X\Delta_{11}^T-X\gamma_{11}X & \Delta_{21}X+X\Delta_{21}^T-X\gamma_{21}X & \cdots & \Delta_{r1}X+X\Delta_{r1}^T-X\gamma_{r1}X \\ \Delta_{21}X+X\Delta_{21}^T-X\gamma_{21}X & \Delta_{22}X+X\Delta_{22}^T-X\gamma_{22}X & \cdots & \Delta_{r2}X+X\Delta_{r2}^T-X\gamma_{r2}^T X \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{r1}X+X\Delta_{r1}^T-X\gamma_{r1}X & \Delta_{r2}X+X\Delta_{r2}^T-X\gamma_{r2}X & \cdots & \Delta_{rr}X+X\Delta_{rr}^T-X\gamma_{rr}X \end{bmatrix} > 0, \tag{19}$$

substituting $M_j = F_jX$ and $\lambda_{ij} = X\gamma_{ij}X$, with $\lambda_{ij} \geq 0$, yields to conditions in Theorem 1, which is equivalent to find $P_d > 0$ for $P_d = Q - \Lambda$ in Lemma 5, this concludes the proof.

Theorem 2. The origin of the system (3) is asymptotically stable under a control of the form (5) if there are proper size matrices $X = X^T > 0$, M_j, $\lambda_{ii} = \lambda_{ii}^T \geq 0$, $\frac{1}{2}\left(\lambda_{ij} - \lambda_{ij}^T\right) > 0$ for all $j<i$, such that :

$$\begin{bmatrix} Q_{11}-\lambda_{11} & Q_{21}^T-\lambda_{21}^T & \cdots & Q_{r1}^T-\lambda_{r1}^T \\ Q_{21}-\lambda_{21} & Q_{22}-\lambda_{22} & \cdots & Q_{r2}^T-\lambda_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r1}-\lambda_{r1} & Q_{r2}-\lambda_{r2} & \cdots & Q_{rr}-\lambda_{rr} \end{bmatrix} > 0. \tag{20}$$

is strictly positive definite for $Q_{ii} = -A_iX - B_iM_i - XA_i^T - M_i^T B_i^T$, $Q_{ij} = \frac{1}{2}\left(-A_iX - B_iM_j - XA_i^T - M_j^T B_i^T\right) + \frac{1}{2}\left(-A_jX - B_jM_i - XA_j^T - M_i^T B_j^T\right)$ with $j < i$ and $i,j \in \{1, 2, \cdots, r\}$. The gains for the controller are $F_j = M_jX^{-1}$.

Proof. It follows directly from previous developments, consider that the elements λ_{ij} with $j < i$ are full matrices and they hold the positive condition $\frac{1}{2}\left(\lambda_{ij} + \lambda_{ij}^T\right) > 0$.

Theorem 3. The origin of the error system (8) with the observer (7) is asymptotically stable if there are proper size matrices $P = P^T > 0$, N_j, $\gamma_{ij} \geq 0$, for all $j<i$ such that

$$\begin{bmatrix} Q_{11}-\gamma_{11} & Q_{21}^T-\gamma_{21}^T & \cdots & Q_{r1}^T-\gamma_{r1}^T \\ Q_{21}-\gamma_{21} & Q_{22}-\gamma_{22} & \cdots & Q_{r2}^T-\gamma_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r1}-\gamma_{r1} & Q_{r2}-\gamma_{r2} & \cdots & Q_{rr}-\gamma_{rr} \end{bmatrix} > 0. \tag{21}$$

is elementwise strictly positive definite with $Q_{ij} = -PA_i + N_jC_i + C_i^T N_j^T - A_i^T P$ and $i,j \in \{1, 2, \cdots, r\}$, where the gains for the observer are computed as $L_j = P^{-1}N_j$.

Proof. It follows directly from previous developments.

Theorem 4. The origin of the error system (8) with the observer (7) is asymptotically stable if there are proper size matrices $P = P^T > 0$, N_j, $\frac{1}{2}\left(\gamma_{ij} - \gamma_{ij}^T\right) \geq 0$, for all $j<i$, such that

$$\begin{bmatrix} Q_{11}-\gamma_{11} & Q_{21}^T-\gamma_{21}^T & \cdots & Q_{r1}^T-\gamma_{r1}^T \\ Q_{21}-\gamma_{21} & Q_{22}-\gamma_{22} & \cdots & Q_{r2}^T-\gamma_{r2}^T \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r1}-\gamma_{r1} & Q_{r2}-\gamma_{r2} & \cdots & Q_{rr}-\gamma_{rr} \end{bmatrix} > 0. \tag{22}$$

is elementwise strictly positive definite with $Q_{ij} = -PA_i + N_jC_i + C_i^T N_j^T - A_i^T P$ and $i,j \in \{1, 2, \cdots, r\}$, where the gains for the observer are computed as $L_j = P^{-1}N_j$.

Proof. It follows directly from previous developments, just consider $\frac{1}{2}\left(\gamma_{ij} + \gamma_{ij}^T\right) > 0$.

The conditions for control design in Theorems 1 and 2 should be at least as good as Lemma 3 and 4, respectively, there is an equivalence given by Lemma 5.

The concept of copositive matrices can be applied to previous results where the Positivstellensatz problem appears, not only to controller and observer design for continuous systems as it was the case for this report. It is neither constrained to be applied only for systems with double-sums, it can be adapted for analyzing higher-order polynomials as shown in Parrilo (2000).

It is important to remark that only the property of positiveness in the membership functions was taken into account $h \geq 0$ in the previous developments, therefore, an appropriate method for including the information of the algebraic constraint $\sum h = 1$ in the stability analysis should reduce the existing gap. To illustrate this fact consider the easier case, which is a nonlinear system with two membership functions h_1 and h_2, the inequalities in Theorem 1 and 2 are sufficient to satisfy stability conditions in the black-region that represents $h_i \geq 0$ in Fig. 1, the region corresponding to the algebraic constraint $h_1+h_2=1$ is indicated with a blue-line and we must satisfy stability conditions only for the intersection of both, which is indicated with a red-line.

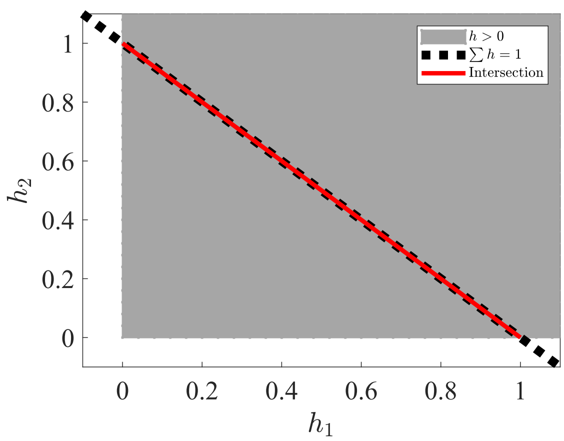


Fig. 1. Illustration of the algebraic condition.

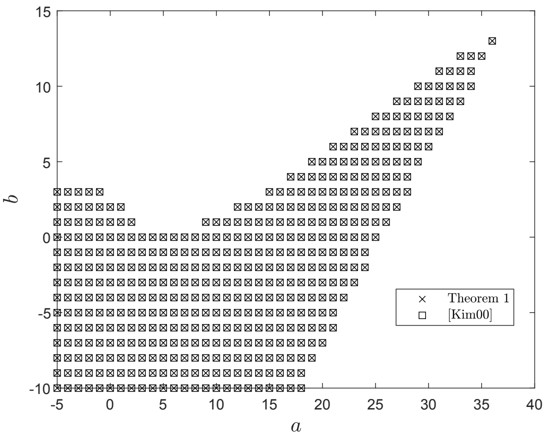


Fig. 2. Comparison between Theorem 1 and Lemma 3.

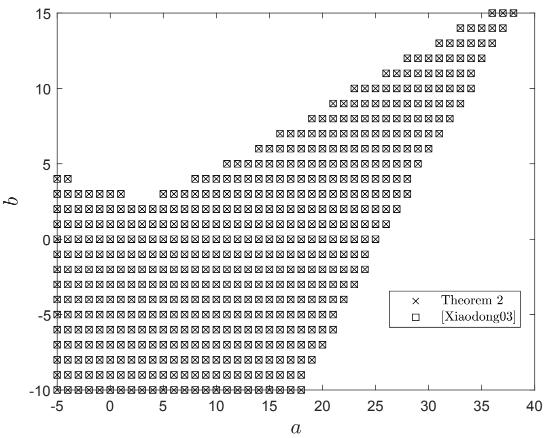
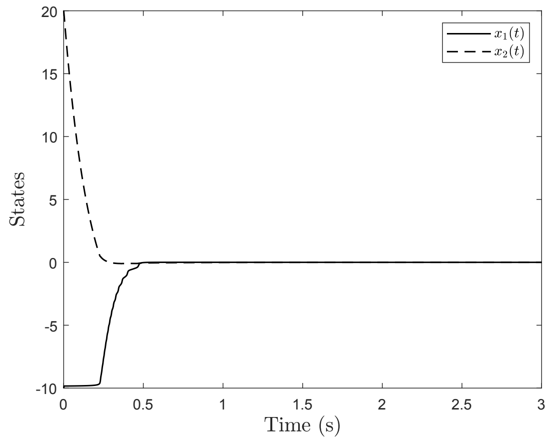
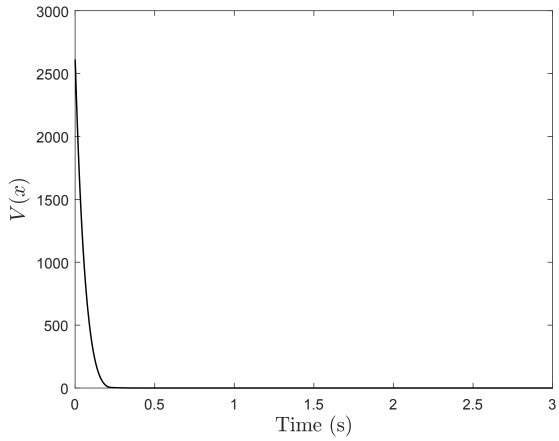


Fig. 3. Comparison between Theorem 2 and Lemma 4.



(a) States of the system.



(b) Time evolution of the Lyapunov function.

Fig. 4. Time evolution of the system in closed-loop.

4. NUMERICAL EXAMPLE

Example 1. Consider the matrices for a nonlinear system given in Fang et al. (2006); Kruszewski et al. (2009) written in a convex representation (3) are $A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}$ and $B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}$, the parameters (a, b) vary within the range $a \in [-10, 40]$ and $b \in [-10, 17]$ and a control law of the form (5) is designed. The feasible points for Theorem 1 and Lemma 3 are indicated with are indicated with $\times$ and $\square$, respectively in Fig. 2. The feasible points for Theorem 2 and Lemma 4 are indicated with are indicated with $\times$ and $\square$, respectively in Fig. 3. These points were obtained via the LMI solvers available in the Robust Control Toolbox in MATLAB R2019b, see Gahinet et al. (1994).

The solution for the point $(a, b) = (38, 15)$ found using Theorem 2, the matrices found are

$$F_1 = [-1.7053 \ 5.1795],$$
$$F_2 = [-0.8528 \ -11.9668],$$
$$F_3 = [-1.8248 \ 49.1549],$$

the matrix associated to the Lyapunov function is

$$P = \begin{bmatrix} 0.0568 & 0.2863 \\ 0.2863 & 6.8039 \end{bmatrix},$$

the solution obtained was simulated with initial conditions $x(0)^T = [-10 \ 20]$ and the membership functions are

$$h_1 = \frac{\cos(10x_1) + 1}{4}, \quad h_2 = \frac{\sin(10x_1) + 1}{4},$$
$$h_3 = -\frac{\sin(10x_1) + \cos(10x_1)}{4} + \frac{1}{2},$$

these functions has been taken from Montagner et al. (2009) and the time evolution of the system with the Lyapunov function obtained is shown in Fig. 4a and Fig. 4b, respectively.

5. CONCLUSION

New conditions for controller and observer design have been established exploiting the concept of copositive matrix and sufficient tests for guaranteeing it. The conditions obtained in Theorem 1 and 2 seems to be as good as Lemmas 3 and 4, respectively, but, no formal proof for it was given for establishing an equivalence.

Future's work might include the second property of the convex sum ($\sum h = 1$) via Finsler's Lemma see Lendek et al. (2018) to improve the obtained conditions or maybe using a different method. Existing results can be applied to get better results, such as considering partitions of the simplex space Kruszewski et al. (2009), stacking convex sums which yields to an increasing in the degree of the polynomial in the Positivstellensatz problem Fang et al. (2006); Sala and Arino (2007) which yields to asymptotically necessary and sufficient conditions, consider shape dependence of the membership functions Bernal et al. (2009); Lendek and Lauber (2016) or allow the candidate Lyapunov function to go beyond quadratic forms maybe using a nonquadratic approach Guerra et al. (2012).

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