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Variance computation for system matrices and transfer function from input/output subspace system identification

Szymon Greś Michael Döhler Laurent Mevel

Univ. Gustave Eiffel, Inria, COSYS-SII, I4S, Campus de Beaulieu, 35042 Rennes, France (e-mail: szymon.gres@inria.fr)

Abstract: The transfer function of a linear system is defined in terms of the quadruplet of matrices (A, B, C, D) that can be identified from input and output measurements. Similarly these matrices determine the state space evolution for the considered dynamical system. Estimation of the quadruplet has been well studied in the literature from both theoretical and practical points of view. Nonetheless, the uncertainty quantification of their estimation errors has been mainly discussed from a theoretical viewpoint. For several output-only and input/output subspace methods, the variance of the (A, C) matrices can be effectively obtained with recently developed first-order perturbation-based schemes. This paper addresses the estimation of the (B, D) matrices, and the remaining problem of the effective variance computation of their estimates and the resulting transfer function. The proposed schemes are validated on a simulation of a mechanical system.

Keywords: Linear time-invariant systems, subspace methods, transfer function, variance computation

1. INTRODUCTION

Subspace-based methods are effective for the identification of linear systems (Benveniste and Fuchs, 1985; Larimore, 1990; Verhaegen and Dewilde, 1992; van Overschee and de Moor, 1994), where they have found applications in many areas of automatic control and mechanical engineering, amongst others. In the underlying state-space model, the quadruplet of matrices (A, B, C, D) defines the evolution of a dynamic system, represented in practice by input/output data. Estimation of the quadruplet is subject of many publications, e.g. (van Overschee and de Moor, 1996; Qin et al., 2005). Some of the work focuses particularly on the identification of (B, D) and subsequently the transfer function (dos Santos and de Carvalho, 2003; Gandino et al., 2013), which is a topic of the current paper.

Subspace methods have favorable statistical properties e.g. are consistent when the noise driving the system is stationary (Deistler et al., 1995; Bauer et al., 1999; Bauer and Jansson, 2000; Knudsen, 2001) or non-stationary (Benveniste and Mevel, 2007). Furthermore, asymptotic normality has been shown and the theoretical expressions for their asymptotic variances were derived e.g. in (Verhaegen, 1993; Viberg et al., 1997; Jansson, 2000; Chiuso and Picci, 2004). Use of those expressions for the actual variance estimation is often problematic in practice, requiring in addition e.g. the estimation of the unknown states and their variances, which are not required to estimate (A, B, C, D). A different approach was proposed by Pintelon et al. (2007); Reynders et al. (2008), where the variance of parameters estimated through subspace identification is computed easily from the sample variances of the underlying data sequences and related sensitivities, based on the Delta method (Casella and L. Berger, 2001). Explicit expressions for the (A, C) matrices have been proposed in (Döhler and Mevel, 2013; Döhler et al., 2014; Mellinger et al., 2016). However, explicit expressions relating the covariance of (B, D) estimates to the covariance of computed data matrices are missing.

The plan of this work is twofold. First, we derive a scheme for the estimation of (B, D) in Section 3, which is a simpler alternative to the *combined* approach from (van Overschee and de Moor, 1996), and a modification of the scheme from (Gandino et al., 2013). The novelty of the proposed method lies in exploiting the theoretical properties of the input/output data matrices to cancel the output noise from the input/output data matrix relation. Second, we extend the current state of the art on the uncertainty quantification to provide an explicit covariance expression for the estimates of (B, D) and the corresponding transfer function in Section 4. Both schemes are validated on a numerical example of a mechanical system in Section 5.

2. SYSTEM MODELING

The dynamic behavior of a linear time-invariant system can be represented by a discrete-time state space model of order n

$$x_{k+1} = Ax_k + Bu_k + w_k \ , \tag{1}$$

$$y_k = Cx_k + Du_k + v_k \ , \tag{2}$$

where $x_k \in \mathbb{R}^n$ are the states, $u_k \in \mathbb{R}^u$ are the known inputs, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times u}$, $C \in \mathbb{R}^{r\times n}$ and $D \in \mathbb{R}^{r\times u}$ are the state transition, input, output and feedthrough matrices, and w_k and v_k are the process and output noises, respectively. In this paper we shall assume the persistence

of the excitation of the input process (Bitmead, 1984), and that there is no feedback from y_k to u_k.

Definition 1. (Data matrix). Let $a_k \in \mathbb{R}^b$ be a discrete signal at time step k. The parameter p defines the 'past' and 'future' data horizons. For $0 \leq i \leq j \leq 2p-1$ the data matrix $\mathcal{A}_{i|j}$ writes

$$\mathcal{A}_{i|j} = \frac{1}{\sqrt{N}} \begin{bmatrix} a_i & a_{i+1} & \dots & a_{i+N-1} \\ a_{i+1} & a_{i+2} & \dots & a_{i+N} \\ \vdots & \vdots & \vdots & \vdots \\ a_j & a_{j+1} & \dots & a_{j+N-1} \end{bmatrix} \in \mathbb{R}^{(j-i+1)b \times N}. \quad (3)$$

From the input and output data, define the data matrices $\mathcal{Y}^- \in \mathbb{R}^{pr \times N}$, $\mathcal{Y}^+ \in \mathbb{R}^{pr \times N}$, $\mathcal{U}^- \in \mathbb{R}^{qu \times N}$, $\mathcal{U}^+ \in \mathbb{R}^{qu \times N}$

$$\mathcal{Y}^- = \mathcal{Y}_{0|p-1},\ \mathcal{Y}^+ = \mathcal{Y}_{p|2p-1}, \quad (4)$$
$$\mathcal{U}^- = \mathcal{U}_{0|p-1},\ \mathcal{U}^+ = \mathcal{U}_{p|2p-1},\ \mathcal{W}^- = \begin{bmatrix} \mathcal{U}^{-T} & \mathcal{Y}^{-T} \end{bmatrix}^T,$$

and define the data covariance matrices as all the cross-product of those data matrices

$$\mathcal{R}_1 = \mathcal{Y}^+ \mathcal{U}^{+T},\ \mathcal{R}_2 = \mathcal{U}^+ \mathcal{U}^{+T}, \quad (5)$$
$$\mathcal{R}_3 = \mathcal{Y}^+ \mathcal{W}^{-T},\ \mathcal{R}_4 = \mathcal{W}^- \mathcal{U}^{+T},\ \mathcal{R}_5 = \mathcal{W}^- \mathcal{W}^{-T}\ .$$

Furthermore, denote respectively the future and past block-row matrices for the system states as

$$\mathcal{X}^- = \mathcal{X}_{0|0},\ \mathcal{X}^+ = \mathcal{X}_{p|p},$$

and future and past output noise matrices as

$$\mathcal{V}^- = \mathcal{V}_{0|p-1},\ \mathcal{V}^+ = \mathcal{V}_{p|2p-1}.$$

Data matrices $\mathcal{Y}^-$ and $\mathcal{Y}^+$ are expressed by recursion of (1) and (2), as in (van Overschee and de Moor, 1996)

$$\mathcal{Y}^- = \Gamma \mathcal{X}^- + H \mathcal{U}^- + \mathcal{V}^-\ , \quad (6)$$
$$\mathcal{Y}^+ = \Gamma \mathcal{X}^+ + H \mathcal{U}^+ + \mathcal{V}^+\ , \quad (7)$$

where $\Gamma \in \mathbb{R}^{pr \times n}$ is the extended observability matrix and

$$H = \begin{bmatrix} D & 0 & 0 & \dots & 0 \\ CB & D & 0 & \dots & 0 \\ CAB & CB & D & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ CA^{p-2}B & CA^{p-3}B & CA^{p-4}B & \dots & D \end{bmatrix} \in \mathbb{R}^{pr \times pu}$$

is used to factor D and B in the next section.

3. SYSTEM IDENTIFICATION

In subspace identification, the system matrices are obtained from a matrix $\mathcal{H}$ that is usually a projection of the data matrices in (4) depending on the chosen identification algorithm, see e.g. (van Overschee and de Moor, 1996). E.g., for N4SID (van Overschee and de Moor, 1994) it is defined as the oblique projection of $\mathcal{Y}^+$ along $\mathcal{U}^+$ on $\mathcal{W}^-$,

$$\mathcal{H} = \mathcal{Y}^+ /_{\mathcal{U}^+} \mathcal{W}^- \in \mathbb{R}^{pr \times N}\ .$$

Then, Γ can be estimated from $\mathcal{H}$ by a Singular Value Decomposition truncated at model order n,

$$\mathcal{H} = \begin{bmatrix} U_\mathrm{s} & U_\mathrm{ker} \end{bmatrix} \begin{bmatrix} D_\mathrm{s} & 0 \\ 0 & D_\mathrm{ker} \end{bmatrix} V^T,\ \Gamma = U_\mathrm{s} D_\mathrm{s}^{1/2}\ , \quad (8)$$

and a least-squares solution for A and C is obtained from the shift invariance property of Γ.

3.1 Estimation of B and D

To obtain (B, D) from the input/output data relation in (7) we use the fact that $\mathrm{E}[\mathcal{V}^+ (\mathcal{U}^+)^T] = 0$. The resulting estimate can be obtained for any choice of $\mathcal{H}$ and is consistent. Multiplying (7) by U_ker^T (see (8)) from the left and $\mathcal{U}^{+T}$ from the right, and taking the expectation, yields

$$U_\mathrm{ker}^T \mathcal{Y}^+ \mathcal{U}^{+T} = U_\mathrm{ker}^T H \mathcal{U}^+ \mathcal{U}^{+T}\ , \quad (9)$$
$$U_\mathrm{ker}^T \mathcal{R}_1 (\mathcal{R}_2)^{-1} = U_\mathrm{ker}^T H\ . \quad (10)$$

Denote $\mathcal{M} = U_\mathrm{ker}^T \mathcal{R}_1 (\mathcal{R}_2)^{-1} \in \mathbb{R}^{(pr-n) \times pu}$ and partition

$$\mathcal{M} = [\mathcal{M}_1 \dots \mathcal{M}_p]\,,\ \mathcal{M}_k \in \mathbb{R}^{(pr-n) \times u},\ k = 1 \dots p\ . \quad (11)$$

Similarly partition $U_\mathrm{ker}^T = [\mathcal{L}_1 \dots \mathcal{L}_p]$, $\mathcal{L}_k \in \mathbb{R}^{(pr-n) \times r}$. Matrices (B, D) follow from the factorization

$$\underbrace{\begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \mathcal{M}_p \end{bmatrix}}_{=\mathcal{M}^v} = \underbrace{\begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \dots & \mathcal{L}_{p-1} & \mathcal{L}_p \\ \mathcal{L}_2 & \mathcal{L}_3 & \dots & \mathcal{L}_p & 0 \\ \mathcal{L}_3 & \mathcal{L}_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{L}_p & 0 & \dots & 0 & 0 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} \mathcal{I}_r & 0 \\ 0 & \underline{\Gamma} \end{bmatrix}}_{=O_s} \begin{bmatrix} D \\ B \end{bmatrix} \quad (12)$$

analogously to (van Overschee and de Moor, 1996) as

$$\begin{bmatrix} D \\ B \end{bmatrix} = (L_s)^\dagger \mathcal{M}^v, \quad (13)$$

where $L_s = L O_s$, $\mathcal{M}^v \in \mathbb{R}^{p(pr-n) \times u}$, $L \in \mathbb{R}^{p(pr-n) \times pr}$, $O_s \in \mathbb{R}^{pr \times (r+n)}$, $\underline{\Gamma} \in \mathbb{R}^{(p-1)r \times n}$ is the matrix Γ without the last block row and $\mathcal{I}_r$ is the $r \times r$ identity matrix. Compared to the *combined* approach from (van Overschee and de Moor, 1996), the proposed approach does not involve approximating the Kalman state sequences, nor minimizing the prediction error, resulting in a simpler algorithm. The algorithm in (Gandino et al., 2013) uses $\mathrm{E}[\mathcal{V}^+ (\mathcal{Y}^-)^T] = 0$ to cancel the noise term in (7). For the case when $\mathcal{U}^+$ is a white noise process, then $\mathrm{E}[\mathcal{U}^+ (\mathcal{Y}^-)^T] = 0$ and the estimation of (B, D) becomes impossible in this case. Moreover, when $u > r$ the product $\mathcal{U}^+ (\mathcal{Y}^-)^T$ is not full row rank and (13) becomes invalid. The proposed approach is free of those shortcomings.

3.2 Computation of G(s)

The transfer function is defined for any complex s as

$$G(s) = C(s\mathcal{I}_n - A)^{-1} B + D \in \mathbb{C}^{r \times u}. \quad (14)$$

Denote the real and imaginary part of a complex variable as $(\cdot)^\mathrm{R}$ and $(\cdot)^\mathrm{I}$, then

$$\begin{bmatrix} G^\mathrm{R}(s) \\ G^\mathrm{I}(s) \end{bmatrix} = \begin{bmatrix} CZ^\mathrm{R} B + D \\ CZ^\mathrm{I} B \end{bmatrix}\ , \quad (15)$$

where $Z = (s\mathcal{I}_n - A)^{-1}$. Since $G(s)$ is complex-valued, it is usually interpreted by its magnitude $m_{G(s)} = |G(s)|$ and phase $p_{G(s)} = \tan^{-1}\left(G^\mathrm{I}(s)/G^\mathrm{R}(s)\right)$ in many engineering applications.

4. COVARIANCE COMPUTATION

Covariance estimates are obtained based on the first-order Delta method, which allows to characterize the probability distribution of a function of an asymptotically Gaussian variable. Let $\hat{\theta}$ be a real-valued vector, which is an estimate of the (unknown) vector θ_*. Let $\hat{\theta}$ be computed on N data samples, and assume that $\hat{\theta}$ satisfies a Central Limit Theorem (CLT), i.e.

$$\sqrt{N}(\hat{\theta} - \theta_*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma_{\theta_*}), \quad (16)$$

where Σ_{θ_*} is the asymptotic covariance of $\hat{\theta}$. Now consider a vector-valued function $g(\theta) \in \mathbb{R}$, such that $g(\theta)$ is differentiable in θ_* with a non-zero derivative $\mathcal{J}^g_{\theta_*} = \frac{\partial g}{\partial \theta} \neq 0$. Then, a first-order Taylor expansion yields

$$g(\hat{\theta}) = g(\theta_*) + \mathcal{J}^g_{\theta_*}(\hat{\theta} - \theta_*) + o(||\hat{\theta} - \theta_*||),$$

and based on (16), the Delta method states that an associated CLT holds for $g(\hat{\theta})$ as (Casella and L. Berger, 2001)

$$\sqrt{N}\left(g(\hat{\theta}) - g(\theta_*)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathcal{J}^g_{\theta_*}\Sigma_{\theta_*}(\mathcal{J}^g_{\theta_*})^T),$$

Consistent estimates of $\mathcal{J}^g_{\theta_*}$ and Σ_{θ_*} are required for the estimation of the asymptotic covariance. A consistent estimate of $\mathcal{J}^g_{\theta_*}$ can be obtained as $\mathcal{J}^g_{\hat{\theta}}$, i.e. by using $\hat{\theta}$ in its computation, and a consistent estimate of Σ_{θ_*} is based on the applied estimation method, e.g. using an underlying sample variance estimate.

Often an estimate of $\mathcal{J}^g_{\theta_*}$ cannot be computed directly, hence it is obtained with the first order perturbation

$$\Delta g \approx \mathcal{J}^g_{\hat{\theta}} \Delta\theta \ ,$$

where $\Delta g = g(\hat{\theta}) - g(\theta_*)$ and $\Delta\theta = \hat{\theta} - \theta_*$. In consequence, the asymptotic covariance of $g(\hat{\theta})$ is consistently approximated by

$$\Sigma_{g(\theta_*)} \approx \mathcal{J}^g_{\hat{\theta}}\Sigma_{\theta_*}(\mathcal{J}^g_{\hat{\theta}})^T \ .$$

4.1 Covariance related to $\mathcal{H}$

The estimation of B and D depends, amongst others, on U_{ker} from (8), and thus their covariance is related to the one of $\mathcal{H}$. For subspace algorithms as in (van Overschee and de Moor, 1996), in which the number of columns of $\mathcal{H}$ equals the number of samples N, there is no finite limit as $N \to \infty$, and a sample covariance estimate would be undefined. Therefore, the matrix $\mathcal{H}$ is replaced by the "square" matrix $\mathcal{H}^s = \mathcal{H}\mathcal{H}^T$, whose properties are equal for the identification of Γ and thus U_{ker}. Then, matrix $\mathcal{H}^s \in \mathbb{R}^{pr \times pr}$ can be easily linked to data covariance matrices such as in (5), whose sample covariance is easy to evaluate (Mellinger et al., 2016).

Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_J$ be the data covariance matrices that the chosen subspace algorithm depends on, and define

$$\text{vec}(\mathcal{R}) = \begin{bmatrix} \text{vec}(\mathcal{R}_1) \\ \vdots \\ \text{vec}(\mathcal{R}_J) \end{bmatrix} \ , \tag{17}$$

and $\Sigma_{\mathcal{R}} = \text{cov}(\text{vec}(\mathcal{R}))$ that can be easily obtained as a sample covariance. Then, first-order perturbations yield

$$\text{vec}(\Delta\mathcal{H}^s) = \mathcal{J}^{\mathcal{H}^s}_{\mathcal{R}}\text{vec}(\Delta\mathcal{R}) \ ,$$

where $\mathcal{J}^{\mathcal{H}^s}_{\mathcal{R}}$ is dependent on the identification algorithm; e.g. for N4SID it holds $\mathcal{H}^s = f(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5)$, and

$$\text{vec}(\Delta\mathcal{H}^s) = \mathcal{J}^{\mathcal{H}^s}_{\mathcal{R}} \begin{bmatrix} \text{vec}(\Delta\mathcal{R}_1) \\ \text{vec}(\Delta\mathcal{R}_2) \\ \text{vec}(\Delta\mathcal{R}_3) \\ \text{vec}(\Delta\mathcal{R}_4) \\ \text{vec}(\Delta\mathcal{R}_5) \end{bmatrix} \ ,$$

where $\mathcal{J}^{\mathcal{H}^s}_{\mathcal{R}}$ is defined in (Mellinger et al., 2016).

Note that while (A, C) can be obtained only from $\mathcal{H}^s$ and thus a relation for their respective covariances follows, this is not the case for (B, D). Hence, the approach of (Mellinger et al., 2016) cannot be simply extended to the covariance computation of (B, D).

4.2 Covariance computation of B and D

Estimates of (B, D) from (13) are a function of data covariance matrices $\mathcal{R}_1$ and $\mathcal{R}_2$, and of U_{ker} that depends on the chosen subspace method, thus on the set $\mathcal{R}$ of covariance matrices in (17). The propagation of the covariances of $\mathcal{R}$ to those of the estimates of (B, D) can be made based on Proposition 2.

Proposition 2. The sensitivity of (D, B) w.r.t. $\mathcal{R}$ yields

$$\text{vec}\left(\Delta\begin{bmatrix} D \\ B \end{bmatrix}\right) = \left(\mathcal{J}^{\mathcal{M}^{v,1}}_{\mathcal{R}} + \mathcal{J}^{(L_s)^\dagger}_{\mathcal{R}}\right)\text{vec}(\Delta\mathcal{R}) + \mathcal{J}^{\mathcal{M}^{v,2}}_{\mathcal{R}_{1,2}}\begin{bmatrix} \text{vec}(\Delta\mathcal{R}_1) \\ \text{vec}(\Delta\mathcal{R}_2) \end{bmatrix} \tag{18}$$

where $(\mathcal{J}^{\mathcal{M}^{v,1}}_{\mathcal{R}}, \mathcal{J}^{\mathcal{M}^{v,2}}_{\mathcal{R}_{1,2}})$ and $\mathcal{J}^{(L_s)^\dagger}_{\mathcal{R}}$ are the respective sensitivities of $\mathcal{M}^v$ and $(L_s)^\dagger$ towards the data covariance matrices.

Proof: Perturbation in B and D is developed as

$$\begin{aligned}\text{vec}\left(\Delta\begin{bmatrix} D \\ B \end{bmatrix}\right) &= \text{vec}\left(\Delta\left(L_s^\dagger\mathcal{M}^v\right)\right) \\ &= \left(\mathcal{I}_u \otimes L_s^\dagger\right)\text{vec}\left(\Delta\mathcal{M}^v\right) \\ &\quad + \left(\mathcal{M}^{vT} \otimes \mathcal{I}_{r+n}\right)\text{vec}\left(\Delta(L_s^\dagger)\right).\end{aligned}$$

The first order perturbation of $\mathcal{M}^v$ is linked to $\mathcal{M}$ in (11) by $\text{vec}(\Delta\mathcal{M}^v) = S_1\text{vec}(\Delta\mathcal{M})$ with the permutation matrix

$$S_1 = \begin{bmatrix} \mathcal{I}_p \otimes e_1^T \otimes \mathcal{I}_{pr-n} \\ \vdots \\ \mathcal{I}_p \otimes e_u^T \otimes \mathcal{I}_{pr-n} \end{bmatrix}$$

where $e_j \in \mathbb{R}^u$ is a unit vector that is 1 at entry j. The first order perturbation of $\mathcal{M}$ in (11) yields

$$\begin{aligned}\text{vec}(\Delta\mathcal{M}) &= \text{vec}\left(\Delta(U_{\text{ker}}^T\mathcal{R}_1\mathcal{R}_2^{-1})\right) \\ &= \left((\mathcal{R}_1\mathcal{R}_2^{-1})^T \otimes \mathcal{I}_{pr-n}\right)\text{vec}\left(\Delta U_{\text{ker}}^T\right) \\ &\quad + \left((\mathcal{R}_2^{-1})^T \otimes U_{\text{ker}}^T\right)\text{vec}(\Delta\mathcal{R}_1) \\ &\quad + \left((\mathcal{R}_2^{-1})^T \otimes (-U_{\text{ker}}^T\mathcal{R}_1\mathcal{R}_2^{-1})\right)\text{vec}(\Delta\mathcal{R}_2),\end{aligned}$$

where $\Delta(\mathcal{R}_2^{-1}) = -\mathcal{R}_2^{-1}\Delta(\mathcal{R}_2)\mathcal{R}_2^{-1}$. $\text{vec}(\Delta U_{\text{ker}}^T)$ was developed in (Viefhues et al., 2018) and writes

$$\text{vec}(\Delta U_{\text{ker}}^T) = \mathcal{J}^{U_{\text{ker}}^T}_{\mathcal{H}^s}\text{vec}(\Delta\mathcal{H}^s) \ ,$$

where $\mathcal{J}^{U_{\text{ker}}^T}_{\mathcal{H}^s} = -(U_sD_s^{-1}V_s^T \otimes U_{\text{ker}}^T)$. Thus

$$\text{vec}\left(\Delta\mathcal{M}^v\right) = \mathcal{J}^{\mathcal{M}^{v,1}}_{\mathcal{R}}\text{vec}(\Delta\mathcal{R}) + \mathcal{J}^{\mathcal{M}^{v,2}}_{\mathcal{R}_{1,2}}\begin{bmatrix} \text{vec}(\Delta\mathcal{R}_1) \\ \text{vec}(\Delta\mathcal{R}_2) \end{bmatrix} \ ,$$

where

$$\begin{aligned}\mathcal{J}^{\mathcal{M}^{v,1}}_{\mathcal{R}} &= S_1\left((\mathcal{R}_1\mathcal{R}_2^{-1})^T \otimes \mathcal{I}_{pr-n}\right)\mathcal{J}^{U_{\text{ker}}^T}_{\mathcal{H}^s}\mathcal{J}^{\mathcal{H}^s}_{\mathcal{R}} \\ \mathcal{J}^{\mathcal{M}^{v,2}}_{\mathcal{R}_{1,2}} &= S_1\left[(\mathcal{R}_2^{-1})^T \otimes U_{\text{ker}}^T \quad (\mathcal{R}_2^{-1})^T \otimes -U_{\text{ker}}^T\mathcal{R}_1\mathcal{R}_2^{-1}\right].\end{aligned}$$

Next, the first order perturbation of $L_s^\dagger$ writes as

$$\text{vec}\left(\Delta(L_s^\dagger)\right) = \mathcal{J}^{L_s^\dagger}_{L_s}\text{vec}(\Delta L_s) \ ,$$

where $\mathcal{J}^{L_s^\dagger}_{L_s}$ is expressed after (Golub and Pereyra, 1973)

$$\mathcal{J}_{L_s}^{L_s^\dagger} = \left(L_s^{\dagger T} \otimes -L_s^\dagger\right) + \left((\mathcal{I}_{p(pr-n)} - L_s L_s^\dagger) \otimes (L_s^{\dagger T} L_s^\dagger)^T\right) P_{p(pr-n),r+n} \ , \tag{19}$$

and $P_{a,b}$ is a permutation matrix such that for $X \in \mathbb{R}^{a,b}$ it holds $\text{vec}(X^T) = P_{a,b}\text{vec}(X)$, after (Döhler and Mevel, 2013). Next, the perturbation of L_s yields

$$\text{vec}(\Delta L_s) = \left(O_s^T \otimes \mathcal{I}_{pr}\right)\text{vec}(\Delta L) + \left(\mathcal{I}_{r+n} \otimes L\right)\text{vec}(\Delta O_s) \ .$$

The expression for L in (12) can be written as

$$L = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \dots & \mathcal{L}_{p-1} & \mathcal{L}_p \\ \mathcal{L}_2 & \mathcal{L}_3 & \dots & \mathcal{L}_p & 0 \\ \mathcal{L}_3 & \mathcal{L}_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{L}_p & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{\text{ker}}^T S_{3,p} \\ U_{\text{ker}}^T S_{3,p-1} \\ \vdots \\ U_{\text{ker}}^T S_{3,1} \end{bmatrix} \ ,$$

where $S_{3,k} = \begin{bmatrix} 0_{(p-k)r,kr} & 0_{(p-k)r,(p-k)r} \\ \mathcal{I}_{kr} & 0_{kr,(p-k)r} \end{bmatrix}$. Thus, $L^T = \begin{bmatrix} S_{3,p}^T U_{\text{ker}} & S_{3,p-1}^T U_{\text{ker}} & \dots & S_{3,1}^T U_{\text{ker}} \end{bmatrix}$ and the first order perturbation of L yields

$$\begin{aligned} \text{vec}(\Delta L) &= P_{pr,p(pr-n)}\text{vec}(\Delta L^T) \\ &= \underbrace{P_{pr,p(pr-n)} \begin{bmatrix} \mathcal{I}_{pr-n} \otimes S_{3,p}^T \\ \mathcal{I}_{pr-n} \otimes S_{3,p-1}^T \\ \vdots \\ \mathcal{I}_{pr-n} \otimes S_{3,1}^T \end{bmatrix}}_{=S_4} \text{vec}(\Delta U_{\text{ker}}) \ . \end{aligned}$$

Next, $\text{vec}(\Delta O_s) = S_5\text{vec}(\Delta\Gamma)$, where $S_5 = \begin{bmatrix} 0_{pr^2,(p-1)rn} \\ (I_n \otimes S_6) \end{bmatrix}$ and $S_6 = \begin{bmatrix} 0_{r,(p-1)r} \\ \mathcal{I}_{(p-1)r} \end{bmatrix} \begin{bmatrix} \mathcal{I}_{(p-1)r} & 0_{(p-1)r,r} \end{bmatrix}$.

Therefore $\text{vec}\left(\Delta(L_s)^\dagger\right)$ writes

$$\text{vec}\left(\Delta(L_s)^\dagger\right) = \mathcal{J}_{\mathcal{R}}^{(L_s)^\dagger}\text{vec}(\Delta\mathcal{R}) = \mathcal{J}_{L_s}^{L_s^\dagger}\mathcal{J}_{\mathcal{H}^s}^{L_s}\mathcal{J}_{\mathcal{R}}^{\mathcal{H}^s}\text{vec}(\Delta\mathcal{R}) \ ,$$

where $\mathcal{J}_{\mathcal{H}^s}^{L_s} = \left(\mathcal{J}_L^{L_s} S_4 \mathcal{J}_{\mathcal{H}^s}^{U_{\text{ker}}^T} + \mathcal{J}_{O_s}^{L_s} S_5 \mathcal{J}_{\mathcal{H}^s}^{\Gamma}\right)$, $\mathcal{J}_L^{L_s} = (O_s^T \otimes \mathcal{I}_{pr})$ and $\mathcal{J}_{O_s}^{L_s} = (\mathcal{I}_{r+n} \otimes L)$, which finishes the proof. □

Finally, since $\mathcal{R}_1$ and $\mathcal{R}_2$ are the first two components of $\mathcal{R}$ (see (17)), the covariance of B and D follows as

$$\Sigma_{D,B} = \mathcal{J}_{\mathcal{R}}^{D,B}\Sigma_{\mathcal{R}}(\mathcal{J}_{\mathcal{R}}^{D,B})^T$$

where $\mathcal{J}_{\mathcal{R}}^{D,B} = \mathcal{J}_{\mathcal{R}}^{\mathcal{M}^{v,1}} + \mathcal{J}_{\mathcal{R}}^{(L_s)^\dagger} + \begin{bmatrix} \mathcal{J}_{\mathcal{R}_{1,2}}^{\mathcal{M}^{v,2}} & 0 \end{bmatrix}$ and 0 is a matrix of zeros of appropriate dimension.

4.3 Covariance computation of $G(s)$

Similar to the section above, the scheme to compute the covariance related to $G(s)$ is general for any input-output identification algorithm.

Corollary 3. A perturbation in $G^{\text{R}}(s)$ and $G^{\text{I}}(s)$ yields

$$\begin{bmatrix} \text{vec}(\Delta G^{\text{R}}(s)) \\ \text{vec}(\Delta G^{\text{I}}(s)) \end{bmatrix} = \begin{bmatrix} \mathcal{J}_A^{G^{\text{R}}} & \mathcal{J}_B^{G^{\text{R}}} & \mathcal{J}_C^{G^{\text{R}}} & \mathcal{I}_{ru} \\ \mathcal{J}_A^{G^{\text{I}}} & \mathcal{J}_B^{G^{\text{I}}} & \mathcal{J}_C^{G^{\text{I}}} & 0_{ru,ru} \end{bmatrix} \begin{bmatrix} \text{vec}(\Delta A) \\ \text{vec}(\Delta B) \\ \text{vec}(\Delta C) \\ \text{vec}(\Delta D) \end{bmatrix}$$

where $\mathcal{J}_A^{G^{\text{R}}} = ((ZB)^T \otimes CZ)^{\text{R}}$, $\mathcal{J}_B^{G^{\text{R}}} = \mathcal{I}_u \otimes (CZ_k^{\text{R}})$, $\mathcal{J}_C^{G^{\text{R}}} = (Z_k^{\text{R}}B)^T \otimes \mathcal{I}_r$, and analogously $\mathcal{J}_A^{G^{\text{I}}} = ((ZB)^T \otimes CZ)^{\text{I}}$, $\mathcal{J}_B^{G^{\text{I}}} = \mathcal{I}_u \otimes (CZ^{\text{I}})$ and $\mathcal{J}_C^{G^{\text{I}}} = (Z^{\text{I}}B)^T \otimes \mathcal{I}_r$.

The derivation of the aforementioned sensitivity matrices is straightforward and omitted for brevity.

In practice $G(s)$ is expressed in terms of its magnitude and phase (cf. Section 3.2) for each component $i = 1, \dots, ru$, denoted by $(\text{vec}(G(s)))_i$. The sensitivity of the respective quantities yields

$$\begin{bmatrix} \Delta m_{G(s)}^i \\ \Delta p_{G(s)}^i \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{m_G^i}^G \\ \mathcal{J}_{p_G^i}^G \end{bmatrix} \begin{bmatrix} (\text{vec}(\Delta G^{\text{R}}(s)))_i \\ (\text{vec}(\Delta G^{\text{I}}(s)))_i \end{bmatrix}$$

where the derivation of $\mathcal{J}_{m_G^i}^G$ and $\mathcal{J}_{p_G^i}^G$ is straightforward and omitted.

5. NUMERICAL VALIDATION

For the numerical validation consider a 6 degrees of freedom (DOF) mechanical system governed by the second order differential equation of motion, which can easily be transformed to state-space model (1)–(2) (Reynders et al., 2008). The system is modeled with spring stiffnesses $k_1 = k_3 = k_5 = 100$, $k_2 = k_4 = k_6 = 200$, mass $m_{1-6} = [1 \dots 6]/20$ and a proportional modal damping matrix, and is excited by a random white noise signal in all DOFs, sampled with a frequency of 50 Hz for 2000 seconds. Three known inputs at DOFs 1, 4 and 6 are considered and the responses at DOFs 1, 2 and 5 are measured. Gaussian white noise with 5% of the standard deviation of the output is added to the measured responses. The computations are performed in a Monte Carlo setup with $m = 1000$ realizations of the described signal.

5.1 Validation of B and D estimates and their covariances

Estimates of B are not unique since they are obtained in an arbitrary basis, thus not directly comparable with the exact B from the numerical model. Therefore, to validate estimates of B from (13) and their corresponding variances from (18), we consider estimates of the product CB and its perturbation-based variances ($\text{vec}\Delta(CB) = (B^T \otimes \mathcal{I}_r)\text{vec}(\Delta C) + (\mathcal{I}_u \otimes C)\text{vec}(\Delta B)$), which are independent of the state space basis.

The estimates of D and CB from the Monte Carlo simulation yield histograms characterizing their underlying distribution. The variance of this empirical distribution

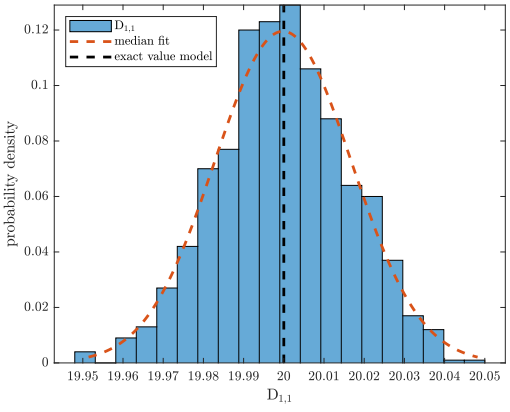

Fig. 1. Comparison of Monte Carlo estimates of $\text{D}_{1,1}$ to exact model value and Delta method distribution fits.

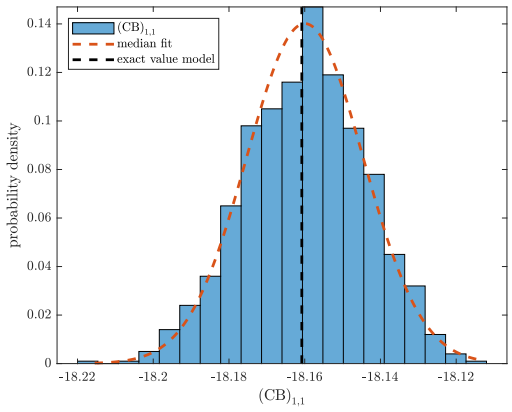

Fig. 2. Comparison of Monte Carlo estimates of $(\mathrm{CB})_{1,1}$ to exact model value and Delta method distribution fits.

Table 1. Comparison of the exact and mean values of D components.

Case	Component	1	2	3
Exact	1	20	0	0
Mean MC		19.9998	0.0001	-0.0003
Exact	2	0	0	0
Mean MC		-0.0003	0.0001	0.0002
Exact	3	0	0	0
Mean MC		0.0004	-0.0001	0.0001

Table 2. Comparison of the exact and mean values of CB components.

Case	Component	1	2	3
Exact	1	-19.0020	0.0110	0.0016
Mean MC		-19.0019	0.0109	0.0014
Exact	2	0.1841	1.2189	0.0056
Mean MC		0.1842	1.2190	0.0055
Exact	3	0.0025	0.3840	0.5317
Mean MC		0.0010	0.3837	0.5318

can be directly compared to variances estimated with Delta method in this paper. Moreover, an average quality of the estimated variances can be illustrated by a normal distribution fit characterized by the mean of the Monte Carlo estimates and the median of Delta method-based variances, which is shown for the first component of D and CB in Figures 1 and 2.

The exact and the mean Monte Carlo values of D and CB are summarized in Tables 1 and 2. The standard deviations obtained from the Monte Carlo simulation σ_{MC} of D and CB, and the mean values of standard deviations estimated with Delta method $\overline{\sigma}_{DM}$ are summarized in Tables 3 and 4. One can observe that the mean estimates of D and CB from Monte Carlo simulation are close to their respective values from the model. Moreover, the median Delta method-based distribution accurately characterizes the empirical distribution of D and CB, and the corresponding mean estimates of the standard deviations are close to the standard deviation of the Monte Carlo experiments. Each Monte Carlo realization yields a different variance estimate, however coefficients of variation of the

Table 3. Comparison of the standard deviations of D components.

Case	Component	1	2	3
σ_{MC}	1	0.0197	0.0103	0.0067
$\overline{\sigma}_{DM}$		0.0199	0.0099	0.0066
σ_{MC}	2	0.0119	0.0059	0.0041
$\overline{\sigma}_{DM}$		0.0118	0.0059	0.0039
σ_{MC}	3	0.0105	0.0055	0.0036
$\overline{\sigma}_{DM}$		0.0107	0.0053	0.0036

Table 4. Comparison of the standard deviations of CB components.

Case	Component	1	2	3
σ_{MC}	1	0.0195	0.0095	0.0065
$\overline{\sigma}_{DM}$		0.0188	0.0094	0.0063
σ_{MC}	2	0.0120	0.0061	0.0042
$\overline{\sigma}_{DM}$		0.0127	0.0064	0.0042
σ_{MC}	3	0.0118	0.0061	0.0040
$\overline{\sigma}_{DM}$		0.0120	0.0060	0.0040

estimated standard deviation are less than 5% for each of the components of D and CB, which is a small value. This validates the algorithms proposed in (13) and (18).

5.2 Validation of $G(s)$ variances

Let $G(s_k)$ be the discrete transfer function evaluated at $s_k = \mathrm{e}^{\mathrm{i}\omega_k T_s}$, where $0 \leq \omega_k \leq \omega_f$ and ω_f is the Nyquist

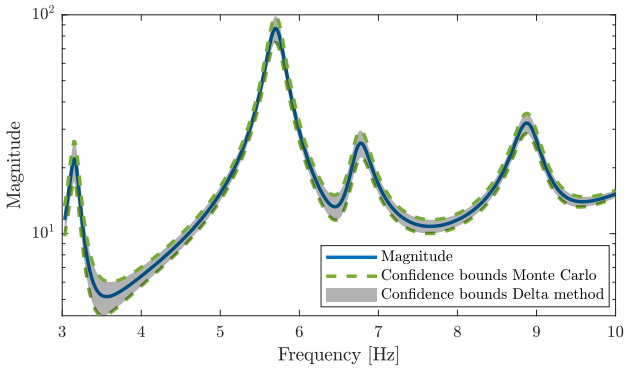

Fig. 3. $m^1_{G(s_k)}$ with zoomed Monte Carlo and Delta method-based confidence intervals.

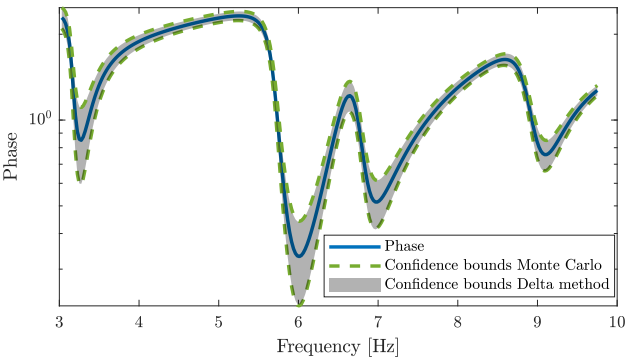

Fig. 4. $p^1_{G(s_k)}$ with zoomed Monte Carlo and Delta method-based confidence intervals.

frequency. For each s_k, the magnitude and phase of the transfer function is plotted with both of its Monte Carlo and Delta based standard deviations, where the latter is zoomed 20 times for illustration purposes, in Figures 3 and 4. The mean of standard deviations computed with the Delta method accurately matches the standard deviation of the Monte Carlo experiment for both the magnitude and the phase of the first component of $G(s_k)$. That validates the proposed uncertainty quantification framework.

6. CONCLUSIONS

In this paper we have proposed a simple input/output matrix relation for the estimation of (B, D) in the context of input/output subspace-based system identification. Based on this relation, an expression to estimate the covariance of (B, D) has been developed, which can be easily evaluated from measurement data. With these results, the covariance of the transfer function estimate has also been derived. The proposed expressions have been validated in Monte Carlo simulations of a mechanical system.

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