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Frequency-Domain Computation of Quadratic-Exponential Cost Functionals for Linear Quantum Stochastic Systems ⋆

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Abstract: This paper is concerned with quadratic-exponential functionals (QEFs) as risk-sensitive performance criteria for linear quantum stochastic systems driven by multichannel bosonic fields. Such costs impose an exponential penalty on quadratic functions of the quantum system variables over a bounded time interval, and their minimization secures a number of robustness properties for the system. We use an integral operator representation of the QEF, obtained recently, in order to compute its infinite-horizon asymptotic growth rate in the invariant Gaussian state when the stable system is driven by vacuum input fields. The resulting frequency-domain formula expresses the QEF growth rate in terms of two spectral functions associated with the real and imaginary parts of the quantum covariance kernel of the system variables. We also discuss the computation of the QEF growth rate using homotopy and contour integration techniques and provide an illustrative numerical example with a two-mode open quantum harmonic oscillator.

Keywords: Linear quantum stochastic systems, quadratic-exponential functionals, frequency-domain representation.

1. INTRODUCTION

Quantum-mechanical adaptation of quadratic-exponential cost functionals, originating from classical risk-sensitive control (BV1985; J1973; W1981), provides a relevant addition to the mean square optimality criteria for linear quantum stochastic systems. Such systems, governed by linear quantum stochastic differential equations (QSDEs) in the framework of the Hudson-Parthasarathy calculus (HP1984; P1992; P2015), are the main subject of linear quantum systems theory (NY2017; P2017) which is concerned with tractable models of open quantum dynamics. In particular, quadratic cost functionals and their minimization provide a natural way to quantify and improve the performance of observers in filtering problems in terms of the mean square discrepancy between the system variables and their estimates (MJ2012).

The quadratic-exponential functional (QEF) (VPJ2018b) (see also (B1996)), which, similarly to its classical predecessors, is organized as the averaged exponential of an integral of a quadratic form of the system variables over a bounded time interval, pertains to important higher-order properties of the quantum system. One of them is related to the worst-case values of mean square costs (VPJ2018b) in the presence of quantum statistical uncertainty, when the actual system-field state differs from its nominal model, but not "too much" in the sense of the quantum relative entropy (OW2010). Another property is concerned with the tail distributions for the quantum system trajectories (VPJ2018a), which corresponds to the classical Cramer type large deviations bounds. These properties involve the QEF in such a way that its minimization makes the behaviour of the open quantum system more robust and conservative. The resulting performance analysis and optimal control problems require methods for computing and minimizing the QEF, which is different from its time-ordered exponential counterpart in the original quantum risk-sensitive control formulation (J2004; J2005).

The development of methods for computing the QEF has been a subject of several recent publications which have developed Lie-algebraic techniques (VPJ2019a), parametric randomization (VPJ2019e) and quantum Karhunen-Loeve expansions (VPJ2019b; VJP2019) for this purpose. These results have led to an integral operator representation of the QEF (VPJ2019c) for open quantum harmonic oscillators (OQHOs) in Gaussian quantum states (P2010). In addition to its relevance to quantum risk-sensitive control, the approach, which has been used in obtaining this representation, has deep connections with operator exponential structures studied in mathematical physics and quantum probability (for example, in the context of operator algebras (AB2018), moment-generating functions for quadratic Hamiltonians (PS2015) and the quantum Lévy area (CH2013; H2018)).

The present paper employs the integral operator representation of the QEF, mentioned above, and establishes an infinite-horizon asymptotic growth rate of the QEF for invariant Gaussian states of stable OQHOs driven by vacuum input fields. We represent the QEF growth rate in frequency domain through the Fourier transforms of the real and imaginary parts of the invariant quantum covariance kernel of the system variables. One of these matrix-valued spectral functions, coming from the two-point commutator kernel, enters the frequency-domain

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formula in composition with trigonometric functions (H2008). This affects the (otherwise meromorphic) structure of the function (whose logarithm is present in the integrand) in comparison with its classical counterpart in the $\mathscr{H}_\infty$-entropy integral (AK1981; MG1990). We take this issue into account when considering a contour integration technique for evaluating the QEF growth rate. For general multimode OQHOs, we obtain a differential equation for the QEF growth rate (as a function of the risk sensitivity parameter), which leads to a numerical algorithm for its computation, similar to the homotopy method (MB1985).

The paper is organized as follows. Section 2 specifies the class of linear quantum stochastic systems under consideration. Section 3 describes the QEF as a finite-horizon system performance criterion and revisits its integral operator representation in the Gaussian case. Section 4 obtains a frequency-domain formula for the infinite-horizon asymptotic growth rate of the QEF in terms of the system transfer function. Section 5 discusses the computation of the QEF growth rate using homotopy and contour integration techniques. Section 6 provides a numerical example of computing the QEF growth rate for a two-mode OQHO. Section 7 makes concluding remarks and outlines further directions of research. Proofs are given in the full version of this paper in (VPJ2019d).

2. OPEN QUANTUM HARMONIC OSCILLATORS

Let $W_1(t), \ldots, W_m(t)$ be an even number of time-varying self-adjoint operators on a subspace $\mathfrak{F}_t$ of a symmetric Fock space $\mathfrak{F}$ (P1992), which form a multichannel quantum Wiener process $W := (W_k)_{1\leqslant k\leqslant m}$ and represent bosonic fields (we will often omit the time argument t for brevity). The increasing family $(\mathfrak{F}_t)_{t\geqslant 0}$ of these subspaces provides a filtration for the Fock space $\mathfrak{F}$ in accordance with its continuous tensor-product structure (PS1972). The quantum Wiener process W satisfies the two-point canonical commutation relations (CCRs)

$$[W(s), W(t)^{\mathrm{T}}] := ([W_j(s), W_k(t)])_{1\leqslant j,k\leqslant m} = 2i\min(s,t)J \quad (1)$$

for all $s,t \geqslant 0$, where $(\cdot)^{\mathrm{T}}$ is the transpose (vectors are organized as columns unless indicated otherwise), $[\alpha, \beta] := \alpha\beta - \beta\alpha$ is the commutator of linear operators, and $i := \sqrt{-1}$ is the imaginary unit. In (1), use is also made of an orthogonal real antisymmetric matrix

$$J := \mathbf{J} \otimes I_{m/2} \quad (2)$$

(so that $J^2 = -I_m$), where $\otimes$ is the Kronecker product, I_r is the identity matrix of order r, and $\mathbf{J} := \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$ spans the one-dimensional subspace of antisymmetric matrices of order 2. In addition to its relation to the second Pauli matrix $-i\mathbf{J}$ (S1994), this matrix specifies the CCRs $[\vartheta, \vartheta^{\mathrm{T}}] = i\mathbf{J}$ for the vector $\vartheta := \begin{bmatrix} q\\ p\end{bmatrix}$ of the quantum mechanical position and momentum operators q and $p := -i\partial_q$ on the Schwartz space (V2002). More complicated CCRs between quantum variables are obtained by using linear combinations of the conjugate position-momentum pairs as building blocks. Such combinations are present in a multimode OQHO, which interacts with external bosonic fields (modelled by the quantum Wiener process W) and is endowed with an even number of time-varying self-adjoint quantum variables $X_1(t), \ldots, X_n(t)$ on the subspace $\mathfrak{H}_t := \mathfrak{H}_0 \otimes \mathfrak{F}_t$ of the system-field tensor-product space

$$\mathfrak{H} := \mathfrak{H}_0 \otimes \mathfrak{F}. \quad (3)$$

Accordingly, $\mathfrak{H}_0$ is a complex separable Hilbert space for the action of the initial system variables $X_1(0), \ldots, X_n(0)$. At every moment of time, the vector $X := (X_k)_{1\leqslant k\leqslant n}$ of system variables of the OQHO satisfies the CCRs

$$[X, X^{\mathrm{T}}] = 2i\Theta \quad (4)$$

as the Heisenberg infinitesimal form of the Weyl CCRs (F1989), specified by a constant real antisymmetric matrix Θ of order n, which is assumed to be nonsingular for what follows. The evolution of the system variables is governed by a linear QSDE

$$\mathrm{d}X = AX\mathrm{d}t + B\mathrm{d}W, \quad (5)$$

driven by the quantum Wiener process W. In accordance with the structure of the system-field interaction model in the quantum stochastic calculus (HP1984; P1992; P2015), the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are parameterized as

$$A = 2\Theta(R + M^{\mathrm{T}}JM), \qquad B = 2\Theta M^{\mathrm{T}} \quad (6)$$

by the energy and coupling matrices $R = R^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{m\times n}$ which specify the system Hamiltonian $\frac{1}{2}X^{\mathrm{T}}RX$ and the vector MX of m system-field coupling operators, with the matrix J given by (2). Due to the parameterization (6), the matrices A, B satisfy the physical realizability (PR) condition (JNP2008)

$$A\Theta + \Theta A^{\mathrm{T}} + BJB^{\mathrm{T}} = 0, \quad (7)$$

which is equivalent to the conservation of the CCR matrix Θ in (4) in time. For what follows, the OQHO is assumed to be stable in the sense that A is Hurwitz. In this case, Θ is a unique solution of (7) as an algebraic Lyapunov equation (ALE) and is given by $\Theta = \int_0^{+\infty} \mathrm{e}^{tA}BJB^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\mathrm{d}t$.

3. QUADRATIC-EXPONENTIAL COST FUNCTIONAL

Feedback connections of linear quantum stochastic systems, arising in quantum control and filtering settings (NJP2009; MJ2012; ZJ2012), are also organized as OQHOs. For a given but otherwise arbitrary time horizon $T > 0$, the performance of such a system over the time interval $[0, T]$ can be described in the risk-sensitive framework in terms of the QEF (VPJ2018a)

$$\Xi_{\theta,T} := \mathbf{E}\mathrm{e}^{\frac{\theta}{2}Q_T} \quad (8)$$

as a cost functional to be minimized. Here, $\mathbf{E}\zeta := \mathrm{Tr}(\rho\zeta)$ is the quantum expectation over an underlying density operator ρ on the system-field space $\mathfrak{H}$ in (3). The risk sensitivity parameter $\theta > 0$ in (8) specifies the severity of exponential penalty imposed on the positive semi-definite self-adjoint quantum variable

$$Q_T := \int_0^T X(t)^{\mathrm{T}}\Pi X(t)\mathrm{d}t = \int_0^T Z(t)^{\mathrm{T}}Z(t)\mathrm{d}t, \quad (9)$$

which depends quadratically on the system variables in (5) over the time interval $[0, T]$. This dependence is parameterized by a real positive definite symmetric matrix Π of order n which relates an auxiliary quantum process Z to the system variables by

$$Z := SX, \qquad S := \sqrt{\Pi}. \quad (10)$$

In fact, Z consists of n system variables of an OQHO with appropriately transformed matrices $S\Theta S$, $S^{-1}RS^{-1}$, MS^{-1}, SAS^{-1}, SB in (4)–(6) in view of the symmetry $S = S^{\mathrm{T}}$. This transformation preserves the nonsingularity of the CCR matrix Θ and the Hurwitz property of the dynamics matrix A. The process Z satisfies the two-point CCRs (VPJ2018a)

$$[Z(s), Z(t)^{\mathrm{T}}] = 2i\Lambda(s-t), \qquad s,t \geqslant 0, \quad (11)$$

with

$$\Lambda(\tau) := \begin{cases} S\mathrm{e}^{\tau A}\Theta S & \text{if } \tau \geqslant 0\\ S\Theta\mathrm{e}^{-\tau A^{\mathrm{T}}}S & \text{if } \tau < 0\end{cases} = -\Lambda(-\tau)^{\mathrm{T}}, \quad (12)$$

from which the one-point CCR matrix of Z is recovered as $S\Theta S = \Lambda(0)$. The two-point CCR kernel (12) specifies a skew self-adjoint integral operator $\mathscr{L}_T : f \mapsto g$ on the Hilbert space $L^2([0,T],\mathbb{C}^n)$ of square integrable $\mathbb{C}^n$-valued functions on $[0,T]$:

$$g(s) := \int_0^T \Lambda(s-t)f(t)\mathrm{d}t, \qquad 0 \leqslant s \leqslant T. \tag{13}$$

The commutation structure of the process Z in (11), (12) (and the related operator (13)) do not depend on a particular system-field state ρ.

In what follows, it is assumed that the stable OQHO under consideration is driven by vacuum fields. In this case, the system variables have a unique invariant multipoint zero-mean Gaussian quantum state (VPJ2018a). This property is inherited by the process Z in (10). The corresponding two-point quantum covariance function

$$\mathbf{E}(Z(s)Z(t)^{\mathrm{T}}) = P(s-t) + i\Lambda(s-t), \qquad s,t \geqslant 0, \tag{14}$$

has the real part

$$P(\tau) = \begin{cases} Se^{\tau A}\Sigma S & \text{if } \tau \geqslant 0 \\ S\Sigma e^{-\tau A^{\mathrm{T}}}S & \text{if } \tau < 0 \end{cases} = P(-\tau)^{\mathrm{T}}, \qquad \tau \in \mathbb{R}. \tag{15}$$

The positive semi-definite symmetric matrix $\Sigma := \mathrm{Re}\mathbf{E}(XX^{\mathrm{T}})$ of order n describes the invariant one-point statistical correlations of the system variables and satisfies the ALE $A\Sigma + \Sigma A^{\mathrm{T}} + BB^{\mathrm{T}} = 0$. The kernel (15) specifies a positive semi-definite self-adjoint integral operator $\mathscr{P}_T : f \mapsto g$ on $L^2([0,T],\mathbb{C}^n)$ as

$$g(s) := \int_0^T P(s-t)f(t)\mathrm{d}t, \qquad 0 \leqslant s \leqslant T. \tag{16}$$

Moreover, the self-adjoint operator $\mathscr{P}_T + i\mathscr{L}_T$ on $L^2([0,T],\mathbb{C}^n)$ is positive semi-definite. Also, both $\mathscr{P}_T$ and $\mathscr{L}_T$ are compact operators (RS1980). By applying the results of (VPJ2019c) to the OQHO in the invariant multipoint Gaussian quantum state, the QEF (8) is represented as

$$\ln \Xi_{\theta,T} = -\tfrac{1}{2}\mathrm{Tr}(\ln\cos(\theta\mathscr{L}_T) + \ln(\mathscr{I} - \theta\mathscr{P}_T\mathscr{K}_{\theta,T})). \tag{17}$$

Here,

$$\mathscr{K}_{\theta,T} := \mathrm{tanhc}(i\theta\mathscr{L}_T) = \mathrm{tanc}(\theta\mathscr{L}_T) \tag{18}$$

is a positive definite self-adjoint operator on $L^2([0,T],\mathbb{C}^n)$, where $\mathrm{tanhc}z := \mathrm{tanc}(-iz)$ is a hyperbolic version of the function $\mathrm{tanc}z := \frac{\tan z}{z}$, with $\mathrm{tanc}0 := 1$ by continuity. The operator $\mathscr{K}_{\theta,T}$ is nonexpanding: $\mathscr{K}_{\theta,T} \preccurlyeq \mathscr{I}$, with $\mathscr{I}$ the identity operator on $L^2([0,T],\mathbb{C}^n)$. With $\mathscr{P}_T\mathscr{K}_{\theta,T}$ being compact (and isospectral to the positive semi-definite self-adjoint operator $\sqrt{\mathscr{K}_{\theta,T}}\mathscr{P}_T\sqrt{\mathscr{K}_{\theta,T}}$), the representation (17) is valid under the condition

$$\theta\lambda_{\max}(\mathscr{P}_T\mathscr{K}_{\theta,T}) < 1, \tag{19}$$

where $\lambda_{\max}(\cdot)$ is the largest eigenvalue. The representation (17) is obtained by applying the results of (VPJ2019c) to the process Z in (9), (10) using its quantum Karhunen-Loeve expansion over an orthonormal eigenbasis of the operator $\mathscr{L}_T$ in (13), provided it has no zero eigenvalues. The latter property is inherited by Z from the system variables under the sufficient condition (VPJ2019c, Theorem 1)

$$\det(BJB^{\mathrm{T}}) \neq 0, \tag{20}$$

with J, B given by (2), (6). Indeed, the corresponding condition $\det(SBJB^{\mathrm{T}}S) \neq 0$ for the process Z is equivalent to (20) since the matrix S in (10) is nonsingular.

4. QEF GROWTH RATE IN THE FREQUENCY DOMAIN

The representation (17) has a *trace-analytic* structure (VP2010) in the sense that $\ln\Xi_{\theta,T} = -\frac{1}{2}\mathrm{Tr}(\varphi(\theta\mathscr{P}_T\mathscr{K}_{\theta,T}) + \psi(\theta\mathscr{L}_T))$, where $\varphi(z) := \ln(1-z)$ and $\psi(z) := \ln\cos z$ are holomorphic functions of $z \in \mathbb{C}$ whose domains contain the spectra of the operators $\theta\mathscr{P}_T\mathscr{K}_{\theta,T}$ (under the condition (19)) and $\theta\mathscr{L}_T$ given by (13), (16), (18). This structure plays an important role in the following theorem on the asymptotic behaviour of (17), as $T \to +\infty$, which uses the Fourier transforms

$$\Phi(\lambda) := \int_{\mathbb{R}} e^{-i\lambda t}P(t)\mathrm{d}t = F(i\lambda)F(i\lambda)^*, \tag{21}$$

$$\Psi(\lambda) := \int_{\mathbb{R}} e^{-i\lambda t}\Lambda(t)\mathrm{d}t = F(i\lambda)JF(i\lambda)^*, \qquad \lambda \in \mathbb{R}, \tag{22}$$

of the covariance and commutator kernels (15), (12), see also (VPJ2019a, Eq. (5.8)). Here, $(\cdot)^* := \overline{(\cdot)}^{\mathrm{T}}$ is the complex conjugate transpose, and

$$F(v) := S(vI_n - A)^{-1}B, \qquad v \in \mathbb{C}, \tag{23}$$

is the transfer function from the incremented input quantum Wiener process W of the OQHO (5) to the process Z in (10). Note that $\Phi(\lambda)$ is a complex positive semi-definite Hermitian matrix, while $\Psi(\lambda)$ is skew Hermitian for any $\lambda \in \mathbb{R}$, with $\Phi + i\Psi$ being the Fourier transform of the quantum covariance kernel $P + i\Lambda$ from (14).

Theorem 1. Suppose the OQHO (5) is driven by vacuum input fields, the matrix A in (6) is Hurwitz, and the matrix B satisfies (20). Also, let the risk sensitivity parameter $\theta > 0$ in (8) satisfy

$$\theta \sup_{\lambda\in\mathbb{R}} \lambda_{\max}(\Phi(\lambda)\mathrm{tanc}(\theta\Psi(\lambda))) < 1, \tag{24}$$

where the functions Φ, Ψ are given by (21)–(23). Then the QEF $\Xi_{\theta,T}$, defined by (8)–(10), has the following infinite-horizon growth rate:

$$\Upsilon(\theta) := \lim_{T\to+\infty}\left(\tfrac{1}{T}\ln\Xi_{\theta,T}\right) = -\tfrac{1}{4\pi}\int_{\mathbb{R}} \ln\det D_\theta(\lambda)\mathrm{d}\lambda, \tag{25}$$

where

$$D_\theta(\lambda) := \cos(\theta\Psi(\lambda)) - \theta\Phi(\lambda)\mathrm{sinc}(\theta\Psi(\lambda)), \tag{26}$$

and $\mathrm{sinc}z := \frac{\sin z}{z}$ (with $\mathrm{sinc}0 := 1$ by continuity). ■

Under the condition (24), $-\ln\det D_\theta(\lambda)$ is a symmetric function of the frequency λ with nonnegative values. From (22), (23), it follows that the Hurwitz property of the matrix A, the nonsingularity of the matrix S in (10) and the condition (20) imply that $\det\Psi(\lambda) = \frac{\det\Pi\det(BJB^{\mathrm{T}})}{|\det(i\lambda - A)|^2} \neq 0$ for all $\lambda \in \mathbb{R}$, which makes the extension $\mathrm{sinc}0 = 1$ irrelevant for the evaluation of $\mathrm{sinc}(\theta\Psi(\lambda))$. However, this extension (and also $\mathrm{tanc}0 = 1$) plays its role in the limiting classical case, when (5) is an SDE driven by a standard Wiener process W (formally with $J = 0$ in (1)), and Z in (10) is a stationary Gaussian diffusion process (GS2004) with zero mean and the spectral density Φ in (21). In this case, the function Ψ vanishes, the condition (24) takes the form

$$\theta < \theta_0 := \frac{1}{\sup_{\lambda\in\mathbb{R}}\lambda_{\max}(\Phi(\lambda))} = \frac{1}{\|F\|_\infty^2} \tag{27}$$

in terms of the $\mathscr{H}_\infty$-norm of the transfer function (23), and the right-hand side of (25) reduces to the $\mathscr{H}_\infty$-entropy integral (AK1981; MG1990)

$$V(\theta) := -\tfrac{1}{4\pi}\int_{\mathbb{R}} \ln\det(I_n - \theta\Phi(\lambda))\mathrm{d}\lambda. \tag{28}$$

In contrast to its classical counterpart, the QEF growth rate (25) in the quantum case depends on both functions Φ, Ψ which constitute the "quantum spectral density" $\Phi + i\Psi$ of the process Z in (10). Furthermore, the condition (24) is substantially nonlinear with respect to θ and, unlike (27), does not admit a closed-form representation. However, since `tanc` on the imaginary axis (that is, `tanhc` on the real axis) takes values in the interval $(0,1]$, then $\lambda_{\max}(\Phi(\lambda)\mathrm{tanc}(\theta\Psi(\lambda))) = \lambda_{\max}(\sqrt{\mathrm{tanc}(\theta\Psi(\lambda))}\Phi(\lambda)\sqrt{\mathrm{tanc}(\theta\Psi(\lambda))}) \leqslant \lambda_{\max}(\Phi(\lambda))$ for any $\lambda \in \mathbb{R}$, whereby the fulfillment of the classical constraint (27) implies (24).

As a function of θ, the QEF growth rate (25) plays an important role in quantifying the large deviations of quantum trajectories (VPJ2018a) and for robustness of the OQHO with respect to state uncertainties described in terms of quantum relative entropy (see (VPJ2018b, Section IV) and references therein). More precisely,

$$\limsup_{T\to+\infty} \big(\tfrac{1}{T}\ln \mathbf{P}_T([2\alpha T,+\infty))\big) \leqslant \inf_{\theta\geqslant 0}(\Upsilon(\theta)-\alpha\theta) \qquad (29)$$

for any $\alpha > 0$, where $\mathbf{P}_T$ is the probability distribution of the self-adjoint quantum variable Q_T in (9) (H2001). Therefore, (29) provides asymptotic upper bounds for the tail probability distribution of Q_T in terms of the QEF growth rate (25). Furthermore,

$$\limsup_{T\to+\infty} \Big(\frac{1}{T}\sup_{\sigma\in\mathfrak{S}_{\varepsilon,T}} \mathbf{E}_\sigma Q_T\Big) \leqslant 2\inf_{\theta>0}\big(\tfrac{1}{\theta}(\varepsilon+\Upsilon(\theta))\big), \qquad (30)$$

where $\mathbf{E}_\sigma Q_T := \mathrm{Tr}(\sigma Q_T)$ is the expectation of the $\mathfrak{H}_T$-adapted quantum variable Q_T in (9) over a density operator σ on the system-field subspace $\mathfrak{H}_T$. Here, the supremum is taken over the set

$$\mathfrak{S}_{\varepsilon,T} := \{\sigma:\ \mathbf{D}(\sigma\|\rho_T)\leqslant \varepsilon T\}, \qquad (31)$$

where the parameter $\varepsilon \geqslant 0$ limits the growth rate of the quantum relative entropy (OW2010)

$$\mathbf{D}(\sigma\|\rho_T) := -\mathbf{H}(\sigma) - \mathbf{E}_\sigma \ln \rho_T \qquad (32)$$

of σ with respect to $\rho_T := \mathfrak{P}_T\rho\mathfrak{P}_T$, with $\mathfrak{P}_T$ the orthogonal projection onto $\mathfrak{H}_T$, and $\mathbf{H}(\sigma) := -\mathbf{E}_\sigma \ln\sigma$ is the von Neumann entropy of σ; cf. (YB2009, Eq. (7)). The density operator σ is interpreted as the actual quantum state, about which it is only known that it belongs to the class (31) of states being not "too far" from the reference state ρ_T as a nominal model. In the framework of this quantum statistical uncertainty description, specified by ε in terms of (32), the left-hand side of (30) is the worst-case quadratic cost growth rate, similar to the robust performance criteria of minimax LQG control (DJP2000; P2006; PJD2000). Therefore, for a suitably chosen $\theta > 0$, the minimization of $\Upsilon(\theta)$ over an admissible range of parameters of the OQHO in the context of risk-sensitive control and filtering problems enhances the large deviations and robust performance bounds (29), (30). The computation of these bounds and the QEF minimization demand techniques for evaluating the functional (25).

5. EVALUATION OF THE QEF GROWTH RATE

The following technique for computing the QEF growth rate (25) resembles the homotopy method for numerical solution of parameter dependent algebraic equations (MB1985) and exploits the specific dependence of $\Upsilon(\theta)$ on the risk sensitivity parameter θ. To this end, with the function D_θ in (26), we associate a function $U_\theta : \mathbb{R} \to \mathbb{C}^{n\times n}$ by

$$U_\theta(\lambda) := -D_\theta(\lambda)^{-1}\partial_\theta D_\theta(\lambda) \qquad (33)$$

for all $\theta > 0$ satisfying (24) (which ensures that $\det D_\theta(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$).

Theorem 2. Under the conditions of Theorem 1, the QEF growth rate $\Upsilon(\theta)$ in (25) satisfies the differential equation

$$\Upsilon'(\theta) = \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{Tr}U_\theta(\lambda)\mathrm{d}\lambda, \qquad (34)$$

with the initial condition $\Upsilon(0) = 0$. Here, the function (33) takes values in the subspace of Hermitian matrices and satisfies a Riccati equation

$$\partial_\theta U_\theta(\lambda) = \Psi(\lambda)^2 + U_\theta(\lambda)^2, \qquad \lambda\in\mathbb{R}, \qquad (35)$$

with the initial condition $U_0 = \Phi$ given by (21). ∎

The proof of Theorem 2 uses, as one of its intermediate steps, a linear second-order ODE

$$D_\theta'' = -\Psi^2\cos(\theta\Psi) + \Phi\Psi\sin(\theta\Psi) = -D_\theta\Psi^2 \qquad (36)$$

for (26) with the initial conditions $D_0 = I_n$, $D_0' = -\Phi$, where $(\cdot)' := \partial_\theta(\cdot)$. The relation (33), which links the quadratically nonlinear ODE (35) with the linear ODE (36), can be regarded as a matrix-valued analogue of the Hopf-Cole transformation (C1951; H1950) converting the viscous Burgers equation to the heat (or diffusion) equation. We also mention an analogy between (33) and the logarithmic transformation in the context of dynamic programming equations for stochastic control (F1982) (see also (VP2010)).

The right-hand side of (34) can be evaluated by numerical integration over the frequency axis and used for computing (25) as

$$\Upsilon(\theta) = \int_0^\theta \Upsilon'(v)\mathrm{d}v = \frac{1}{4\pi}\int_{\mathbb{R}\times[0,\theta]} \mathrm{Tr}U_v(\lambda)\mathrm{d}\lambda\mathrm{d}v. \qquad (37)$$

In particular, (34) yields

$$\Upsilon'(0) = \frac{1}{4\pi}\int_{\mathbb{R}}\mathrm{Tr}\Phi(\lambda)\mathrm{d}\lambda = \tfrac{1}{2}\|F\|_2^2 = \tfrac{1}{2}\mathbf{E}(Z(0)^{\mathrm{T}}Z(0)), \qquad (38)$$

which, in accordance with (8), reproduces the LQG cost for the process Z in (10) for the stable OQHO in the invariant Gaussian state. In (38), we have also used the $\mathscr{H}_2$-norm of the transfer function (23) which factorizes the spectral density (21). In addition to its role for the computation of Υ, the function Υ' admits the following representation (see also (VPJ2018a, Theorem 1)):

$$\Upsilon'(\theta) = \tfrac{1}{2}\lim_{T\to+\infty}\big(\tfrac{1}{T}\mathbf{E}_{\theta,T}Q_T\big), \qquad (39)$$

where $\mathbf{E}_{\theta,T}\zeta := \mathrm{Tr}(\rho_{\theta,T}\zeta)$ is the quantum expectation over a modified density operator $\rho_{\theta,T} := \frac{1}{\Xi_{\theta,T}}\mathrm{e}^{\frac{\theta}{4}Q_T}\rho\,\mathrm{e}^{\frac{\theta}{4}Q_T}$. Therefore, (39) relates Υ' to the asymptotic growth rate of the weighted average of the quantum variable Q_T in (9) rather than its exponential moment.

Another approach to evaluating the QEF growth rate (25) is provided by contour integration. More precisely, consider the $\mathbb{C}^{n\times n}$-valued function

$$E_\theta(s) := \cos(\theta\mho(s)) - \theta\Gamma(s)\mathrm{sinc}(\theta\mho(s)), \qquad (40)$$

which is defined in terms of the rational (and hence, meromorphic) functions

$$\Gamma(s) := F(s)F(-s)^{\mathrm{T}}, \qquad (41)$$

$$\mho(s) := F(s)JF(-s)^{\mathrm{T}}, \qquad s\in\mathbb{C}, \qquad (42)$$

associated with the transfer function (23). Since (40)–(42) are related to (26), (21), (22) as $D_\theta(\lambda) = E_\theta(i\lambda)$, $\Phi(\lambda) = \Gamma(i\lambda)$, $\Psi(\lambda) = \mho(i\lambda)$ for all $\lambda\in\mathbb{R}$, then (25) admits the representation

$$\Upsilon(\theta) = -\frac{1}{4\pi i}\int_{i\mathbb{R}}\ln\det E_\theta(s)\mathrm{d}s$$
$$= \frac{1}{4\pi i}\lim_{r\to+\infty}\oint_{C_r}\ln\det E_\theta(s)\mathrm{d}s, \qquad (43)$$

where the last integral is over the counterclockwise oriented contour in Fig. 1. Here, use is made of the asymptotic behaviour

$$E_\theta(s) = I_n + \frac{\theta}{s^2}SBB^{\mathrm{T}}S + o(s^{-2}) \qquad (44)$$

of the function (40), as $s\to\infty$, due to the transfer function F in (23) being strictly proper. Therefore, the contribution from the semicircular part $C_r\bigcap\mathbb{C}_+ = \{s\in\mathbb{C}_+:\ |s| = r\}$ of the contour C_r in (43) indeed vanishes asymptotically: $\int_{C_r\bigcap\mathbb{C}_+}\ln\det E_\theta(s)\mathrm{d}s \sim 2i\theta\mathrm{Tr}(\Pi BB^{\mathrm{T}})\frac{1}{r}$, as $r\to+\infty$, where $\mathbb{C}_+ := \{s\in\mathbb{C}:\ \mathrm{Re}s > 0\}$

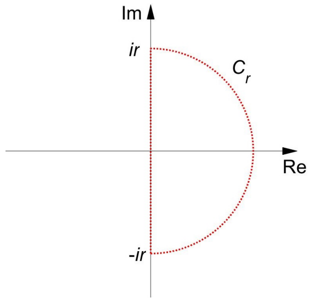

Fig. 1. The counterclockwise oriented contour C_r in (43) consisting of an arc of radius r (centered at the origin) and a line segment of the imaginary axis with the endpoints $\pm ir$.

is the open right half-plane, and the relation $\det(I_n + N) = 1 + \mathrm{Tr} N + o(N)$, as $N \to 0$, is used together with the identity $\mathrm{Tr}(SBB^{\mathrm{T}}S) = \mathrm{Tr}(S^2BB^{\mathrm{T}}) = \mathrm{Tr}(\Pi BB^{\mathrm{T}})$ which follows from the structure of the matrix S in (10). However, application of the residue theorem (S1992) to (43) is complicated by the nature of singularities of the function $\det E_\theta$ (considered in $\mathbb{C}_+$). Note that the corresponding function $\det(I_n - \theta\Phi)$ in the classical counterpart (28) is rational, thus simplifying the evaluation of the integral. This observation can be combined with the Maclaurin series expansions of the trigonometric functions, which allows (26) to be approximated as

$$\begin{aligned} D_\theta &= I_n - \tfrac{1}{2}\theta^2\Psi^2 - \theta\Phi(I_n - \tfrac{1}{6}\theta^2\Psi^2) + o(\theta^3) \\ &= I_n - \theta\Phi - \tfrac{1}{2}\theta^2(I_n - \tfrac{\theta}{3}\Phi)\Psi^2 + o(\theta^3) \end{aligned} \quad (45)$$

as $\theta \to 0$. Substitution of (45) into (25) leads to the approximate computation of the QEF growth rate as a perturbation of its classical counterpart (28):

$$\begin{aligned} \Upsilon(\theta) =& V(\theta) \\ &+ \frac{\theta^2}{8\pi}\int_{\mathbb{R}} \mathrm{Tr}((I_n - \theta\Phi(\lambda))^{-1}(I_n - \tfrac{\theta}{3}\Phi(\lambda))\Psi(\lambda)^2)\mathrm{d}\lambda \\ &+ o(\theta^3), \qquad \text{as } \theta \to 0. \end{aligned} \quad (46)$$

Since the integrand in (46) is a rational function of the frequency λ, whose continuation to the closed right half-plane $(i\mathbb{R})\bigcup\mathbb{C}_+$ has no poles on the imaginary axis under the condition (27), the correction term is amenable to calculation via its residues in $\mathbb{C}_+$. In view of $\Psi(\lambda)^2 \prec 0$ for all $\lambda \in \mathbb{R}$, the relation (46) also implies that $\Upsilon(\theta) < V(\theta)$ for all sufficiently small $\theta > 0$.

6. NUMERICAL EXAMPLE WITH A TWO-MODE OQHO

Consider a two-mode OQHO, whose $n = 4$ system variables consist of two position-momentum pairs, which have the CCR matrix $\Theta := \frac{1}{2}\mathbf{J} \otimes I_2$ and are driven by $m = 6$ quantum Wiener processes. The state-space matrices A, B in (6) and the weighting matrix Π in (9) are given by

$$A := \begin{bmatrix} -5.8100 & -1.6357 & 0.2062 & -3.1331 \\ 4.0006 & 0.1377 & 5.3578 & -0.5514 \\ 1.1223 & -3.0351 & -5.7830 & 4.4308 \\ 2.7957 & -0.8671 & -2.2443 & -0.0737 \end{bmatrix}, \quad (47)$$

$$B := \begin{bmatrix} -0.4698 & 0.5026 & 1.9107 & -1.0020 & 1.8676 & -1.0523 \\ 0.8036 & -0.0727 & -1.9520 & 2.4997 & -1.2066 & -0.7074 \\ -0.1061 & -0.1776 & 0.9175 & -0.3621 & -0.2116 & 2.3771 \\ -2.2158 & -1.3753 & -1.2109 & -0.8576 & 0.3423 & 1.1991 \end{bmatrix},$$

$$\Pi := \begin{bmatrix} 3.2123 & 3.5111 & 1.3912 & -1.8097 \\ 3.5111 & 10.6258 & 3.7561 & -3.7850 \\ 1.3912 & 3.7561 & 3.3244 & -0.5456 \\ -1.8097 & -3.7850 & -0.5456 & 1.9349 \end{bmatrix}.$$

In this example, the threshold (27) is $\theta_0 = 0.0908$. The graph of the function $-\ln\det D_\theta$ from (25), (26) for $\theta = 0.9\theta_0 = 0.0817$ is shown in Fig. 2 along with its high-frequency asymptote. The results of numerical computation of the QEF growth rate

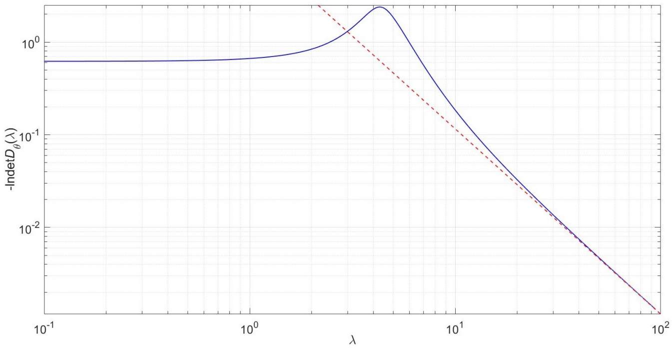

Fig. 2. The graph of the function $-\ln\det D_\theta(\lambda)$ for positive frequencies $\lambda > 0$ (solid line). The dashed line represents the high-frequency asymptote $\frac{\theta}{\lambda^2}\mathrm{Tr}(\Pi BB^{\mathrm{T}})$, as $\lambda \to \infty$, following from (44).

using (37) and Theorem 2 are shown in Fig. 3. The numerical

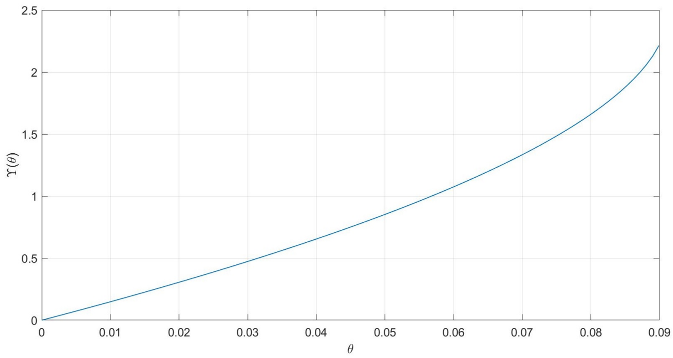

Fig. 3. The graph of the QEF growth rate (25) as a function of the risk sensitivity parameter θ.

integration (over positive frequencies in view of the symmetry of the integrand) employed a combination of a mesh of step size 0.005 for a low-frequency range $[0, 100]$ and the high-frequency asymptote for $\lambda > 100$. The choice of the cutoff frequency was based on the spectrum $\{-3.4734 \pm 2.6849i, -2.2911 \pm 4.1584i\}$ and the operator norm $\|A\| = 9.4475$ of the matrix (47). The integration over θ was carried out with step size $0.01\theta_0 = 9.08 \times 10^{-4}$.

7. CONCLUSION

We have established a frequency-domain formula for the infinite-horizon QEF growth rate at the invariant Gaussian state of a stable multimode OQHO driven by multichannel vacuum fields. This representation involves the quantum spectral density, whose parts are expressed in terms of the transfer function of the system. We have obtained a differential equation for the QEF growth rate as a function of the risk sensitivity parameter and outlined its computation using a homotopy technique. A contour integration approach has also been discussed for this purpose along with a more complicated nature of singularities in compositions of trigonometric and matrix-valued rational functions. The latter requires the development of novel spectral factorization techniques (and state-space equations) for this class of computational problems which go beyond the standard application of the residue theorem to rational functions. The results of the paper provide a solution of the risk-sensitive robust performance analysis problem for linear quantum stochastic systems, which will be applied in future publications to coherent and measurement-based control and filtering settings for such systems.

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