Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Distributed Algorithm for Higher-Order Integrators to Track Average of Unbounded Signals

Arijit Sen * Soumya Ranjan Sahoo * Mangal Kothari **

* *Department of Electrical Engineering, Indian Institute of Technology Kanpur (e-mail: ajsen, srsahoo@iitk.ac.in).*
** *Department of Aerospace Engineering, Indian Institute of Technology Kanpur (e-mail: mangal@iitk.ac.in).*

Abstract:
In this work, we design an algorithm for a group of higher-order integrators aiming to track the average of multiple time-varying and possibly unbounded reference signals. The existing literature has studied distributed average tracking (DAT) for higher-order systems in the presence of bounded or Lipschitz-type reference signals. In such DAT algorithms, each agent requires the knowledge of global bounds on signals for bounded references and state-dependent control gains for unbounded references. Addressing these issues, we propose a DAT algorithm for a group of higher-order integrators in the presence of time-varying references that can possibly be unbounded. The highest derivative of references become equal, asymptotically. Agents use neighbors' data obtained from the local communication framework that makes the current algorithm distributed in nature. In contrast to existing work, our DAT algorithm uses constant gains to reduce high control effort, which may be caused due to state-dependent gains. Using numerical example, the performance of the current algorithm is compared with the existing state-of-the-art. This reveals the superiority of the proposed algorithm.

Keywords: Cooperative control, distributed average tracking, linear control theory, multi-agent.

1. INTRODUCTION

In a multi-agent system (MAS) with each agent having different reference signals, a distributed average tracking (DAT) algorithm makes the agents to track the average of all reference signals. Applications of DAT algorithms have got immense attention in the area of distributed estimation techniques, such as Kalman filtering (Olfati-Saber, 2007; Bai et al., 2011), sensor fusion (Spanos et al., 2005), merging of feature-based maps (Aragues et al., 2012). In an MAS, different physical states of the agents also need to track the time-varying references to ensure consensus (Sen et al., 2019b; Singh et al., 2020), distributed optimization (Gharesifard and Cortés, 2013; Rahili and Ren, 2016), formation control(Sen et al., 2019a), cooperative tracking (Sen et al., 2018), distributed state coordination (Song et al., 2015), containment control Sen et al. (2020). Solving a DAT problem is more challenging than a consensus or distributed tracking problem due to the presence of multiple time-varying references. In this work, we propose a DAT algorithm for multiple higher-order integrators with different time-varying, possibly unbounded references.

Various DAT algorithms are designed for multiple agents with the motion model described by single integrators (Freeman et al., 2006; Nosrati et al., 2012; Chen et al., 2012), double integrators (Chen et al., 2014; Ghapani et al., 2017), and second-order nonlinear systems (Zhao et al., 2018a). To achieve DAT, the reference signals in these mentioned works need to satisfy different constraints, such as signals with constant amplitude (Freeman et al., 2006), bounded magnitude (Nosrati et al., 2012; Ghapani et al., 2017), bounded acceleration (Chen et al., 2014; Zhao et al., 2018a), bounded time derivative (Chen et al., 2012). Recently, the DAT algorithms for a group of single integrators relaxed the assumptions regarding the bounds on each reference signal (Sun et al., 2019; George and Freeman, 2019). In these works, deviations among the references must be bounded throughout the time to guarantee the ultimate boundedness of the tracking error. The algorithm in (Ghapani et al., 2018) uses discontinuous structure to achieve DAT for second-order nonlinear systems irrespective of any constraints on reference signals.

The aforementioned works consider either first or second-order agent dynamics. As mentioned in (Zhao et al., 2017, 2018b), there are various events where the motion models need to be expressed by using higher-order dynamics rather than a first or second-order system. For example, a higher-order integrator can express the sudden change of heading angles during flocking of birds or swarm-on-swarm scenarios (Ren et al., 2007), and maneuvering motion of mobile robots or unmanned helicopters (Rezaee and Abdollahi, 2015). Since agents need to track the average of multiple time-varying references generated by different dynamical systems, it is difficult to extend a first or second-order DAT algorithm to higher-order agents.

DAT algorithms for higher-order systems are studied for linear (Liu et al., 2017; Zhao et al., 2017) and nonlinear

Copyright lies with the authors

agents (Zhao et al., 2018b). Under the constraints of input saturations, DAT problem is solved for multiple agents with stable open-loop dynamics (Liu et al., 2017). Later, the assumption on agents' open-loop dynamics is relaxed where the algorithms use non-smooth function and solve respective algebraic Riccati equations (ARE) to guarantee DAT in the presence of bounded (Zhao et al., 2017) and Lipschitz-like (Zhao et al., 2018b) reference inputs. Although the algorithm in (Zhao et al., 2018b) ensures DAT with unbounded and Lipschitz-like references, the control effort may become significant due to the presence of state-dependent control gains. In addition, solving an ARE becomes complex as the order of agents increases.

Therefore, motivated by these, we propose an algorithm with constant control gains for a group of higher-order integrators to track the average of different time-varying references. This algorithm comprises a distributed averaging filter and a decentralized control law. Each agent uses an undirected communication framework for information exchange. As compared to the existing literature, the take-away points of our proposed DAT algorithm are:

- There is no bound on the different time-varying references. However, the highest derivative of each reference becomes equal asymptotically.
- The proposed distributed averaging filter is formed by using single integrator models to ease the setting of the convergence rate with the choice of suitable gains.
- Unlike the earlier works (Zhao et al., 2017, 2018b), the proposed DAT does not require any additional adaptive update laws to remove the requirements of global information.
- Contrary to the DAT algorithms in (Zhao et al., 2018b), the gains in our algorithm are positive constants and independent of agents' states. For this, the control inputs can not become very large due to increasing magnitudes of agents' states.

The rest of this paper's structure is stated as follows: we formally introduce the problem in Section 2, along with some key aspects of graph theory. Section 3 presents the DAT algorithm for a group of higher-order integrators. We show that the tracking errors converge to zeros if the highest derivative of reference time-varying signal for all agents become equal asymptotically. We compare the performance of the current and existing protocol through numerical simulations in Section 4. In Section 5, we conclude the current work.

2. FORMULATION OF THE PROBLEM WITH NECESSARY PRELIMINARIES

This work assumes that each agent of the MAS follows the dynamics of a dth-order integrator

$$\begin{aligned}\dot{x}_{1i}(t)&= x_{2i}(t)\\ \dot{x}_{2i}(t)&= x_{3i}(t)\\ &\vdots\\ \dot{x}_{di}(t)&= u_i(t), \qquad i=1,2,\ldots,N\end{aligned} \tag{1}$$

where the control input is $u_i(t) \in \mathbb{R}$ and internal states are $x_{1i}(t), x_{2i}(t), \ldots, x_{di}(t) \in \mathbb{R}$ for agent i.

We suppose that every agent has a time-varying reference signal generated by the following dynamics:

$$\begin{aligned}\dot{p}_{1i}(t)&= p_{2i}(t)\\ \dot{p}_{2i}(t)&= p_{3i}(t)\\ &\vdots\\ \dot{p}_{di}(t)&= r_i(t), \qquad i=1,2,\ldots,N\end{aligned} \tag{2}$$

where $p_{1i}(t), p_{2i}(t), \ldots, p_{di}(t) \in \mathbb{R}$ are reference states and $r_i(t) \in \mathbb{R}$ is input to the reference dynamics (2).

The objective of the current work is to develop a distributed algorithm for the MAS consisting of agents with dynamics (1) to guarantee average tracking of the reference signals generated by (2). Thus, the objective of the problem statement can be mathematically expressed as

$$\lim_{t\to\infty} |x_{ki}(t) - (1/N)\sum_{i=1}^{N} p_{ki}(t)| = 0, \tag{3}$$

$\forall\ k = 1,2,\ldots,d$, and $\forall\ i=1,2,\ldots,N$. For convenience, t is removed from the variable in the rest of this paper.

An information flow framework among N-agents can be emulated by a graph. This makes it important to state certain notions of graph theory which will be used later.

2.1 Graph theory

An undirected graph $\mathscr{G}_N = (\mathscr{V}_N, \mathscr{E}_N)$ represents the information flow framework in an MAS. The sets $\mathscr{V}_N$ and $\mathscr{E}_N$ denote the node set and edge set, respectively. The existence of a directed edge $\mathfrak{e}_{ij} \in \mathscr{E}_N$ implies that agent i has the information of agent j. If $\mathfrak{e}_{ij} \in \mathscr{N}_i$, agent j is a neighbor of agent i where $\mathscr{N}_i$ is the neighbor set of agent i. The adjacency $\mathscr{A}_N$ and Laplacian $\mathscr{L}_N$ matrices associated with $\mathscr{G}_N$ are: $\mathscr{A}_N := [a_{ij}]_{N\times N}$, for $a_{ij}\begin{cases} > 0, & \text{if } j \in \mathscr{N}_i \\ = 0, & \text{otherwise,}\end{cases}$ and $\mathscr{L}_N := [l_{ij}]_{N\times N}$, for $l_{ij} = \begin{cases}\sum_{j=1}^{N} a_{ij}, & \text{for } i=j \\ -a_{ij}, & \text{otherwise.}\end{cases}$ The graph $\mathscr{G}_N$ being undirected, $\mathfrak{e}_{ji} \in \mathscr{E}_N \Leftrightarrow \mathfrak{e}_{ij} \in \mathscr{E}_N,\ \forall\, i,j \in \mathscr{V}_N$. Thus, $\mathscr{A}_N$ and $\mathscr{L}_N$ are symmetric. When $\mathscr{G}_N$ is connected, $\mathscr{L}_N$ is positive semi-definite satisfying $\mathscr{L}_N \mathbf{1}_N = \mathbf{0}_N$ and $\mathbf{1}_N^\top \mathscr{L}_N = \mathbf{0}_N^\top$ (Olfati-Saber and Murray, 2004), where $\mathbf{1}_N = [1,\ldots,1]^\top, \mathbf{0}_N = [0,\ldots,0]^\top \in \mathbb{R}^N$. The assumption on $\mathscr{G}_N$ is stated next.

Assumption 1. The graph $\mathscr{G}_N$ is fixed, undirected, and connected.

Having the preliminary knowledge of graph theory, we are now prepared to present the DAT algorithm.

3. DAT ALGORITHM FOR HIGHER-ORDER INTEGRATORS

The proposed DAT algorithm has two parts. In the first part, we design a distributed filter to estimate the average of the reference signals. The second part comes up with the control input u_i to guarantee average tracking. For i-th agent, proposed distributed filter is

$$\begin{aligned}\dot{s}_{1i}&= s_{2i} - c_1 \sum_{j\in\mathscr{N}_i} a_{ij}\,(s_{1i} - s_{1j})\\ \dot{s}_{2i}&= s_{3i} - c_2 \sum_{j\in\mathscr{N}_i} a_{ij}\,(s_{2i} - s_{2j})\\ &\vdots\\ \dot{s}_{di}&= -c_d \sum_{j\in\mathscr{N}_i} a_{ij}\,(s_{di} - s_{dj}) + r_i,\end{aligned} \tag{4}$$

where $c_k \in \mathbb{R}^+$ are filter gains, and $s_{ki} \in \mathbb{R}$ are the filter states satisfying $s_{ki}(0) = p_{ki}(0), \forall\ k = 1, \ldots, d$. Since $\mathscr{G}_N$ is undirected, $a_{ij} = a_{ji}\ \forall\ i, j \in \mathscr{V}_N$.

Assumption 2. The reference signals r_i for all agents become equal asymptotically, that implies, $\lim_{t\to\infty} \|r_i(t) - r_j(t)\| = 0,\ \forall\ i, j \in \mathscr{V}_N$.

Lemma 1. For an MAS with N agents, the distributed filter (4) ensures $\lim_{t\to\infty} \|s_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\| = 0$, under Assumptions 1-2, and $s_{ki}(0) = p_{ki}(0),\ \forall\ k = 1, \ldots, d$ and $\forall\ i \in \mathscr{V}_N$.

Proof. Proof is provided in Appendix A. □

Next, we use the states of (4) to design the following control input for i-th agent to establish DAT in higher-order systems:

$$u_i = -\sum_{k=1}^{d}\mu_k\left(x_{ki} - s_{ki}\right) + r_i, \tag{5}$$

where $\mu_1, \mu_2, \ldots, \mu_d$ are the gains such that $[\lambda^d + \mu_d\lambda^{d-1} + \cdots + \mu_2\lambda + \mu_1]$ is a Hurwitz polynomial. We discuss the stability under the control law (5) with distributed filter (4) in the next theorem.

Theorem 1. For an MAS with agents having dynamics (1) and their respective reference dynamics (2), the DAT algorithm comprising (4) and (5) ensures $\lim_{t\to\infty} \|x_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\| = 0$, in the presence of Assumptions 1-2 when $s_{ki}(0) = p_{ki}(0),\ \forall\ k = 1, \ldots, d$ and $\forall\ i \in \mathscr{V}_N$.

Proof. Lemma 1 establishes $\lim_{t\to\infty} \|s_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\| = 0,\ \forall\ k = 1, \ldots, d$, and $\forall\ i \in \mathscr{V}_N$. Next, we show DAT of higher-order integrators using the control law (5). The proof is done for agent i. Same holds for others as well.

Higher-order agents (1) under the control law (5) become

$$\begin{aligned}\dot{x}_{1i} &= x_{2i}\\ \dot{x}_{2i} &= x_{3i}\\ &\vdots\\ \dot{x}_{di} &= -\sum_{k=1}^{d}\mu_k\left(x_{ki} - s_{ki}\right) + r_i.\end{aligned} \tag{6}$$

For $k = 1, \ldots, d$, we define $e_{ki} := \left(x_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\right)$ and $\epsilon_{ki} := \left(s_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\right)$, to obtain the error dynamics

$$\begin{aligned}\dot{e}_{1i} &= e_{2i}\\ \dot{e}_{2i} &= e_{3i}\\ &\vdots\\ \dot{e}_{di} &= -\sum_{k=1}^{d}\mu_k e_{ki} + \sum_{k=1}^{d}\mu_k\epsilon_{ki} + \left(r_i - \frac{1}{N}\sum_{i=1}^{N} r_i\right).\end{aligned} \tag{7}$$

Supposing $e_i := [e_{1i}, e_{2i}, \ldots, e_{di}]^\top$, (7) is rewritten as

$$\dot{e}_i = \mathfrak{B}e_i + \mathfrak{H}(\epsilon_{1i}, \epsilon_{2i}, \ldots, \epsilon_{di}, r, t), \tag{8}$$

where $\mathfrak{B} = \begin{bmatrix} 0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\mu_1 & -\mu_2 & \cdots & -\mu_d\end{bmatrix}$ and $\mathfrak{H} = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\left\{\sum_{k=1}^{d}\mu_k\epsilon_{ki} + \left(r_i - \frac{1}{N}\sum_{i=1}^{N} r_i\right)\right\}$. With $\mathfrak{H}$ as the input to (8), we show the stability of the zero input system

$$\dot{e}_i = \mathfrak{B}e_i. \tag{9}$$

Characteristics equation of $\mathfrak{B}$ is

$$\lambda^d + \mu_d\lambda^{d-1} + \cdots + \mu_2\lambda + \mu_1 = 0. \tag{10}$$

Using Routh-Hurwitz criteria (Chapter 6.3, Shinners (1998)), gains μ_i's can be chosen such that the characteristic polynomial in (10) is Hurwitz. Therefore, zero input dynamics (9) becomes globally exponentially stable.

The error dynamics (8) is input-to-state stable as its zero input dynamics (9) is exponentially stable (Lemma 4.6, Khalil (2002)). Thus, in the presence of a bounded $\mathfrak{H}(t)$,

$$\|e_i(t)\| \le \Phi\left(\|e_i(\mathfrak{T})\|, (t - \mathfrak{T})\right) + \Psi\left(\sup_{\mathfrak{T}\le\tau\le t}\|\mathfrak{H}(\tau)\|\right),$$

for $t \ge \mathfrak{T} \ge 0$, where $\Phi(.)$ and $\Psi(.)$ are class $\mathcal{KL}$ and class $\mathcal{K}$ functions, respectively (Definition 4.7, Khalil (2002)). Let $\varsigma > 0$ be any arbitrary constant. For this ς, we can always find a $\sigma > 0$ satisfying $\Psi(\sigma) \le \frac{\varsigma}{2}$. From Lemma 1 and Assumption 2, we have $\lim_{t\to\infty}\|\mathfrak{H}(t)\| = 0$. Therefore, there exists a $\gamma_1 > 0$ corresponding to this σ, such that $\|\mathfrak{H}(t)\| \le \sigma,\ \forall\ t \ge \gamma_1$. Hence, $\forall\ t \ge \mathfrak{T} \ge \gamma_1$, we have

$$\begin{aligned}\|e_i(t)\| &\le \Phi\left(\|e_i(\mathfrak{T})\|, (t - \mathfrak{T})\right) + \Psi\left(\sup_{\mathfrak{T}\le\tau\le t}\|\mathfrak{H}(\tau)\|\right)\\ &\le \Phi\left(\|e_i(\mathfrak{T})\|, (t - \mathfrak{T})\right) + \Psi\left(\sigma\right)\\ \Rightarrow \|e_i(t)\| &\le \Phi\left(\bar{e}, (t - \gamma_1)\right) + \varsigma/2,\end{aligned}$$

where $\|e_i(\mathfrak{T})\| \le \bar{e}$. As $\lim_{t\to\infty}\Phi\left(\bar{e}, (t - \mathfrak{T})\right) = 0$, there exists a $\gamma_2 > 0$ such that

$$\Phi\left(\bar{e}, (t - \mathfrak{T})\right) \le \varsigma/2, \quad \forall\ t \ge \gamma_2.$$

Therefore, for any given ς, there always be a $\gamma := \max\{\gamma_1, \gamma_2\} > 0$ such that

$$\|e_i(t)\| \le \varsigma, \quad \forall\ t \ge \gamma > 0.$$

Since $\lim_{t\to\infty}\Phi\left(\bar{e}, (t - \mathfrak{T})\right) = 0$ and $\lim_{t\to\infty}\|\mathfrak{H}(t)\| = 0$, the given constant $\varsigma \to 0$ as $\gamma \to \infty$. Hence, for the error dynamics (8), $\lim_{t\to\infty}\|e_i(t)\| = 0$. Therefore, under Assumptions 1-2, the proposed DAT algorithm (4)-(5) guarantees $\lim_{t\to\infty}\|x_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\| = 0$ with $s_{ki}(0) = p_{ki}(0)$ $\forall\ k = 1, \ldots, d$ and $\forall\ i \in \mathscr{V}_N$. □

Remark 1. If difference among the reference signals are always bounded, the tracking errors become ultimately bounded. A similar proof can be found in (Sun et al., 2019).

Remark 2. If $r_i(t)$ are same for all agents, $\mathfrak{F} = \mathbf{0}_N$ (defined in (A.2)) and $\mathfrak{H} = \mathbf{0}_d$. Thus, the higher-order integrators under the proposed DAT algorithm become exponentially stable with minimum convergence rate $\kappa = \max\{\lambda_{\mathfrak{B}}^{\max}, -\min_k\{c_k\}\lambda_2(\mathscr{L}_N)\}$ where $\lambda_{\mathfrak{B}}^{\max}$ is the maximum eigenvalue of $\mathfrak{B}$ and $\lambda_2(\mathscr{L}_N)$ is minimum positive eigenvalue of $\mathscr{L}_N$.

In the following section, we compare the efficacy of the proposed DAT algorithm with existing algorithm in (Zhao et al., 2018b) through numerical examples.

4. NUMERICAL SIMULATIONS

For simulation, we take an MAS with 5 agents, each being 3rd-order integrator, that is $N = 5$ and $d = 3$. The information flow framework $\mathscr{G}_5$ is shown in Fig. 1. Weights of all non-zero a_{ij} are equal to 1. The chosen initial values of the agents' states and reference signals are tabulated in Table 1. The reference signals are $r_i = i \times e^{-(10\times t\times i)} + \sin\left(\frac{\pi}{2}t\right)$ for $i = 1, 2, \ldots, 5$.

The gains of the distributed filter (4) and control law (5) are $c_1 = 1,\ c_2 = 2,\ c_3 = 3$, and $\mu_1 = \frac{16}{9},\ \mu_2 = \frac{44}{9},\ \mu_3 = 4$.

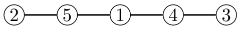


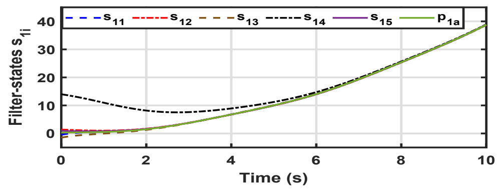
Fig. 1. Information exchage framework $\mathscr{G}_5$.

Agent (i)	$x_{1i}(0)$	$x_{2i}(0)$	$x_{3i}(0)$	$p_{1i}(0)$	$p_{2i}(0)$	$p_{3i}(0)$
1	10	-1.5	0.2	-0.5	0.35	-0.04
2	12	-2.2	-0.6	1.4	-0.24	-0.075
3	-15	3	1	-1.5	0.3	0.1
4	14	-2.4	-0.75	1.2	0.22	-0.06
5	-5	3.5	-0.4	1	-0.15	0.2

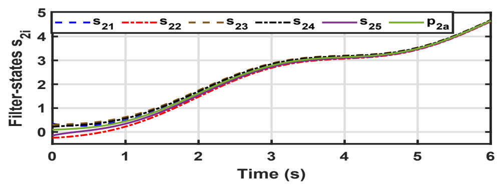
Table 1. Values of the states at $t = 0$.

Asymptotic convergence of s_{ki} to $\frac{1}{5}\sum_{i=1}^{5} p_{ki}$ is shown in Fig. 2 for $k = 1, 2, 3$, and $\forall\ i \in \mathscr{G}_5$. The estimates s_{1i}, s_{2i}, and s_{3i}'s merge with the respective average signals approximately at $t = 8$ s, 4.5 s, and 0.8 s, respectively (within $\pm 2\%$ tolerance band). As we set $c_3 > c_2 > c_1$, the filter states s_{3i}'s converge first, followed by s_{2i}'s and s_{1i}'s. Figure 3 depicts the trajectories of states x_{ki} for all agents. We observe that the state trajectories of all agents converge to average of respective reference states (within $\pm 2\%$ tolerance band) at $t \approx 10$ s. In these figures, $p_{ka} := \frac{1}{5}\sum_{i=1}^{5} p_{ki}$, for $k = 1, 2, 3$. The control efforts for the agents are shown in Fig. 4. Due to the presence of constant control gains, the control efforts are bounded for chosen reference $r(t)$.

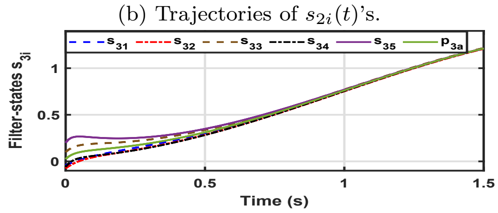
Therefore, agents' states and filtered variables converge asymptotically to the average of the respective reference states under assumptions 1 and 2. Next, we will compare the performance of DAT algorithm in (Zhao et al., 2018b) with the discussed numerical results.



(a) Trajectories of $s_{1i}(t)$'s.



(b) Trajectories of $s_{2i}(t)$'s.

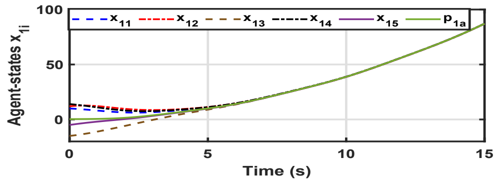


(c) Trajectories of $s_{3i}(t)$'s.

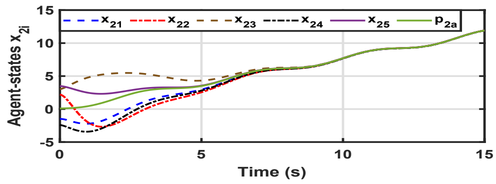
Fig. 2. Evolution of $s_i(t)$'s under proposed filter (4). Note that $p_{ka} := \frac{1}{5}\sum_{i=1}^{5} p_{ki}$ for $k = 1, 2, 3$.

Performance comparison with (Zhao et al., 2018b)

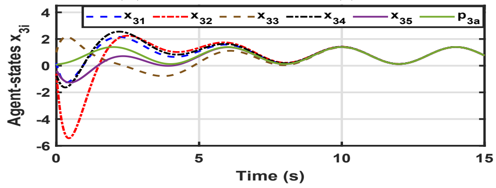
For the same set of agents with identical initial conditions and reference input $r_i(t)$, we check the performance of the DAT algorithm for higher-order agents proposed in (Zhao et al., 2018b). We consider the maximum control input as the benchmark to compare the performance while keeping the time of convergence to approximately 10 s (within 2%



(a) Trajectories of $x_{1i}(t)$'s.



(b) Trajectories of $x_{2i}(t)$'s.



(c) Trajectories of $x_{3i}(t)$'s.

Fig. 3. Convergence of agents' states under algorithm (4)-(5). Note that $p_{ka} := \frac{1}{5}\sum_{i=1}^{5} p_{ki}$ for $k = 1, 2, 3$.

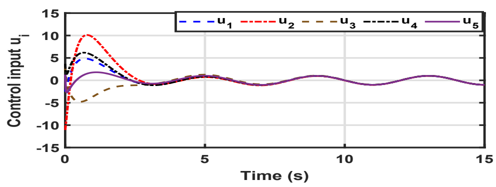


Fig. 4. Agents' control input under DAT algorithm (4)-(5).

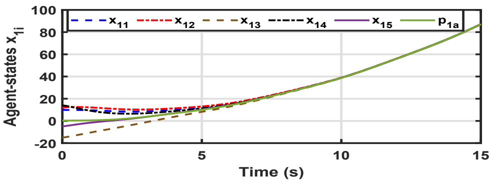
tolerance band) for both DAT algorithms. From Fig. 5, we notice that the states of all agents track their respective average approximate at $t = 10$ s. However, the magnitude of control inputs under existing protocol from (Zhao et al., 2018b) are very large as observed from Fig. 6.

Hence, from Fig. 4 and 6, it is observed that the same agents with identical references and initial conditions require much lesser control effort to obtain DAT under proposed algorithm (4)-(5) as compared to the existing one in (Zhao et al., 2018b).

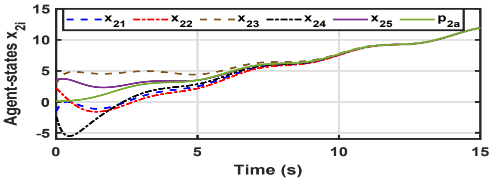
In addition, Fig. 6 also shows that magnitude of the control input are growing with increasing values of agents' states. The presence of state-dependent gains causes such behaviour in control trajectory. Since our proposed DAT algorithm (4)-(5) contains constant gains, the control input u_i will not grow with for increasing state values.

5. CONCLUSION

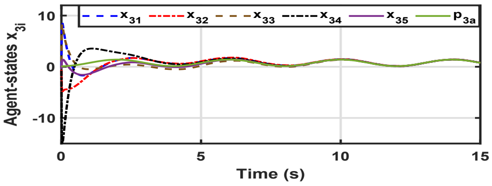
We have presented a DAT algorithm for a group of higher-order integrators. The algorithm consists of a distributed averaging filter and a control input to track the average of the time-varying references which are generated from a higher-order dynamical model. This algorithm also relaxes the need for knowledge about global information without using additional update equations for adaptive gains. We have established the asymptotic stability of the MAS under the proposed DAT algorithm using the notions of



(a) Trajectories of $x_{1i}(t)$'s.



(b) Trajectories of $x_{2i}(t)$'s.



(c) Trajectories of $x_{3i}(t)$'s.

Fig. 5. Convergence of agents' states under the DAT algorithm proposed in (Zhao et al., 2018b). Note that $p_{ka} := \frac{1}{5}\sum_{i=1}^{5} p_{ki}$ for $k = 1, 2, 3$.

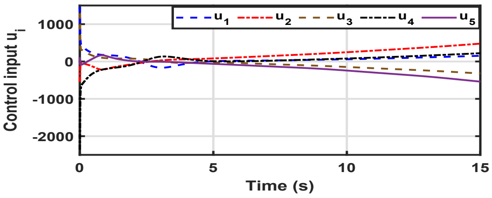


Fig. 6. Agents' control input under proposed DAT algorithm proposed in (Zhao et al., 2018b).

input-to-state stability. Instead of state-dependent-gains, we have taken constant positive gains to avoid the magnitude growth of the control effort with increasing state values. The numerical example has presented to show the efficacy of the proposed algorithm. A comparison with an existing DAT algorithm demonstrates a better performance of our DAT algorithm.

Appendix A. PROOF OF LEMMA 1

Define the error variables $\epsilon_{ki} := \left(s_{ki} - \frac{1}{N}\sum_{i=1}^{N} p_{ki}\right)$, $\forall\, k = 1, \ldots, d$. Thus, the error dynamics for (4) is written as

$$\begin{aligned}
\dot{\epsilon}_{1i} &= \epsilon_{2i} - c_1 \sum_{j\in\mathscr{N}_i} a_{ij}\left(\epsilon_{1i} - \epsilon_{1j}\right)\\
\dot{\epsilon}_{2i} &= \epsilon_{3i} - c_2 \sum_{j\in\mathscr{N}_i} a_{ij}\left(\epsilon_{2i} - \epsilon_{2j}\right)\\
&\ \vdots\\
\dot{\epsilon}_{di} &= -c_d \sum_{j\in\mathscr{N}_i} a_{ij}\left(\epsilon_{di} - \epsilon_{dj}\right) + \left(r_i - \frac{1}{N}\sum_{i=1}^{N} r_i\right).
\end{aligned} \tag{A.1}$$

Let $\epsilon_k := [\epsilon_{k1}, \epsilon_{k2}, \ldots, \epsilon_{kN}]^\top \in \mathbb{R}^N$, $\forall\, k = 1, \ldots, d$, $\epsilon = \left[\epsilon_1^\top, \epsilon_2^\top, \ldots, \epsilon_d^\top\right]^\top \in \mathbb{R}^{dN}$, and $r = [r_1, r_2, \ldots, r_N]^\top \in \mathbb{R}^N$. Thus, the matrix representation of (A.1) is

$$\begin{aligned}
\dot{\epsilon}_1 &= -c_1 \mathscr{L}_N \epsilon_1 + \epsilon_2\\
\dot{\epsilon}_2 &= -c_2 \mathscr{L}_N \epsilon_2 + \epsilon_3\\
&\ \vdots\\
\dot{\epsilon}_d &= -c_d \mathscr{L}_N \epsilon_d + \mathfrak{F}(t),
\end{aligned} \tag{A.2}$$

where $\mathfrak{F}(t) = \left(r - \frac{1}{N}\mathbf{1}_N^\top r \mathbf{1}_N\right)$. For the following system:

$$\dot{\epsilon}_d = -c_d \mathscr{L}_N \epsilon_d + \mathfrak{F}(t), \tag{A.3}$$

and the disagreement variable $\widetilde{\epsilon}_d := \left(\epsilon_d - \frac{1}{N}\mathbf{1}_N^\top \epsilon_d(0)\mathbf{1}_N\right)$, we get

$$\dot{\widetilde{\epsilon}}_d = -c_d \mathscr{L}_N \widetilde{\epsilon}_d + \mathfrak{F}(t), \tag{A.4}$$

with zero input error dynamics for (A.4) as

$$\dot{\widetilde{\epsilon}}_d = -c_d \mathscr{L}_N \widetilde{\epsilon}_d. \tag{A.5}$$

The zero input disagreement dynamics (A.5) is globally exponentially stable with minimum convergence rate, $-c_d\lambda_2(\mathscr{L}_N)$ (Theorem 8, Olfati-Saber and Murray (2004)). Note that $\lambda_2(\mathscr{L}_N)$ is the minimum positive eigenvalue of $\mathscr{L}_N$ for an undirected and connected graph $\mathscr{G}_N$.

As the zero input system (A.5) is globally exponentially stable, the system (A.4) is input-to-state stable (Lemma 4.6, Khalil (2002). Hence, there exists a class $\mathcal{KL}$ function Θ and a class $\mathcal{K}$ function Ω such that $\forall\, t \geq \mathfrak{T} \geq 0$

$$\|\widetilde{\epsilon}_d(t)\| \leq \Theta\left(\|\widetilde{\epsilon}_d(\mathfrak{T})\|, (t-\mathfrak{T})\right) + \Omega\left(\sup_{\mathfrak{T}\leq\tau\leq t}\|\mathfrak{F}(\tau)\|\right)$$

where $\mathfrak{F}$ is bounded (Definition 4.7, Khalil (2002)). For any given $\varrho > 0$, there always exists a $\varphi > 0$ for which the class $\mathcal{K}$ function $\Omega(.)$ satisfies $\Omega(\varphi) \leq \frac{\varrho}{2}$. From Assumption 2, $\lim_{t\to\infty}\|\mathfrak{F}(t)\| = 0$. Therefore, there is a $\eta_1 > 0$ for this φ satisfying $\|\mathfrak{F}(t)\| \leq \varphi$, $\forall\, t \geq \eta_1$. Thus, we set $\mathfrak{T} \geq \eta_1$ to obtain the bound on $\|\widetilde{\epsilon}_d(t)\|$, $\forall\, t \geq \mathfrak{T} \geq \eta_1$ as follows:

$$\begin{aligned}
\|\widetilde{\epsilon}_d(t)\| &\leq \Theta\left(\|\widetilde{\epsilon}_d(\mathfrak{T})\|, (t-\mathfrak{T})\right) + \Omega\left(\sup_{\mathfrak{T}\leq\tau\leq t}\|\mathfrak{F}(\tau)\|\right)\\
&\leq \Theta\left(\|\widetilde{\epsilon}_d(\mathfrak{T})\|, (t-\mathfrak{T})\right) + \Omega\left(\varphi\right)\\
\Rightarrow \|\widetilde{\epsilon}_d(t)\| &\leq \Theta\left(\overline{\epsilon}, (t-\mathfrak{T})\right) + \varrho/2,
\end{aligned}$$

where $\overline{\epsilon} > 0$ is a finite constant such that $\overline{\epsilon} \geq \|\widetilde{\epsilon}_d(\mathfrak{T})\|$. Since $\lim_{t\to\infty}\Theta\left(\overline{\epsilon}, (t-\mathfrak{T})\right) = 0$, there exists a $\eta_2 > 0$, such that

$$\Theta\left(\overline{\epsilon}, (t-\mathfrak{T})\right) \leq \varrho/2, \quad \forall\, t \geq \eta_2.$$

Hence, for any arbitrary ϱ, there exists a $\eta := \max\{\eta_1, \eta_2\}$ satisfying

$$\|\widetilde{\epsilon}_d(t)\| \leq \varrho, \quad \forall\, t \geq \eta > 0.$$

Since, $\lim_{t\to\infty}\Theta\left(\overline{\epsilon}, (t-\mathfrak{T})\right) = 0$ and $\lim_{t\to\infty}\|\mathfrak{F}(t)\| = 0$, the value of the arbitrary constant ϱ will become 0 as $\eta \to \infty$. Therefore, for the system (A.4), $\|\widetilde{\epsilon}_d(t)\|$ converges to 0 asymptotically. Hence, $\lim_{t\to\infty}\|\epsilon_d - \frac{1}{N}\mathbf{1}_N^\top \epsilon_d(0)\mathbf{1}_N\| = 0$.

From (A.1), since $\mathscr{G}_N$ is undirected, we have $\forall\, t \geq 0$

$$\sum_{i\in\mathscr{V}_N}\dot{\epsilon}_{ki}(t) = \sum_{i\in\mathscr{V}_N}\epsilon_{k+1,i}(t), \quad \text{if } k \neq d, \text{ and } \sum_{i\in\mathscr{V}_N}\dot{\epsilon}_{di}(t) = 0.$$

For the choice of initial filter states, $s_{ki}(0) = p_{ki}(0)$,

$$\sum_{i\in\mathscr{V}_N}\epsilon_{ki}(0) = \sum_{i\in\mathscr{V}_N}s_{ki}(0) - \sum_{i\in\mathscr{V}_N}p_{ki}(0) = 0$$

for $k = 1, \ldots, d$, $\forall\, i \in \mathscr{V}_N$. Hence, $\forall\, t \geq 0$

$$\sum_{i\in\mathscr{V}_N}\dot{\epsilon}_{di}(t) = 0 \Rightarrow \sum_{i\in\mathscr{V}_N}\epsilon_{di}(t) = \sum_{i\in\mathscr{V}_N}\epsilon_{di}(0) = 0.$$

As $\sum_{i\in\mathscr{V}_N}\dot{\epsilon}_{ki}(t) = \sum_{i\in\mathscr{V}_N}\epsilon_{k+1,i}(t)$ for $k \neq d$, so

$$\begin{aligned}
\sum_{i\in\mathscr{V}_N}\dot{\epsilon}_{d-1,i}(t) &= \sum_{i\in\mathscr{V}_N}\epsilon_{d,i}(t) = 0\\
\Rightarrow \sum_{i\in\mathscr{V}_N}\epsilon_{d-1,i}(t) &= \sum_{i\in\mathscr{V}_N}\epsilon_{d-1,i}(0) = 0.
\end{aligned}$$

Therefore, in a similar way, it can be shown that

$$\sum_{i\in\mathscr{V}_N}\epsilon_{ki}(t)=0,\ \forall\ k=1,\ldots,d,\ \forall\ t\geq 0.$$

Since, $\lim_{t\to\infty}\|\epsilon_d-\frac{1}{N}\mathbf{1}_N^\top\epsilon_d(0)\mathbf{1}_N\|=0$, we get $\lim_{t\to\infty}\|\epsilon_d\| = 0$. As the separation principle holds, dynamics of ϵ_{d-1} associated with (A.2) asymptotically becomes

$$\dot{\epsilon}_{d-1}=-c_{d-1}\mathscr{L}_N\epsilon_{d-1}.$$

Using same analysis, we can conclude that $\lim_{t\to\infty}\|\epsilon_{d-1}-\frac{1}{N}\mathbf{1}_N^\top\epsilon_{d-1}(0)\mathbf{1}_N\|=0$. Since $\sum_{i\in\mathscr{V}_N}\epsilon_{d-1,i}(t)=0,\ \forall\ t\geq 0$, we have $\lim_{t\to\infty}\|\epsilon_{d-1}\|=0$. Similarly, convergence of $\lim_{t\to\infty}\|\epsilon_k\|=0$ for $k=1,2,\ldots,d-2$, can also be obtained. This concludes that under Assumption 1-2, and $s_{ki}(0)=p_{ki}(0),\ k=1,\ldots,d,\ \forall i\in\mathscr{V}_N,\ \lim_{t\to\infty}\|s_{ki}-\frac{1}{N}\sum_{i=1}^N p_{ki}\|=0.$ □

REFERENCES

Aragues, R., Cortés, J., and Sagues, C. (2012). Distributed consensus on robot networks for dynamically merging feature-based maps. *IEEE Transactions on Robotics*, 28(4), 840–854.

Bai, H., Freeman, R., and Lynch, K. (2011). Distributed kalman filtering using the internal model average consensus estimator. In *Proceedings of American Control Conference*, 1500–1505. IEEE.

Chen, F., Cao, Y., and Ren, W. (2012). Distributed average tracking of multiple time-varying reference signals with bounded derivatives. *IEEE Transactions on Automatic Control*, 57(12), 3169–3174.

Chen, F., Ren, W., Lan, W., and Chen, G. (2014). Distributed average tracking for reference signals with bounded accelerations. *IEEE Transactions on Automatic Control*, 60(3), 863–869.

Freeman, R., Yang, P., and Lynch, K. (2006). Stability and convergence properties of dynamic average consensus estimators. In *Proceedings of the 45th Conference on Decision and Control*, 338–343. IEEE.

George, J. and Freeman, R. (2019). Robust dynamic average consensus algorithms. *IEEE Transactions on Automatic Control*, 64(11), 4615–4622.

Ghapani, S., Rahili, S., and Ren, W. (2018). Distributed average tracking of physical second-order agents with heterogeneous unknown nonlinear dynamics without constraint on input signals. *IEEE Transactions on Automatic Control*, 64(3), 1178–1184.

Ghapani, S., Ren, W., Chen, F., and Song, Y. (2017). Distributed average tracking for double-integrator multi-agent systems with reduced requirement on velocity measurements. *Automatica*, 81, 1–7.

Gharesifard, B. and Cortés, J. (2013). Distributed continuous-time convex optimization on weight-balanced digraphs. *IEEE Transactions on Automatic Control*, 59(3), 781–786.

Khalil, H.K. (2002). Nonlinear systems, 3rd edition. *Prentice Hall*.

Liu, H., Chen, F., Xiang, L., and Lan, W. (2017). Distributed average tracking with input saturation. *Nonlinear Dynamics*, 90(4), 2827–2839.

Nosrati, S., Shafiee, M., and Menhaj, M. (2012). Dynamic average consensus via nonlinear protocols. *Automatica*, 48(9), 2262–2270.

Olfati-Saber, R. (2007). Distributed kalman filtering for sensor networks. In *46th IEEE Conference on Decision and Control*, 5492–5498. IEEE.

Olfati-Saber, R. and Murray, R. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9), 1520–1533.

Rahili, S. and Ren, W. (2016). Distributed continuous-time convex optimization with time-varying cost functions. *IEEE Transactions on Automatic Control*, 62(4), 1590–1605.

Ren, W., Moore, K., and Chen, Y. (2007). High-order and model reference consensus algorithms in cooperative control of multivehicle systems. *Journal of Dynamic Systems, Measurement, and Control*, 129(5), 678–688.

Rezaee, H. and Abdollahi, F. (2015). Average consensus over high-order multiagent systems. *IEEE Transactions on Automatic Control*, 60(11), 3047–3052.

Sen, A., Sahoo, S., and Kothari, M. (2018). A novel distributed algorithm for consensus under digraph topology with uncertain target information. In *Annual American Control Conference (ACC)*, 1665–1670. IEEE.

Sen, A., Sahoo, S., and Kothari, M. (2019a). Circumnavigation on multiple circles around a non-stationary target with desired angular spacing. *IEEE Transactions on Cybernetics*, DOI:10.1109/TCYB.2019.2935839.

Sen, A., Sahoo, S., and Kothari, M. (2019b). A discontinuous consensus algorithm with neighbor counting. In *2019 18th European Control Conference (ECC)*, 180–185. IEEE.

Sen, A., Sahoo, S., and Kothari, M. (2020). Containment using incomplete agent information over a digraph. *IEEE Control Systems Letters*, DOI:10.1109/LCSYS.2020.2988162.

Shinners, S. (1998). *Modern control system theory and design, 3nd Edition*. John Wiley & Sons.

Singh, B., Sen, A., and Sahoo, S. (2020). Min-consensus for heterogeneous higher-order integrators under switching digraph. *IEEE Control Systems Letters*, 4(3).

Song, Q., Yu, W., Cao, J., and Liu, F. (2015). Reaching synchronization in networked harmonic oscillators with outdated position data. *IEEE Transactions on Cybernetics*, 46(7), 1566–1578.

Spanos, D., Olfati-Saber, R., and Murray, R. (2005). Distributed sensor fusion using dynamic consensus. In *IFAC World Congress*.

Sun, S., Chen, F., and Ren, W. (2019). Distributed average tracking over weight-unbalanced directed graphs. In *American Control Conference (ACC)*, 1400–1405. IEEE.

Zhao, Y., Liu, Y., Li, Z., and Duan, Z. (2017). Distributed average tracking for multiple signals generated by linear dynamical systems: An edge-based framework. *Automatica*, 75, 158–166.

Zhao, Y., Liu, Y., Wen, G., and Huang, T. (2018a). Finite-time distributed average tracking for second-order nonlinear systems. *IEEE Transactions on Neural Networks and Learning systems*, 30(6), 1780–1789.

Zhao, Y., Liu, Y., Wen, G., Yu, X., and Chen, G. (2018b). Distributed average tracking for lipschitz-type of nonlinear dynamical systems. *IEEE Transactions on Cybernetics*, 49(12), 4140–4152.