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Constrained Trajectory Planning for Second-Order Chained Form Systems Using Time Polynomials ⋆

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Abstract: This paper deals with time polynomial based trajectory planning for differentially flat affine dynamical systems that can be written as a chain of second-order controlled subsystems. An analytical approach is proposed to account for state and input constraints by adjusting the standard third-order time polynomial based considerations. For a point-to-point motion planning problem the constraints are met by properly selecting the time of motion value or/and initial or final values of some of the state variables. As an illustrative example trajectory planning for a 3-DoF Delta pick and place robot is considered.

Keywords: Nonlinear control, Trajectory planning, Control of constrained systems, Parallel robots.

1. INTRODUCTION

Motion planning is one of the important problems for dynamical systems in robotics, unmanned aerial vehicles flight control and other control theory application areas, see e.g. Faiz et al. (2001), Martin et al. (2003), Biagiotti and Melchiorri (2008), Ryu and Agrawal (2010), Tang et al. (2011), Mohseni and Fakharian (2015), Mohseni and Fakharian (2016), Jond et al. (2016), Richter et al. (2016), Belinskaya and Chetverikov (2016), Golubev et al. (2017), Fetisov (2017), Belinskaya and Chetverikov (2018), Golubev et al. (2019), Golubev et al. (2019a).

A common way to construct reference trajectories is to deal with time polynomials which proved to be an effective tool, especially for differentially flat dynamical systems, see Zhevnin and Krishchenko (1981), Martin et al. (2003), Ryu and Agrawal (2010), Tang et al. (2011), Faulwasser et al. (2014). However, it is worthwhile to note that the time polynomial based approach, at least as it is often used, does not allow to explicitly account for constraints imposed on state variables and controls. Though, many authors were focused on adapting motion planning considerations relying upon polynomials to be applicable under various state and input constraints. Certain numerical procedures to meet geometrical, velocity and acceleration constraints are discussed in Faiz et al. (2001), Mohseni and Fakharian (2015), Mohseni and Fakharian (2016), Richter et al. (2016). Analytical ideas of constrained trajectory planning for mechanical systems can be found e.g. in the monograph Biagiotti and Melchiorri (2008) and in the paper Jond et al. (2016). In Golubev et al. (2017) for a class of affine dynamical systems polynomial-based trajectory generation was shown to meet the state constraints by proper selection of the time of motion.

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Nevertheless, in general case the task of polynomial trajectory generation under constraints still remains relevant and is not yet fully solved.

In this paper, we consider control-affine dynamical systems that can be written as a cascade of second-order subsystems of the form

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= f_i(t, x_i) + g_i(t, x_i) u_i, \end{aligned} \tag{1}$$

where $x_i = (x_{1i}, x_{2i})^{\mathrm{T}} \in \mathbb{R}^2$, $x = (x_1^{\mathrm{T}}, \ldots, x_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2m}$ is the state vector, $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$ is the control input, $f_i(t, x_i)$ and $g_i(t, x_i)$ are continuous in t and locally Lipschitz in x_i, $g_i(t, x_i) \neq 0$ for all $x_i \in \mathbb{R}^2$ and $t \geq 0$, $i \in \overline{1, m}$. Here the presence of the independent variable t in the functions $f_i(t, x_i)$ and $g_i(t, x_i)$ is implicit and due to their possible dependence on the x_k variables for $k \neq i$.

Without loss of generality let us think of t as time. Note that a fully actuated mechanical system with m degrees of freedom usually can be rendered as the nonlinear cascade (1).

Let us fix arbitrary initial $x_{1i}(0) = x_{0i}$, $x_{2i}(0) = \dot{x}_{0i}$ and desired final $x_{1i}(T) = x_{*i}$, $x_{2i}(T) = \dot{x}_{*i}$ values of the state variables x_i, $i \in \overline{1, m}$. Then, the motion planning problem in question is to construct a reference trajectory $x = x_r(t)$ in state space of the system (1) and the appropriate control law $u = u_r(t)$ to satisfy both the above boundary conditions and state and input constraints of the form

$$|x_{1i}(t)| \leq B_i, \ |x_{2i}(t)| \leq N_i, \ |u_i(t)| \leq L_i, \quad t \in [0, T], \tag{2}$$

where B_i, N_i and L_i are some given positive bounds, $i = \overline{1, m}$.

We assume that one has freedom to select the final value $T > 0$ of the independent variable t or/and at least some of the initial $x_{1i}(0)$, $x_{2i}(0)$ or final $x_{1i}(T)$, $x_{2i}(T)$ values of state variables x_i for each $i = \overline{1, m}$.

The paper is structured as follows. Third-order time polynomial based motion planning for a second-order subsystem of the form (1) subject to the state and input constraints is analyzed in section 2. Final time value T selection considerations are suggested to meet the constraints. It is shown that if the value of T belongs to a specific time interval which depends on the initial and final values of the x_i state variables the constraints are readily met. Section 3 propagates the results of section 2 to the whole chain of subsystems (1). Additionally, trajectory planning from a zero velocity point to a zero velocity point is addressed. As an illustrative example, trajectory construction for a 3-DoF Delta parallel robot is discussed in section 3. Finally, section 4 concludes with some remarks.

2. MOTION PLANNING FOR A SECOND-ORDER SUBSYSTEM

We start with reference trajectory construction for one of the cascade (1) subsystems that corresponds to some fixed i value. In view of the constraints (2) the initial and final values of the state variables x_{1i}, x_{2i} must satisfy

$$\max\{|x_{0i}|, |x_{*i}|\} \le B_i,\ \max\{|\dot{x}_{0i}|, |\dot{x}_{*i}|\} \le N_i. \quad (3)$$

Let the point $(x_{*i}, \dot{x}_{*i})$ on the phase plane (x_{1i}, x_{2i}) be reachable from the initial state $(x_{0i}, \dot{x}_{0i})$. Then, without loss of generality suppose that for the initial and final values of the state variables x_{1i}, x_{2i} the following inequalities hold:

$$\dot{x}_{0i}\dot{x}_{*i} \ge 0,\ \dot{x}_{0i}^2 + \dot{x}_{*i}^2 \ne 0,\ (\dot{x}_{0i} + \dot{x}_{*i})(x_{*i} - x_{0i}) > 0. \quad (4)$$

Indeed, if the conditions (4) are not satisfied, to connect the points $(x_{0i}, \dot{x}_{0i})$ and $(x_{*i}, \dot{x}_{*i})$ on the phase plane one can always consider a set of intermediate points. Then, by the choice of their coordinates the relevant versions of (4) can be readily fulfilled for intermediate trajectories connecting each pair of the points. For instance, if $\dot{x}_{0i}^2 + \dot{x}_{*i}^2 = 0$ holds, at least one intermediate point $(x_{li}, \dot{x}_{li})$ is introduced, with its coordinates satisfying the conditions

$$\begin{aligned} &x_{li} \in (\min\{x_{0i}, x_{*i}\}, \max\{x_{0i}, x_{*i}\}),\\ &0 < |\dot{x}_{li}| \le N_i,\ \dot{x}_{li}(x_{*i} - x_{0i}) > 0. \end{aligned} \quad (5)$$

As a t–parametrized curve that connects the points $(x_{0i}, \dot{x}_{0i})$ and $(x_{*i}, \dot{x}_{*i})$ on the phase plane (x_{1i}, x_{2i}) consider phase graphic $\bar{p}_i(t) = (p_i(t), \dot{p}_i(t))$, $t \in [0, T]$, of the polynomial

$$p_i(t) = x_{0i} + \dot{x}_{0i}t + c_{1i}t^2 + c_{2i}t^3, \quad (6)$$

with the coefficients c_{1i}, c_{2i} being found from the conditions $p_i(T) = x_{*i}$, $\dot{p}_i(T) = \dot{x}_{*i}$ and written as

$$\begin{aligned} c_{1i} &= -\big((2\dot{x}_{0i} + \dot{x}_{*i})T + 3(x_{0i} - x_{*i})\big)/T^2,\\ c_{2i} &= \big((\dot{x}_{0i} + \dot{x}_{*i})T + 2(x_{0i} - x_{*i})\big)/T^3. \end{aligned} \quad (7)$$

To comply with the considered constraints (2) the polynomial (6) and its time derivative are required to satisfy respectively the inequalities $|p_i(t)| \le B_i$ and $|\dot{p}_i(t)| \le N_i$ for all $t \in [0, T]$. To that end, by proper values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ such that the inequalities (3), (4) hold it is sufficient to provide the monotonicity property of the functions $p_i(t)$ and $\dot{p}_i(t)$ on the interval $t \in [0, T]$. Moreover, if the conditions (3), (4) are satisfied, the polynomial (6) and its time derivative are monotonic for all $t \in [0, T]$ if and only if the second-order time derivative

$$\ddot{p}_i(t) = 2c_{1i} + 6c_{2i}t \quad (8)$$

holds its sign for all $t \in (0, T)$.

Notice that in case when $c_{2i} \ne 0$ the function $\ddot{p}_i(t)$ given by (8) becomes zero at $t = -c_{1i}/(3c_{2i})$. Hence, since $\ddot{p}_i(t)$ is linear in t it has the same sign for all $t \in (0, T)$ if and only if $c_{2i} = 0$ or

$$c_{2i} \ne 0,\ -c_{1i}/(3c_{2i}) \le 0 \quad (9)$$

or

$$c_{2i} \ne 0,\ -c_{1i}/(3c_{2i}) \ge T. \quad (10)$$

2.1 Final time value selection

Let us first use the freedom to select the final value $T > 0$ of the independent variable t to guarantee that the constraints are met. Introduce the final time values

$$T_{2i} = \frac{3(x_{*i} - x_{0i})}{2\dot{x}_{0i} + \dot{x}_{*i}},\ T_{3i} = \frac{3(x_{*i} - x_{0i})}{\dot{x}_{0i} + 2\dot{x}_{*i}} \quad (11)$$

and define the interval

$$I_{Ti} = [\min\{T_{2i}, T_{3i}\}, \max\{T_{2i}, T_{3i}\}].$$

Then, the following result can be proved.

Theorem 1. Let the values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{i*}$ satisfy the conditions (4). The polynomial (6) with its coefficients given by (7) and its time derivative are monotonic functions of t on the interval $t \in [0, T]$ if and only if $T \in I_{Ti}$.

A direct corollary of theorem 1 reads as follows.

Theorem 2. Let the values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ satisfy the conditions (3) and (4). Then for any value of $T \in I_{Ti}$ the polynomial (6) with the coefficients c_{1i}, c_{2i} given by (7) and its time derivative satisfy for all $t \in [0, T]$ the constraints $|p_i(t)| \le B_i$ and $|\dot{p}_i(t)| \le N_i$, respectively.

Next, analyze a constraint $|\ddot{p}_i(t)| \le Q_i$, $t \in [0, T]$, on the second order time derivative (8) of the polynomial (6). Notice that the function $\ddot{p}_i(t)$ given by (8) due to its linearity in t is monotonic on the interval $t \in [0, T]$ for all values of x_{0i}, $\dot{x}_{0i}$, x_{*i}, $\dot{x}_{*i}$ and $T > 0$. Therefore, for the inequality $|\ddot{p}_i(t)| \le Q_i$ to hold for all $t \in [0, T]$ it is necessary and sufficient that the following condition is met:

$$\begin{aligned} &\max\{|\ddot{p}_i(0)|, |\ddot{p}_i(T)|\}\\ &\qquad = \max\{2|c_{1i}|, 2|c_{1i} + 3c_{2i}T|\} \le Q_i. \end{aligned} \quad (12)$$

Examine $\ddot{p}_i(0) = 2c_{1i}$ and $\ddot{p}_i(T) = 2c_{1i} + 6c_{2i}T$ as functions of the final time value $T \in I_{Ti}$. The first and second order derivatives of $\ddot{p}_i(0)$ with respect to T become zero at $T = 2T_2$ and $T = 3T_2$, respectively. Similarly, the first and second order derivatives of $\ddot{p}_i(T)$ with respect to T equal to zero at respectively $T = 2T_3$ and $T = 3T_3$. Then, one can show that if conditions of theorem 1 hold the inequalities $2T_2 > T_3$ and $2T_3 > T_2$ are always true.

Hence, to find maximum absolute values of $\ddot{p}_i(0)$ and $\ddot{p}_i(T)$ as functions of $T \in I_{Ti}$ it is sufficient to compare their values at $T = T_2$ and $T = T_3$.

If $T = T_2$ one gets $c_{1i} = 0$, $c_{2i} = (\dot{x}_{*i} - \dot{x}_{0i})/(3T_2^2)$ and $\ddot{p}_i(0) = 0$. Thus, the maximum absolute value of second order time derivative of the polynomial (6) on the interval $t \in [0, T_2]$ is achieved at $t = T_2$ and is the following

$$\ddot{p}_i(T_2) = 6c_{2i}T_2 = \frac{2(\dot{x}_{*i} - \dot{x}_{0i})(2\dot{x}_{0i} + \dot{x}_{*i})}{3(x_{*i} - x_{0i})}. \quad (13)$$

In case of $T = T_3$ the equalities $c_{1i} + 3c_{2i}T_3 = 0$, $c_{1i} = (\dot{x}_{*i} - \dot{x}_{0i})/T_3$ and $\ddot{p}_i(T_3) = 0$ hold. So, the maximum absolute value of second order time derivative of the polynomial (6) on the interval $t \in [0, T_3]$ is obtained at $t = 0$ and is written as

$$\ddot{p}_i(0) = 2c_{1i} = \frac{2(\dot{x}_{*i} - \dot{x}_{0i})(\dot{x}_{0i} + 2\dot{x}_{*i})}{3(x_{*i} - x_{0i})}. \tag{14}$$

Finally, the above considerations are summarized in the following statement.

Theorem 3. Let the values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ satisfy the conditions (4) and

$$\max\left\{\left|\frac{2(\dot{x}_{*i} - \dot{x}_{0i})(2\dot{x}_{0i} + \dot{x}_{*i})}{3(x_{*i} - x_{0i})}\right|, \left|\frac{2(\dot{x}_{*i} - \dot{x}_{0i})(\dot{x}_{0i} + 2\dot{x}_{*i})}{3(x_{*i} - x_{0i})}\right|\right\} \le Q_i. \tag{15}$$

Then for any value of $T \in I_{Ti}$ the second order time derivative of the polynomial (6) with the coefficients c_{1i}, c_{2i} given by (7) satisfies for all $t \in [0, T]$ the constraint $|\ddot{p}_i(t)| \le Q_i$.

Remark 1. Theorem 3 presents sufficient conditions for the boundedness with prescribed bounds property of the polynomial (6) second order time derivative. These conditions are given in terms of the final time value T and those of the values x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ that are not fixed by the problem statement and can be varied. In case of a mechanical system of the form (1) theorem 3 gives sufficient conditions for the acceleration boundedness property of the constructed reference trajectory.

Notice that the feedforward control law $u_i = u_i(t)$ that moves the second-order subsystem of the nonlinear cascade (1) in question along the constructed reference trajectory $x_{1i} = p_i(t)$, $x_{2i} = \dot{p}_i(t)$, $t \in [0, T]$, can be readily written using nonlinear dynamics inversion as

$$u_i = \frac{1}{g_i(t, \bar{p}_i(t))}\big(-f(t, \bar{p}_i(t)) + \ddot{p}_i(t)\big),\ t \in [0, T]. \tag{16}$$

Henceforth, if the constraints $|p_i(t)| \le B_i$, $|\dot{p}_i(t)| \le N_i$ and $|\ddot{p}_i(t)| \le Q_i$ are satisfied for all $t \in [0, T]$ then from the continuity property of the functions $f_i(t, x_i)$ and $g_i(t, x_i)$ follows that the condition $|u_i(t)| \le L_i$ holds for all $t \in [0, T]$ with some $L_i > 0$. Moreover, any given control bound L_i can be guaranteed by properly adjusting the bounds B_i, N_i and Q_i.

2.2 Final x_{2i} value selection

Notice that for a pre-given value of $T > 0$ the condition $T \in I_{Ti}$ in theorems 1–3 can be satisfied by properly selecting those of the x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ values that are not fixed by the problem statement. In this case, one can take suitable T_{2i}, T_{3i} and consider the equalities (11) as a system of linear algebraic equations with respect to the relevant initial and final values of the state variables x_{1i}, x_{2i}.

For a fixed value of $T > 0$ let us provide the monotonicity property of the polynomial (6) with the coefficients (7) and its time derivative on the interval $t \in [0, T]$ by choosing, for instance, the final value $x_{2i}(T) = \dot{x}_{*i}$ of the x_{2i} state variable. Let

$$\bar{\dot{x}}^1_{*i} = \frac{2(x_{*i} - x_{0i})}{T} - \dot{x}_{0i},\ \bar{\dot{x}}^2_{*i} = \frac{3(x_{*i} - x_{0i})}{T} - 2\dot{x}_{0i},$$
$$\bar{\dot{x}}^3_{*i} = \frac{3(x_{*i} - x_{0i})}{2T} - \frac{\dot{x}_{0i}}{2}.$$

and

$$I_{xi} = [\min\{\bar{\dot{x}}^2_{*i}, \bar{\dot{x}}^3_{*i}\}, \max\{\bar{\dot{x}}^2_{*i}, \bar{\dot{x}}^3_{*i}\}].$$

In view of (7) the inequalities (9) combined with $c_{2i} = 0$ can be written as $\bar{\dot{x}}^2_{*i} \le \dot{x}_{*i} \le \bar{\dot{x}}^1_{*i}$ if $(x_{*i} - x_{0i})/T \le \dot{x}_{0i}$ or $\bar{\dot{x}}^1_{*i} \le \dot{x}_{*i} \le \bar{\dot{x}}^2_{*i}$ when $(x_{*i} - x_{0i})/T \ge \dot{x}_{0i}$. Similarly, the conditions (10) due to (7) take the form $\bar{\dot{x}}^1_{*i} < \dot{x}_{*i} \le \bar{\dot{x}}^3_{*i}$ if $(x_{*i} - x_{0i})/T < \dot{x}_{0i}$ or $\bar{\dot{x}}^3_{*i} \le \dot{x}_{*i} < \bar{\dot{x}}^1_{*i}$ in case of $(x_{*i} - x_{0i})/T > \dot{x}_{0i}$. Thereby, the following result holds.

Theorem 4. Let the values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$ satisfy the conditions (4). The polynomial (6) with its coefficients given by (7) and its time derivative are monotonic functions of t on the interval $t \in [0, T]$ if and only if $\dot{x}_{*i} \in I_{xi}$. Here $I_{xi} = [\bar{\dot{x}}^2_{*i}, \bar{\dot{x}}^3_{*i}]$ if $(x_{*i} - x_{0i})/T \le \dot{x}_{0i}$ and $I_{xi} = [\bar{\dot{x}}^3_{*i}, \bar{\dot{x}}^2_{*i}]$ when $(x_{*i} - x_{0i})/T \ge \dot{x}_{0i}$.

In terms of the considered constraints (2) theorem 4 reads as below.

Theorem 5. Let the values of x_{0i}, $\dot{x}_{0i}$, x_{*i}, $\dot{x}_{*i}$ and $T > 0$ satisfy the conditions (3), (4) and $\dot{x}_{*i} \in I_{xi}$. Then the polynomial (6) with the coefficients c_{1i}, c_{2i} given by (7) and its time derivative satisfy for all $t \in [0, T]$ the constraints $|p_i(t)| \le B_i$ and $|\dot{p}_i(t)| \le N_i$, respectively.

Next, similarly to the final time value selection considerations of the previous subsection let us analyze $\ddot{p}_i(0) = 2c_{1i}$ and $\ddot{p}_i(T) = 2c_{1i} + 6c_{2i}T$ in the inequality (12) as functions of $\dot{x}_{i*} \in I_{xi}$. Then, to find the maximum absolute values of $\ddot{p}_i(0)$ and $\ddot{p}_i(T)$ as functions of the $\dot{x}_{i*}$ variable due to their linearity in $\dot{x}_{i*}$ it is sufficient to compare their values at $\dot{x}_{*i} = \bar{\dot{x}}^2_{*i}$ and $\dot{x}_{*i} = \bar{\dot{x}}^3_{*i}$.

If $\dot{x}_{*i} = \bar{\dot{x}}^2_{*i}$ one has $c_{1i} = 0$, $\ddot{p}_i(0) = 0$ and $T = T_2$. Therefore, the maximum absolute value of second order time derivative of the polynomial (6) on the interval $t \in [0, T]$ is reached at $t = T$ and is given by (13).

For $\dot{x}_{*i} = \bar{\dot{x}}^3_{*i}$ the equalities $c_{1i} + 3c_{2i}T = 0$, $\ddot{p}_i(T) = 0$ and $T = T_3$ are true. Hence, the maximum absolute value of the polynomial (6) second order time derivative on the interval $t \in [0, T]$ is obtained at $t = 0$ and coincides with (14).

Consequently, the following counterpart of theorem 3 can be formulated.

Theorem 6. Let the values of x_{0i}, $\dot{x}_{0i}$, x_{*i}, $\dot{x}_{*i}$ and $T > 0$ satisfy the conditions (4), (15) and $\dot{x}_{i*} \in I_{xi}$. Then the second order time derivative of the polynomial (6) with the coefficients c_{1i}, c_{2i} given by (7) satisfies for all $t \in [0, T]$ the constraint $|\ddot{p}_i(t)| \le Q_i$.

3. MOTION PLANNING FOR THE CHAIN OF SUBSYSTEMS

Consider now the trajectory planning problem for the whole cascade given by (1). The following theorem is a corollary of the theorems 2 and 3.

Theorem 7. Let the values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$, $i \in \overline{1, m}$, satisfy the conditions (3), (4), (15) and the interval $I_T = I_{T1} \cap \ldots \cap I_{Tm}$ be non-empty. Then for any value of $T \in I_T$ for all $i \in \overline{1,m}$ the polynomials (6) with the coefficients c_{1i}, c_{2i} given by (7) and their first and second order time derivatives satisfy for all $t \in [0,T]$ the constraints $|p_i(t)| \leq B_i$, $|\dot{p}_i(t)| \leq N_i$ and $|\ddot{p}_i(t)| \leq Q_i$, $i \in \overline{1,m}$, respectively.

Notice that if for the considered values of x_{0i}, $\dot{x}_{0i}$ and x_{*i}, $\dot{x}_{*i}$, $i \in \overline{1,m}$, the set I_T is empty then to apply the above theorem one has to adjust relevant initial or final state variables values so that I_T becomes non-empty.

3.1 Trajectory planning from a zero velocity point to a zero velocity point

In particular, let us analyze the case when $\dot{x}_{0i} = \dot{x}_{*i} = 0$ for all $i = \overline{1,m}$. Then, the above theorems from sections 2 and 3 cannot be directly applied to the trajectory planning since they all require that for each $i = \overline{1,m}$ at least one of the values $\dot{x}_{0i}$ and $\dot{x}_{*i}$ is nonzero.

Still, one can always make the above theorems applicable when constructing motion from the initial state $(x_{01}, 0, \ldots, x_{0m}, 0)^{\mathrm{T}}$ to the final state $(x_{*1}, 0, \ldots, x_{*m}, 0)^{\mathrm{T}}$ by introducing at least one intermediate point $(x_{l1}, \dot{x}_{l1}, \ldots$ $\ldots, x_{lm}, \dot{x}_{lm})^{\mathrm{T}}$ in state space of the system (1). Here the values of x_{li} and $\dot{x}_{li}$, $i = \overline{1,m}$, satisfy the conditions (5).

To connect the points $(x_{01}, 0, \ldots, x_{0m}, 0)^{\mathrm{T}}$ and $(x_{l1}, \dot{x}_{l1}, \ldots$ $\ldots, x_{lm}, \dot{x}_{lm})^{\mathrm{T}}$ in state space of the system (1) within a time interval $[0, T_l]$ one can use polynomials (6) that take the form

$$p_i(t) = x_{0i} + c_{1i}t^2 + c_{2i}t^3, \tag{17}$$

where $t \in [0, T_l]$, $i = \overline{1,m}$, and the coefficients c_{1i}, c_{2i} are written as below

$$\begin{aligned} c_{1i} &= -\big(\dot{x}_{li}T_l + 3(x_{0i} - x_{li})\big)/T_l^2, \\ c_{2i} &= \big(\dot{x}_{li}T_l + 2(x_{0i} - x_{li})\big)/T_l^3. \end{aligned} \tag{18}$$

Next, to construct a system trajectory from the intermediate state $(x_{l1}, \dot{x}_{l1}, \ldots \ldots, x_{lm}, \dot{x}_{lm})^{\mathrm{T}}$ to the final state $(x_{*1}, 0, \ldots, x_{*m}, 0)^{\mathrm{T}}$ on a time interval $[T_l, T_l + T_*]$ we use polynomials

$$\tilde{p}_i(t) = x_{li} + \dot{x}_{li}(t - T_l) + \tilde{c}_{1i}(t - T_l)^2 + \tilde{c}_{2i}(t - T_l)^3, \tag{19}$$

where $t \in [T_l, T_l + T_*]$, $i = \overline{1,m}$, and

$$\begin{aligned} \tilde{c}_{1i} &= -\big(2\dot{x}_{li}T_* + 3(x_{li} - x_{*i})\big)/T_*^2, \\ \tilde{c}_{2i} &= \big(\dot{x}_{li}T_* + 2(x_{li} - x_{*i})\big)/T_*^3. \end{aligned} \tag{20}$$

By theorem 7 one takes the values of T_l, T_*, x_{li} and $\dot{x}_{li}$, $i = \overline{1,m}$, to satisfy the inequalities

$$\begin{aligned} \frac{3(x_{li} - x_{0i})}{\dot{x}_{li}} &\leq (\geq) T_l \leq (\geq) \frac{3(x_{li} - x_{0i})}{2\dot{x}_{li}}, i = \overline{1,m}, \\ \frac{3(x_{*i} - x_{li})}{2\dot{x}_{li}} &\leq (\geq) T_* \leq (\geq) \frac{3(x_{*i} - x_{li})}{\dot{x}_{li}}, i = \overline{1,m}. \end{aligned}$$

Notice that to avoid discontinuity of the second order time derivatives (accelerations) at $t = T_l$ one has to satisfy the conditions $\ddot{p}_i(T_l) = 2c_{1i} + 6c_{2i}T_l = 2\tilde{c}_{1i} = \ddot{\tilde{p}}_i(T_l)$ for all $i = \overline{1,m}$. For instance, the choice

$$T_l = \frac{3(x_{li} - x_{0i})}{2\dot{x}_{li}}, \; T_* = \frac{3(x_{*i} - x_{li})}{2\dot{x}_{li}} \tag{21}$$

results in $\ddot{p}_i(T_l) = \ddot{\tilde{p}}_i(T_l) = 0$ for all $i = \overline{1,m}$.

Then, for any fixed values of $T_l > 0$ and $T_* > 0$ consider the equalities (21) as a system of linear algebraic equations with respect to the x_{li} and $\dot{x}_{li}$, $i = \overline{1,m}$, variables. Hereby one gets

$$x_{li} = \frac{x_{*i}T_l + x_{0i}T_*}{T_l + T_*}, \; i = \overline{1,m}, \tag{22}$$

$$\dot{x}_{li} = \frac{3(x_{*i} - x_{0i})}{2(T_l + T_*)}, \; i = \overline{1,m}. \tag{23}$$

Notice that the values of $\dot{x}_{li}$ given by (23) satisfy the constraints $|\dot{x}_{li}| \leq N_i$, $i = \overline{1,m}$, if T_l, T_* are such that the following inequalities hold

$$\frac{3|x_{*i} - x_{0i}|}{2(T_l + T_*)} \leq N_i, \; i = \overline{1,m}. \tag{24}$$

In this case, the conditions (5) are always satisfied for x_{li} and $\dot{x}_{li}$, $i = \overline{1,m}$, calculated as (22) and (23), respectively.

Finally, to provide boundedness of the functions $\ddot{p}_i(t)$ and $\ddot{\tilde{p}}_i(t)$ with the prescribed bounds $\pm Q_i$, $i = \overline{1,m}$, for all $t \in [0, T_l]$ and $t \in [T_l, T_l + T_*]$, respectively, the constraints (15) take the form

$$\begin{aligned} &\max\left\{\frac{4\dot{x}_{li}^2}{3|x_{li} - x_{0i}|}, \frac{4\dot{x}_{li}^2}{3|x_{li} - x_{*i}|}\right\} \\ &= \max\left\{\frac{3|x_{*i} - x_{0i}|}{T_l(T_l + T_*)}, \frac{3|x_{*i} - x_{0i}|}{T_*(T_l + T_*)}\right\} \leq Q_i, \; i = \overline{1,m}. \end{aligned} \tag{25}$$

Example. Consider motion planning for a 3-DoF Delta parallel robot, with its dynamics written as (see e.g. Olsson (2009), Golubev et al. (2017))

$$\ddot{\theta} = A^{-1}(\theta)(\tau - C(\theta, \dot{\theta})\dot{\theta} - G(\theta)), \tag{26}$$

where $\theta = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}} \in \mathbb{R}^3$ is the vector of angles between the three upper arms and the base plate, $\tau \in \mathbb{R}^3$ is the vector of torques acting on the upper arms, $A(\theta)$ is the inertia matrix, $C(\theta, \dot{\theta})$ stands for the centrifugal and Coriolis forces, $G(\theta)$ accounts for the gravity forces. The robot travelling plate (end effector) position is described by the vector $X = (x, y, z)^{\mathrm{T}}$ of three Cartesian coordinates that can be transformed into the θ angles using the inverse kinematics, see Olsson (2009).

Notice that a typical control problem for a pick and place Delta robot is to move the end effector from a starting point $X_0 = (x_0, y_0, z_0)^{\mathrm{T}}$ to some terminal point $X_* = (x_*, y_*, z_*)^{\mathrm{T}}$ in the space. Using the robot inverse kinematics, see Olsson (2009), the initial $X = X_0$ and the desired final $X = X_*$ end effector positions are written in the θ space as $\theta = \theta_0 = (\theta_{01}, \theta_{02}, \theta_{03})^{\mathrm{T}}$ and $\theta = \theta_* = (\theta_{*1}, \theta_{*2}, \theta_{*3})^{\mathrm{T}}$, respectively.

Then, the motion planning problem in question is to find the control law $\tau = \tau_r(t)$ that moves the robot upper arms from the initial angular position $\theta(0) = \theta_0$ with $\dot{\theta}(0) = 0$ to the final angular position $\theta(T) = \theta_*$ with $\dot{\theta}(T) = 0$ along a reference trajectory $\theta = \theta_r(t)$ in the θ space. The constructed reference trajectory $\theta = \theta_r(t)$ is required to meet the geometrical $|\theta_{ir}(t)| \leq \pi/2$, angular velocity $|\dot{\theta}_{ir}(t)| \leq N_i$ and angular acceleration $|\ddot{\theta}_{ir}(t)| \leq Q_i$ constraints for all $t \in [0, T]$, where N_i and Q_i are some given positive bounds, $i = \overline{1,3}$. Here the final time value $T > 0$ can be fixed by the problem statement or passed over to be defined later.

Let us take $T_l = T_* = T/2$. To connect the initial state

$$(\theta_1(0), \dot{\theta}_1(0), \theta_2(0), \dot{\theta}_2(0), \theta_3(0), \dot{\theta}_3(0))^{\mathrm{T}} = (\theta_{01}, 0, \theta_{02}, 0, \theta_{03}, 0)^{\mathrm{T}}$$

with an intermediate state

$$(\theta_1(T_l), \dot{\theta}_1(T_l), \theta_2(T_l), \dot{\theta}_2(T_l), \theta_3(T_l), \dot{\theta}_3(T_l))^{\mathrm{T}} = (\theta_{l1}, \dot{\theta}_{l1}, \theta_{l2}, \dot{\theta}_{l2}, \theta_{l3}, \dot{\theta}_{l3})^{\mathrm{T}} \quad (27)$$

in phase space of the system (26) one can use polynomials $p_i(t)$, $i = \overline{1,3}$, given by (17), (18), where the x-labeled values are replaced with the relevant θ-labeled ones. Here, in (27) the values of θ_{li}, $\dot{\theta}_{li}$, $i = \overline{1,3}$, are given by the fomulae (22) and (23), respectively, written in terms of the θ-labeled variables.

Then, to construct a trajectory from the intermediate state (27) to the final state

$$(\theta_1(T), \dot{\theta}_1(T), \theta_2(T), \dot{\theta}_2(T), \theta_3(T), \dot{\theta}_3(T))^{\mathrm{T}} = (\theta_{*1}, 0, \theta_{*2}, 0, \theta_{*3}, 0)^{\mathrm{T}}$$

one takes proper polynomials $\tilde{p}_i(t)$, $i = \overline{1,3}$, of the form (19), (20) with the replaced x-labeled values.

Here the final time value $T > 0$ has to be consistent with the velocity and acceleration bounds, i.e. the relevant inequalities (24) and (25) written as

$$-N_i \leq \frac{3(\theta_{*i} - \theta_{0i})}{2T} \leq N_i, \ i = \overline{1,3},$$

and

$$-Q_i \leq \frac{6(\theta_{*i} - \theta_{0i})}{T^2} \leq Q_i, \ i = \overline{1,3},$$

respectively, are required to be satisfied.

Finally, the constructed reference trajectory is as below

$$\theta_i = \theta_{ir}(t) = \begin{cases} p_i(t), & \text{if } 0 \leq t \leq T/2, \\ \tilde{p}_i(t), & \text{if } T/2 \leq t \leq T, \end{cases}$$

$i = \overline{1,3}$, with the feedforward control law given by

$$\tau = \tau_r(t) = A(\theta_r(t))\ddot{\theta}_r(t) + C(\theta_r(t), \dot{\theta}_r(t))\dot{\theta}_r(t) + G(\theta_r(t)), \quad (28)$$

where $t \in [0, T]$.

Figures 1 – 4 show numerical simulation results for system (26) with the control law (28). The following initial $\theta_0 = (-0.9106, -0.2294, 0.2568)^{\mathrm{T}}$ and final $\theta_* = (-0.516, 1.04, 0.4541)^{\mathrm{T}}$ upper arms angles were considered, with the angle values corresponding to the travelling plate initial $X_0 = (0.2, 0.1, -0.5)^{\mathrm{T}}$ and final $X_1 = (0.4, -0.2, -0.6)^{\mathrm{T}}$ Cartesian coordinates, respectively. Angular velocity and acceleration bounds were taken as below $N_i = 2\ rad/s$ and $Q_i = 8\ rad/s^2$, $i = \overline{1,3}$.

4. CONCLUSION

In this paper, point-to-point motion planning for a chain of second-order controlled subsystems was considered. The contributions of the current paper are time polynomial based analytical considerations to meet the state and input constraints. Third-order time polynomials were used to construct the reference trajectories. It was shown that the constraints are met by proper selection of the time of motion or/and e.g. final values of the state variables which by their virtue are position time derivatives, i.e. velocities. Within the suggested approach, trajectory planning from a zero velocity point to a zero velocity point was analyzed.

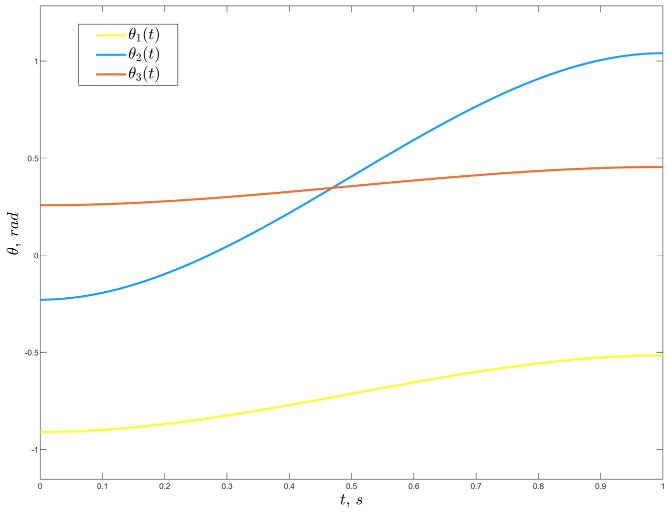

Fig. 1. Reference angular trajectory $\theta_r(t)$ (rad) versus time (s)

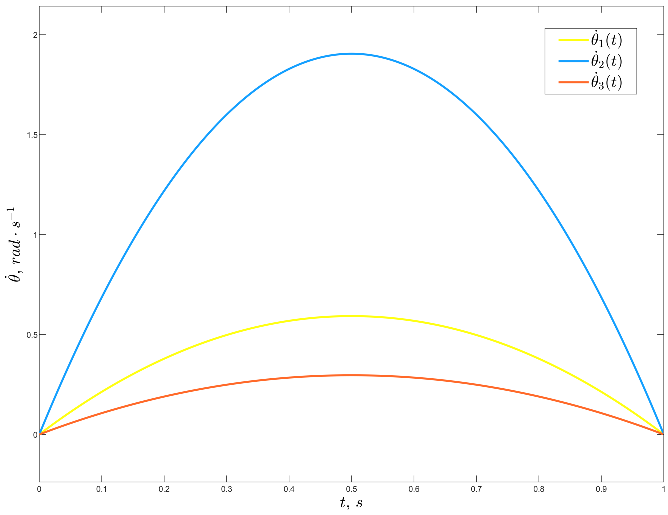

Fig. 2. Reference angular velocites $\dot{\theta}_r(t)$ (rad/s) versus time (s)

It is worth to stress that the current paper was focused on the motion planning problem. To find stabilizing feedback control laws which track the constructed reference trajectories and account for the state and input constraints one can readily use the integrator backstepping considerations based on barrier Lyapunov functions, see e.g. Ngo et al. (2005), Golubev et al. (2019a). Additionally, let us note that to construct time polynomial based reference trajectories for differentially flat dynamical systems which model underactuated mechanical plants, polynomials that have order more than three are likely to be required. So, future research can be focused on extending the obtained results to higher order time polynomial based trajectories.

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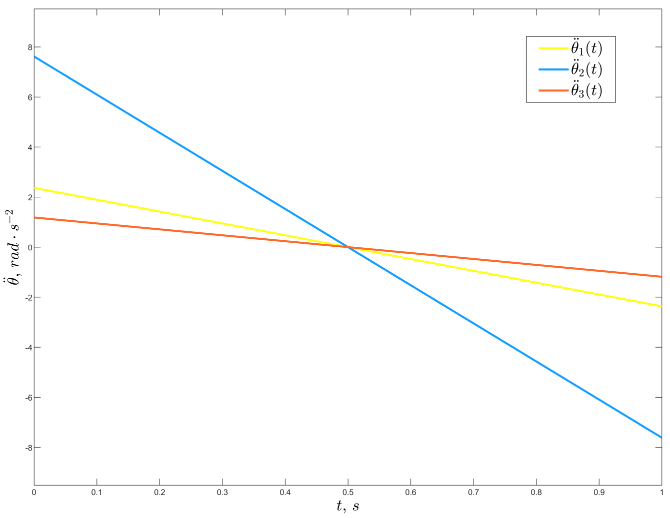

Fig. 3. Reference angular accelerations $\ddot{\theta}_r(t)$ (rad/s^2) versus time (s)

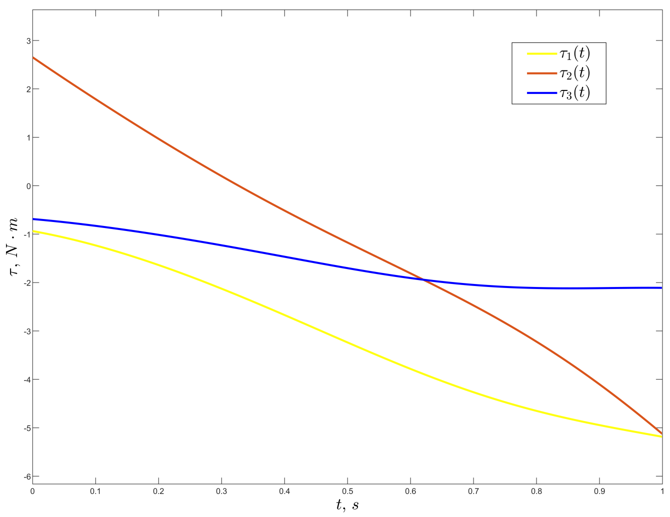

Fig. 4. Control law $\tau_r(t)$ $(N \cdot m)$ versus time (s)

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