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Disturbance Induced Synchronization in Networked Oscillatory System

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Abstract: The problem of synchronization of oscillators that are not coupled directly but are connected to a common disturbance has been studied in this work. Mathematical formalisms to obtain sufficient coupling gain for synchronization of a complex network of oscillators forced by a common disturbance have been developed. The Lyapunov stability approach using quadratic Lyapunov function and non-smooth Lyapunov function has been used for the purpose. The networked oscillatory systems considered are formed by Van der Pol and Fitzhugh Nagumo oscillators independently. These oscillators are diffusively coupled to common disturbance. The comparison of analytical approaches has been made, and the results are validated through numerical simulations.

Keywords: Disturbance induced synchronization, Van der Pol, Fitzhugh Nagumo, Oscillators.

1. INTRODUCTION

The phenomena of synchronization bear relevance in several systems such as physical systems, life sciences, and social systems. The synchronization of oscillators in the networked oscillatory system is possible even when all the oscillators are not directly coupled but coupled diffusively via a common disturbance. The development of mathematical formalism to explain synchronization in such a network is rather a difficult task. It becomes even more challenging when oscillators forming the network are of second-order type. Efforts have been made in this work to obtain sufficient coupling gain, which gives synchronization of all the oscillators when they are equally affected or forced by a common disturbance. The synchronization thus arrived is known as disturbance induced synchronization in the second-order oscillatory system.

In Camacho (2004), the authors could show that for some values of the parameters , there exists two modes namely in-phase and out of phase modes of synchronization when two Van der Pol oscillators are coupled via bath. Also, oscillators exhibit stability in both modes but for out of phase mode the stability exists for wider range of parameters as compared to in-phase mode. It is found that the stability of in-phase mode is reduced due to effect of bath. The authors have used Floquet theory and numerical integration to obtain the results. In Rompala (2007), the authors have investigated the dynamics of two identical oscillators, which are not coupled directly but coupled through the third oscillator. They could prove that the two oscillators possessed the in-phase mode in which two oscillators show similar behavior. The authors have used a variable expansion perturbation method to get a slow flow. It was then analyzed through algebra known as MACSYMA and numerical bifurcation software AUTO. In Belykh (2008), the authors could show that in a system of two Van der Pol duffing oscillators having Huygens coupling, there existed two chaotic phase synchronized modes simultaneously. The authors found that the structure of basins for these modes. The main regimes of complete and phase synchronization could be obtained, and it could be proved that the onset of such synchronization depends on initial conditions. Their system approximates the Huygens synchronization of two mechanical clocks that hang from common support. In Ahmed (2016), authors have focused on the problem of synchronization of genetic oscillators which undergo cell division due to common entraining inputs. The paper proposes sufficiency conditions for phase synchronization of oscillators. The sufficiency conditions for synchronization have been derived in the space of the phase response curve. Authors have proved that the presence of common entrainment input yields the boundedness of synchronization error. The results have been validated through numerical simulations. In Vlasov (2016), the authors could show the existence of remote synchronization in a network of Kuramoto oscillators in the presence of the repulsive mean-field. In remote synchronization central node serves as a transmitter and avail information to all connected nodes. They could observe the effect of changes in the natural frequency of the hub node, on the synchronization of the connected oscillators. Authors have discussed the applications of such networks in real-world problems and in neuronal networks in the brain. The motivation for this work is drawn from the model of two human eyes, which are linked to the brain via the pineal gland. Two eyes are not directly connected to each other but operate in synchrony due to changes in concentration of the melatonin hormone, which is responsible for regulating sleep and wake cycles in humans. Thus melatonin hormone provides bath coupling between the eyes, Camacho (2004).

It is, therefore, in present work, the problem of synchronization of oscillators, which are not directly coupled but are coupled to a common disturbance, has been addressed. It is aimed to determine sufficient value of coupling gain required by individual oscillators with the common disturbance, that yields synchronization of all the connected oscillators. For this, we have used Lyapunov stability analysis using quadratic Lyapunov function, and the non-smooth Lyapunov function to compute the sufficient condition for synchronization. The main contributions of present work are

(i) Occurrence of synchronization in independent oscillators not coupled directly but, coupled via a common disturbance.
(ii) Sufficient coupling gains to achieve synchronization in such a network using the Quadratic Lyapunov function approach and Non-smooth Lyapunov function approach have been derived for independent systems comprising of second-order Van der Pol and Fitzhugh Nagumo oscillators.
(iii) The outcomes for these approaches for Van der Pol and Fitzhugh Nagumo oscillators have been verified by simulation results.

In section 2, the standard second-order oscillators viz, Van der Pol, and Fitzhugh Nagumo oscillators are explained. In section 3, the definition of synchronization is given. In section 4, synchronization analysis is carried out using a quadratic Lyapunov function approach. In section 5, synchronization analyses have been carried out using the non-smooth Lyapunov function. Next, in section 6, derived results have been verified by numerical simulations and concluded in section 7.

2. DYNAMICS OF OSCILLATORS

This section explains the state-space representation of standard relaxation oscillators viz. Van der Pol and Fitzhugh Nagumo oscillators.

2.1 Van der Pol oscillator

Van der Pol oscillator is extensively used to model oscillations in electronic circuits and biological sciences. It may be represented in its two-dimensional form as under,

$$\dot{x_i} = \mu(x_i - y_i - \frac{1}{3}x_i^3) \quad (1)$$

$$\dot{y_i} = \frac{1}{\mu}x_i \quad (2)$$

x_i and y_i are the state variables, and μ is a system parameter. Fig.1. shows simulation of (1) and (2) using ode45 in Matlab for $\mu = 1$ and $(-1, 3)$ taken as initial conditions.

2.2 Fitzhugh Nagumo oscillator

It is a second-order oscillator model of a neuron. It is widely used in neuroscience to explain neuronal behavior. Its dynamics is given as,

$$\dot{v_i} = c[v_i - w_i - \frac{1}{3}v_i^3] + I \quad (3)$$

$$\dot{w_i} = d[a + bv_i - w_i] \quad (4)$$

v_i is the membrane potential, w_i represent recovery variable, $a = 2, b = 1, c = 1$ and $d = 0.1$ are the positive constants and $I = 2$ is the external current. Fig.2. oscillatory response of neuronal membrane potential for initial conditions (-1,4).

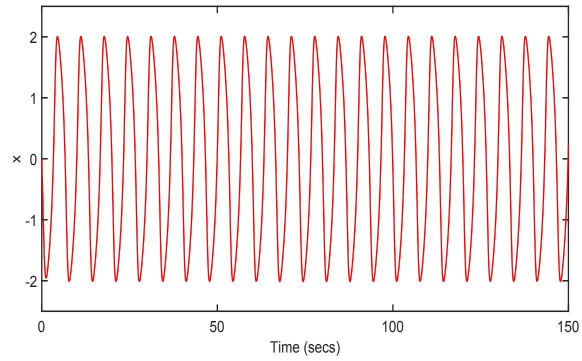

Fig. 1. Oscillations of Van der Pol oscillator

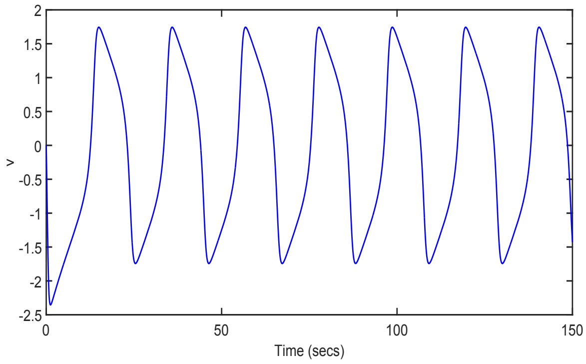

Fig. 2. Oscillations of Fitzhugh Nagumo oscillators

2.3 Oscillators coupled to a common disturbance source

In this section, we explain the dynamics of oscillators, which are coupled to a common disturbance source. Fig.3. shows two oscillators that are not directly coupled but are coupled to a common disturbance source. The dynamics of Van der Pol oscillator in such a network connection be given as, The dynamics of Van der Pol oscillators coupled

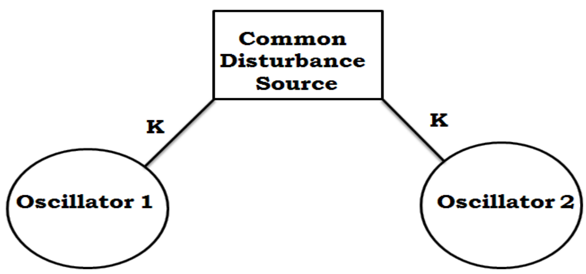

Fig. 3. Two oscillators coupled to a common disturbance source with uniform coupling gain K.

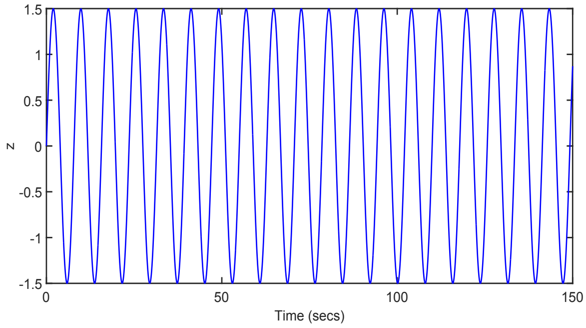

Fig. 4. Common forcing disturbance $z = Asin\omega t$ for A= 1.5, $\omega = 0.8$.

to a common disturbance in a diffusive manner be given as,

$$\dot{x}_i = \mu[x_i - y_i - \frac{x_i^3}{3}] + K(z - x_i) \quad (5)$$

$$\dot{y}_i = \frac{1}{\mu}x_i \qquad \forall i = 1,2 \quad (6)$$

Also, the dynamics of Van der Pol oscillators with both states coupled is given as,

$$\dot{x}_i = \mu[x_i - y_i - \frac{x_i^3}{3}] + K(z - x_i) \quad (7)$$

$$\dot{y}_i = \frac{1}{\mu}x_i + K(z - x_i) \qquad \forall i = 1,2 \quad (8)$$

The common disturbance is,

$$z = Asin\omega t \quad (9)$$

Similarly, the dynamical equations of Fitzhugh Nagumo oscillator coupled to a common disturbance source be given as,

$$\dot{v}_i = c[v_i - w_i - \frac{v_i^3}{3}] + I + K(z - v_i) \quad (10)$$

$$\dot{w}_i = d[a + bv_i - w_i] \qquad \forall i = 1,2 \quad (11)$$

3. DEFINITION OF SYNCHRONIZATION

3.1 Definition III.1

The oscillators are said to synchronize if the difference between their corresponding states becomes zero asymptotically Joshi (2016), Joshi (2017), Joshi (2020). Here, the Van der Pol oscillators are said to be synchronized if,

$$(x_j(t) - x_i(t)) = 0 \text{ as } t \to \infty \;\forall \text{ i, j} = (1,\dots,\text{N}) \quad (12)$$

$$(y_j(t) - y_i(t)) = 0 \text{ as } t \to \infty \;\forall \text{ i, j} = (1,\dots,\text{N}) \quad (13)$$

Similarly, Fitzhugh Nagumo oscillators are said to be synchronized if,

$$(v_j(t) - v_i(t)) = 0 \text{ as } t \to \infty \;\; \forall \text{i, j} = (1,\dots,\text{N}) \quad (14)$$

$$(w_j(t) - w_i(t)) = 0 \text{ as } t \to \infty \;\; \forall \text{i, j} = (1,\dots,\text{N}) \quad (15)$$

4. QUADRATIC LYAPUNOV FUNCTION APPROACH

In this section, sufficient coupling gain for synchronization of each type of second-order benchmark oscillators has been derived using quadratic Lyapunov function based stability analysis when the oscillators are not directly coupled to each other but are coupled to a common oscillatory bounded disturbance.

4.1 Van der Pol oscillators

Theorem 4.1: Consider system dynamics (5) and (6), such that z is bounded oscillatory disturbance then, Van der Pol oscillators independently coupled with the common forcing disturbance synchronize asymptotically if, $K > \mu$.

Proof: Let,

$$E_{12} = \frac{1}{2}[(x_1 - x_2)^2 + \mu^2((y_1 - y_2)^2]$$

$$\dot{E}_{12} = (x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + (y_1 - y_2)(\dot{y}_1 - \dot{y}_2)$$

$$\dot{E}_{12} = (\mu - K)(x_1 - x_2)^2 - \frac{\mu}{3}(x_1 - x_2)^2 \times [\frac{(x_1)^2}{2} + \frac{(x_2)^2}{2} + \frac{(x_1 + x_2)^2}{2}]$$

Now, E_{12} is negative semi-definite if

$$(\mu - K)(x_1 - x_2)^2 < 0$$

$$\Rightarrow K > \mu$$

It is therefore, $\dot{E}_{12}$ is negative semi-definite if each of the coupling gain $K > \mu$. This ensures, $(x_1 - x_2) \to 0$ and the boundedness of $(y_1 - y_2)$. Consider set $A = \Big\{(y_1 - y_2) \in R \quad \forall 1,2|\dot{S} = 0\Big\}$ and let B be the largest invariant set contained in A where $(x_1 - x_2) = 0, (y_1 - y_2) = 0, (\dot{x}_1 - \dot{x}_2) = 0$ and $(\dot{y}_1 - \dot{y}_2) = 0$ as per (5) to (9). Using Lasalle's invariance principle, all trajectories $(\dot{x}_1 - \dot{x}_2), (\dot{y}_1 - \dot{y}_2) \quad \forall i = 1,2$ converge to B as $t \to \infty$. Thus, two linearly coupled Van der Pol oscillators which are coupled to common forcing disturbance synchronize asymptotically as $t \to \infty$.

4.2 Fitzhugh Nagumo oscillators

Theorem 4.2: Consider system dynamics (10) and (11), such that z is bounded oscillatory disturbance then, Fitzhugh Nagumo oscillators independently coupled with the common forcing disturbance synchronize asymptotically if, $K > c$.

Proof: Refer Appendix I.

5. NON-SMOOTH LYAPUNOV FUNCTION APPROACH

In this section, sufficient coupling gain for synchronization of each type of second-order benchmark oscillators has been derived using non-smooth Lyapunov function based stability analysis when the oscillators are not directly coupled to each other but are coupled to a common oscillatory bounded disturbance.

5.1 Van der Pol oscillators

Theorem 5.1: Consider system dynamics (7) and (8), such that z is bounded oscillatory disturbance then, Van der Pol oscillators independently coupled with the common forcing disturbance synchronize asymptotically if, $K > 2\mu$.

Proof: Consider two non-smooth Lyapunov functions,

$$V(x(t)) = [max|x_i - x_j||i, j\epsilon(1,2)] \quad (16)$$

$$V(y(t)) = [max|y_i - y_j||i, j\epsilon(1,2)] \quad (17)$$

Now, upper Dini-derivatives are given as,

$$D^+V(x(t)) = \limsup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h} = \dot{x}_{mx} - \dot{x}_{lx}$$

$$D^+V(y(t)) = \limsup_{h\to 0^+} \frac{V(y(t+h)) - V(y(t))}{h} = \dot{y}_{my} - \dot{y}_{ly}$$

Where, $mx\epsilon I_{max}(x(t))$, $lx\epsilon I_{min}(x(t))$, $my\epsilon I_{max}(y(t))$ and $ly\epsilon I_{min}(y(t))$ are indices with properties that,

$$\dot{x}_{mx}(t) = max[\dot{x}_{mx'}(t)|mx'\epsilon I_{max}(x(t))]$$
$$\dot{x}_{lx}(t) = min[\dot{x}_{lx'}(t)|lx'\epsilon I_{min}(x(t))]$$
$$\dot{y}_{my}(t) = max[\dot{y}_{my'}(t)|my'\epsilon I_{max}(y(t))]$$
$$\dot{w}_{ly}(t) = min[\dot{y}_{ly'}(t)|ly'\epsilon I_{min}(y(t))]$$

Upper Dini-derivative along difference dynamics are sequentially given as,

$$D^+V(x(t)) = \mu(x_{mx}(t) - x_{lx}(t)) + \mu(y_{lx}(t) - y_{mx}(t))$$
$$-\frac{\mu}{3}(x^3_{mx}(t) - x^3_{lx}(t)) + K(x_{lx}(t)) - (x_{mx}(t)))$$
$$\Rightarrow D^+V(x(t)) \leq (\mu - K)(x_{mx}(t) - x_{lx}(t)) +$$
$$\mu(y_{my}(t) - y_{ly}(t)) - \frac{\mu}{3}(x^3_{mx}(t) - x^3_{lx}(t))$$
$$D^+V(y(t)) = \frac{(x_{my} - x_{ly})}{\mu} + K(y_{my}(t) - y_{ly}(t))$$
$$D^+V(y(t)) = \frac{(x_{mx} - x_{lx})}{\mu} + K(y_{my}(t) - y_{ly}(t))$$

Let,

$$D_1^+ = D^+V(x(t)) + \mu^2 D^+V(y(t))$$
$$D_1^+ \leq (2\mu - K)(x_{mx}(t) - x_{lx}(t))$$
$$+ (\mu - K)(y_{my}(t) - y_{ly}(t))$$

Thus, if $K > 2\mu$, oscillators which are coupled to common disturbance, synchronize asymptotically.

5.2 *Fitzhugh Nagumo oscillators*

Theorem 5.2: Consider system dynamics (10) and (11), such that z is bounded oscillatory disturbance then, Fitzhugh Nagumo oscillators independently coupled with the common forcing disturbance synchronize asymptotically if, $K > (c + cb)$.

Proof: Refer Appendix II.

6. SIMULATIONS

Using (5) (6) and (7),(8) for $\mu = 1$ and two Van der Pol oscillators coupled to common forcing disturbance with coupling gains K=1.1 ($> \mu$) and 2.1 ($> 2\mu$) in accordance with the derived results as per Theorem 4.1 and 5.1 respectively which were introduced at t=50 secs., it was observed synchronization in among Van der Pol oscillators is achieved (Fig.5 and Fig.6). Similarly, Using (10), (11) and for parameters $a = 2, b = 1, c = 1, d = 0.1$ and $I = 2$ two Fitzhugh Nagumo oscillators which were independently coupled to a common forcing oscillatory disturbance synchronized asymptotically when coupling gains K=1.1 ($> c$), as per Theorem 4.2 and K=2.1 ($> c + cb$) as per Theorem 5.2, were introduced at t= 50 secs (Fig.7 and Fig.8). The initial conditions for all the cases were [(1,2),(-3,1)].

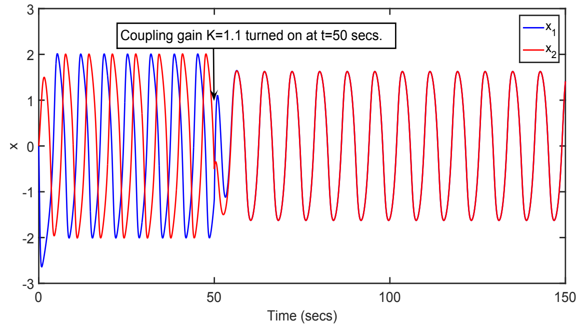

Fig. 5. Synchronization of two Van der Pol oscillators for μ=1, coupled to a common forcing with coupling gain K=1.1 introduced at t=50 secs.

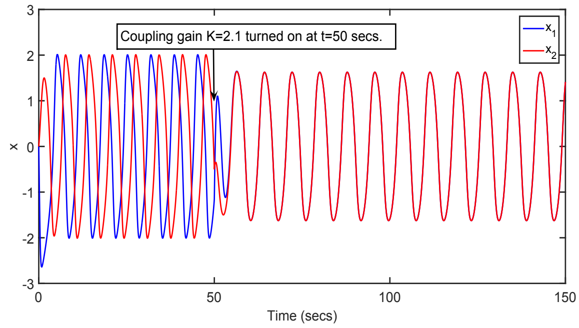

Fig. 6. Synchronization of two Van der Pol oscillators for μ=1, coupled to a common forcing with coupling gain K=2.1 introduced at t=50 secs.

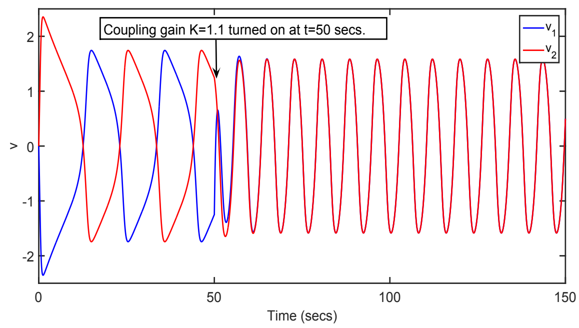

Fig. 7. Synchronization of two Fitzhugh Nagumo oscillators for c=1, coupled diffusively to a common forcing disturbance with coupling gain K=1.1 introduced at t=50 secs.

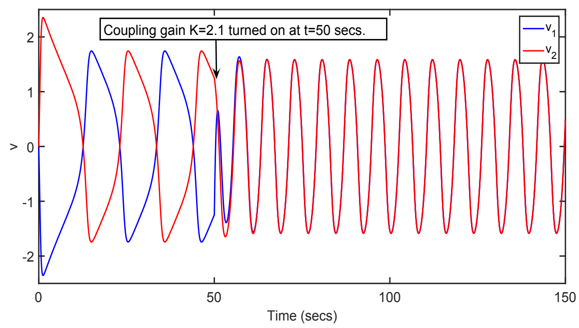

Fig. 8. Synchronization of two Fitzhugh Nagumo oscillators for c=1, coupled diffusively to a common forcing disturbance with coupling gain K=2.1 introduced at t=50 secs.

7. CONCLUSION

This work focuses on developing mathematical formalisms to understand the synchronization of standard second-order oscillators when they are coupled to common forcing disturbance diffusively. Here Lyapunov stability approach using quadratic Lyapunov function and non-smooth Lyapunov function has been used to derive sufficient coupling gain for networked system built-up of either Van der Pol or Fitzhugh Nagumo oscillators with the common forcing disturbance. It was observed that quadratic Lyapunov function-based analysis provides less conservative results. However, both the approaches provide analytical justification of the phenomena of synchronization in second-order benchmark oscillators, when they are independently coupled to a common disturbance.

The prospective areas for future research include the analysis of higher-order oscillator models in the same topological framework, effects of time-delays, heterogeneity of multiple agents, different topologies, and different analytical approaches for achieving better results.

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8. APPENDIX I: PROOF OF THEOREM 4.2

Let, a positive definite Lyapunov function be given as,

$$E_{12} = \frac{1}{2}[(v_1 - v_2)^2 + \frac{c}{db}(w_1 - w_2)^2$$

$$\Rightarrow \dot{E}_{12} = (v_1 - v_2)(\dot{v}_1 - \dot{v}_2) + \frac{c}{db}(w_1 - w_2)(\dot{w}_1 - \dot{w}_2)$$

$$\Rightarrow \dot{E}_{12} \leq (c - K)(v_1 - v_2)^2 - \frac{c}{3}[\frac{v_1}{2} + \frac{v_2}{2} + \frac{(v_1 + v_2)^2}{2}] - \frac{c}{b}(w_1 - w_2)^2$$

Now $\dot{E}_{12} < 0$ i.e. negative definite if,

$$(c - K)(v_1 - v_2)^2 < 0 \Rightarrow K > c$$

Thus, Fitzhugh Nagumo oscillators which are independently coupled to a common disturbance source synchronize asymptotically if $K > c$.

9. APPENDIX II: PROOF OF THEOREM 5.2

Consider two non-smooth Lyapunov functions $V : R^N \to R$ such that,

$$V(v(t)) = [max|v_i - v_j||i, j\epsilon(1, ..., N)]$$
$$V(w(t)) = [max|w_i - w_j||i, j\epsilon(1, ..., N)]$$

Now, upper Dini-derivatives are given as,

$$D^+V(v(t)) = \limsup_{h \to 0^+} \frac{V(v(t+h)) - V(v(t))}{h} = \dot{v}_{mv} - \dot{v}_{lv}$$

$$D^+V(w(t)) = \limsup_{h \to 0^+} \frac{V(w(t+h)) - V(w(t))}{h} = \dot{w}_{mw} - \dot{w}_{lw}$$

Where, $mv\ \epsilon I_{max}(v(t))$, $lv\ \epsilon I_{min}(v(t))$, $mw\epsilon I_{max}(w(t))$, $lw\epsilon I_{min}(w(t))$ are indices with properties that,

$$\dot{v}_{mv}(t) = max[\dot{v}_{mv'}(t)|mv'\epsilon I_{max}(v(t))]$$
$$\dot{v}_{lv}(t) = min[\dot{v}_{lv'}(t)|lv'\epsilon I_{min}(v(t))]$$
$$\dot{w}_{mw}(t) = max[\dot{w}_{mw'}(t)|mw'\epsilon I_{max}(w(t))]$$
$$\dot{w}_{lw}(t) = min[\dot{w}_{lw'}(t)|lv'\epsilon I_{min}(w(t))]$$

The upper Dini-derivatives along the difference dynamics are sequentially given as,

$$D^+V(v(t)) = c(v_{mv}(t) - v_{lv}(t)) + c(w_{lv}(t) - w_{mv}(t)) - \frac{c}{3}(v^3_{mv}(t) - v^3_{lv}(t)) - K(z - v_{mv}(t)) - (z - v_{lv}(t))$$

$$D^+V(v(t)) = (c - K)(v_{mv}(t) - v_{lv}(t)) + c(w_{lv}(t) - w_{mv}(t)) - \frac{c}{3}(v^3_{mv}(t) - v^3_{lv}(t))$$

$$D^+V(w(t)) = db(v_{mw} - v_{lw}) - d(w_{mw} - w_{lw})$$

Let,

$$D_2^+ = D^+V(v(t)) + \frac{c}{d}D^+V(w(t))$$

We know, $(v_{mw}(t) - v_{lw}(t))$ can have maximum value which is equal to $(v_{mv}(t) - v_{lv}(t))$ also,$(w_{mw}(t) + w_{lw}(t)) \geq (w_{lv}(t) - w_{mv}(t))$, it is therefore,

$$D_2^+ \leq (c + cb - K)(v_{mv}(t) - v_{lv}(t)) - \frac{c}{3}(v^3_{mv}(t) - v^3_{lv}(t)) + c[w_{lv}(t) - w_{mv}(t) - w_{mw}(t) - w_{lw}(t)]$$

Now, $D_2^+ < 0$ if,

$$K > (c + cb)$$

Thus, independent Fitzhugh Nagumo oscillators which are diffusively coupled to a common disturbance source synchronize asymptotically as $t \to \infty$.