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On the Stratonovich approach for a satellite dynamics

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Abstract: In contrast to a vector non-linear stochastic differential equation (SDE) describing the satellite dynamics under the 'fluctuating aerodynamic torque', this paper analyses a second-order fluctuation equation for the radial perturbation about the given orbit. The second-order fluctuation equation for the radial perturbation has found its application for the satellite orbital stability. After accomplishing a phase space formulation, we arrive at the two-dimensional SDE. Most notably, the inaccurate choice of stochastic integral describing the satellite stochastic dynamics will have influence on their estimation, stability and control. For this reason, we develop a noise equation of the satellite dynamics in the Stratonovich setting. The satellite dynamics in the Stratonovich sense can be expressed equivalently in the Itô setting by accounting additional correction terms in the system non-linearity term of the SDE. This paper develops the estimation theory of satellite dynamics via the Stratonovich calculus. The analytic findings are useful to the trajectory estimation of the orbiting satellite under the influence of atmospheric dust perturbations, where the observations are not available.

Keywords: Satellite dynamics, stochastic systems, Stratonovich differential, the atmospheric density

1. INTRODUCTION

In satellite mechanics, the two-body problem describes the motion of an orbiting particle, i.e. the Planet-Sun system. One can arrive at the two-body model by writing the Lagrangian of the orbiting particle in combination with the Euler-Lagrange equation. After accounting the effect of 'stochastic dust particles', an astronomical phenomenon, the governing equation leads to a system of two-coupled second-order fluctuation equations (Sharma and Parthasarathy, 2007). Various two-body models and their exposition can be found in the famous book, i.e. *Foundations of Mechanics*, authored by (Abraham and Marsden, 1987). Another appealing model in satellite mechanics is described by a multi-dimensional SDE, a six-dimensional, see Bierbaum et al. (2002) and Montenbruck and Gill, (2000). More notably, random initial conditions and small random perturbation effects give rise to the concept of the SDE. Unaccounting the effect of small perturbation effects, inaccurate choice of the stochastic interpretation and the inaccurate choice of noise models will lead to the inaccurate estimated state trajectory, which will have the following consequences: (i) a poor decision about the positioning of the orbiting satellite (ii) inaccurate satellite orbit determination (Montenbuck and Gill, 2000) leading to the false military threat (Kessler and Cicci, 1997) (iii) poor control algorithmic procedures, which will have influence on the satellite orbit stability. The motion of satellites with magnetic elements is influenced by the geomagnetic (Sagirow, 1972) field in two ways: (i) interaction between the magnetic rod and the geomagnetic field (ii) interaction between the eddy current, which is produced in the shell of the satellite, geomagnetic field. Under fluctuating geomagnetic field, the orbiting satellite can be regarded as a stochastic differential system resulting from the electromagnetic theory and the theory of stochastic processes. Notably, the satellite dynamics under the influence of the fluctuating aerodynamic torque is formalized using a higher-dimensional non-linear SDE (Sagirow, 1970). It becomes harder to analyse the higher-dimensional non-linear stochastic differential system, since it involves the matrix-vector format as well as the closed-form solution is not possible. As a result of this, it becomes imperative to explore a lower dimensional SDE describing the effect of random perturbations on the orbiting satellites. Fortunately, one such model, a second-order fluctuation equation for the pitch motion has proven useful for the satellite orbital stability problem (Kloeden and Platen, 1991, p. 262). After accomplishing the phase space formulation for the second-order fluctuation, we are led to the two-dimensional SDE. The white noise-driven stochastic model is popular, but informal. There are two formal stochastic interpretations, the Itô and Stratonovich. It is a famous stochastic process quotation that the computers are the Itô and the circuits are the Stratonovich, see Rathore and Sharma, (2018) and Victor and Chirikjian, (2019). Filtering and control of physical models involving the 'Stratonovich differential' are relatively very scarce. This paper does that.

Most notably, in this paper, we wish to develop a noise equation of a satellite dynamics in the Stratonovich setting, an alternative interpretation. The satellite dynamics in the Stratonovich sense can be expressed equivalently in the Itô setting by accounting additional correction terms in the system nonlinearity term of the stochastic differential system that contribute to the accuracy of the noise equation.

Secondly, this paper utilizes the theory of Stratonovich SDE. Some experimentations as well as guesswork to develop and examine the efficacy of an estimation algorithm for the satellite dynamics. This paper explains in some detail the stochastic stability of the satellite motion using the Stratonovich differential as well as highlights the mathematical intractability associated with asymptotic stability in probability for the stochastic problem of concern here. The estimation theory, stability conditions and control of dynamical systems in the Stratonovich setting are known in literature are relatively scarce. Since this paper unfolds the estimation procedure for a satellite dynamics in the Stratonovich setting, aerodynamists, practitioners will find the paper revealing.

Notations: This paper adopts the notation $\langle \ \rangle$ for the conditional expectation of the random variable.

2. AN ESTIMATION ALGORITHM FOR A STOCHASTIC SATELLITE DYNAMICS

This section explains briefly the motion of the orbiting satellite under the influence of gravity gradient and fluctuating aerodynamic torque. The satellite dynamics under the influence of randomly varying atmospheric density is, see Sagirow, (1970).

$$\frac{dD}{dt} + \omega \times D = M_{ext}, \tag{1}$$

where the term $'D'$ has an interpretation as moment of the momentum of the orbiting satellite, ω is the vector angular velocity of the satellite, the sign $\times$ denotes the vector product and M_{ext} has two components: (i) gravity gradient torque (ii) the fluctuating aerodynamic torque $\frac{1}{2}gv^2(f(\phi,\ \varphi,\ \lambda)+K\omega)$, where $f(\phi,\ \varphi,\ \lambda)$, $K\omega$ are the vector non-linear functions. Note that the atmospheric density g has the stochastic character,

$$g = g_o(1+\delta\, \dot{B}_t), \tag{2}$$

where g_o is the deterministic atmospheric density, δ is the noise intensity and B_t is the Brownian motion. The stochastic character is attributed to the Earth-Moon interaction, solar activity etc. The satellite xyz-fixed axes and orbital axes are associated with the Euler angles, the roll angle φ, the yaw angle ϕ, the pitch angle λ. As a result of equations (1)-(2), we have

$$\frac{dD}{dt} + \omega \times D = \frac{1}{2} g_0 (1+\delta \dot{B}_t) v^2 (f(\phi,\ \varphi,\ \lambda)+K\omega), \tag{3}$$

Consider the roll and yaw angles are at the equilibrium points and the x-axis is an axis symmetry of the satellite. After a simple calculation, equation (3) reduces to the second-order non-linear stochastic pitch motion, see equation (3.24) of Sagirow (1970, p. 56), i.e.

$$\ddot{\lambda} + \sin\lambda + \eta_0 \dot{\lambda} - l \sin 2\lambda + \delta_0 (\sin\lambda + \eta_0 \dot{\lambda})\dot{B}_t = 0, \tag{4}$$

where η_0 depends on the satellite geometry and aerodynamic constants, δ_0 depends on the satellite geometry and l is the gravity gradient. Interestingly, equation (4) is an immediate consequence of the notion of the time normalization. After rearranging the terms of equation (4), adopting more familiar and convenient notations for the phase space analysis, we arrive at (Kloeden and Platen, 1991, p. 262),

$$\ddot{\lambda}_t + b(1+a\dot{B}_t)\dot{\lambda}_t + (1+a\dot{B}_t)\sin\lambda_t - c\sin 2\lambda_t = 0, \tag{5}$$

where $b=\eta_0, a=\delta_0, c=l.$ For convenience, we have adopted the notation $\dot{B}_t$ for the white noise process in lie of the notation ξ_t adopted in (Kloeden and Platen, 1991). The standard approach to analyse stochastic differential systems is to derive the conditional moment evolution equation, where conditional mean and variance become the special case. After accomplishing the phase space formulation of equation (5), we get a formal stochastic interpretation, i.e.

$$d\lambda_t = f(t,\ \lambda_t)\,dt + G(t,\ \lambda_t)dB_t, \tag{6}$$

The ith component of the above SDE can be stated as

$$d\lambda_i = f_i(t,\ \lambda_t)\,dt + \sum_{\gamma} G_{i\gamma}(t,\ \lambda_t)dB_\gamma, \tag{7}$$

where

$$\lambda_t = (\lambda_1, \lambda_2)^T,\ f(t,\lambda_t) = (\lambda_2, -b\lambda_2 - \sin\lambda_1 + c\sin 2\lambda_1)^T,$$
$$G(t,\lambda_t) = (0, -ab\lambda_2 - a\sin\lambda_1)^T.$$

In the Itô theory, the term $dB_t = \dot{B}_t\,dt$, where $\dot{B}_t$ is the white noise process, The term dB_t is treated as a rigorous mathematical object. The Itô calculus has demonstrated surprising power for stochastic problems arising from diverse field. In the Stratonovich setting, equation (7) becomes

$$d\lambda_i = f_i(t,\ \lambda_t)\,dt + \sum_{\gamma} G_{i\gamma}(t,\ \lambda_t) \circ dB_\gamma, \tag{8}$$

where $'\circ'$ denotes the Stratonovich differential, a linear operator. The Stratnovich SDE can be further replaced with Itô SDE with additional correction terms in the system non-linearity term, see Jazwinski, (1970, p. 119) and Stratonovich, (1966), we get

$$d\lambda_i = (f_i(t,\ \lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial \lambda_k})dt$$

$$+\sum_{\gamma} G_{i\gamma}(t,\lambda_t)dB_{\gamma}. \quad (9)$$

Furthermore, the estimation theory, stability and control of the Stratonovich SDE can be accomplished by considering equation (9) in lieu of equation (7). After considering the Stratonovich setting, the additional correction term $0.5a^2b^2\lambda_2 + 0.5a^2b\sin\lambda_1$ contributes to the satellite dynamics considered here. Thus, the satellite dynamics in Stratonovich setting can be further recast as,

$$d\lambda_1 = \lambda_2 dt, \quad (10)$$

$$d\lambda_2 = (-b\lambda_2 - \sin\lambda_1 + c\sin 2\lambda_1 + 0.5a^2b^2\lambda_2 + a^2b\sin\lambda_1)dt + (-ab\lambda_2 - a\sin\lambda_1)dB_t, \quad (11)$$

The additional correction term $0.5a^2b^2\lambda_2 + 0.5a^2b\sin\lambda_1$ of the satellite dynamics is a special case of the Stratonovich correction term $\frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial\lambda_k}$. Thanks to result stated in Jazwinski (1970, pp. 136-137), we state the following exact evolution equations:

$$d\langle\lambda_i(t)\rangle = \left\langle f_i(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial\lambda_k}\right\rangle dt, \quad (12)$$

$$\begin{aligned}
dP_{ij} &= d\langle\lambda_i\lambda_j\rangle - d\langle\lambda_i\rangle\langle\lambda_j\rangle \\
&= (\left\langle (f_i(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial\lambda_k})\lambda_j\right\rangle \\
&- \left\langle (f_i(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial\lambda_k})\right\rangle\langle\lambda_j\rangle \\
&+ \left\langle \lambda_i (f_j(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{j\gamma}(t,\lambda_t)}{\partial\lambda_k})\right\rangle \\
&- \langle\lambda_i\rangle\left\langle (f_j(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{j\gamma}(t,\lambda_t)}{\partial\lambda_k})\right\rangle \\
&+ \left\langle (GG^T)_{ij}(t,\lambda_t)\right\rangle)dt,
\end{aligned} \quad (13)$$

where,

$$\langle\lambda_i(t)\rangle = E(\lambda_i(t)|\lambda_{t_0},t_0),\ P_{ij} = \langle\lambda_i\lambda_j\rangle - \langle\lambda_i\rangle\langle\lambda_j\rangle$$

$$= E(\lambda_i\lambda_j|\lambda_{t_0},t_0) - E(\lambda_i|\lambda_{t_0},t_0)E(\lambda_j|\lambda_{t_0},t_0),$$
$$1 \le i \le n,\ 1 \le j \le n.$$

For the sake of generality, this paper utilizes the term conditional mean and variance that become the mean and variance for the deterministic initial state. A system of exact evolution equations, equation (12) and (13) is not convenient from for numerical experiments. For this reasons, we utilize the conditional mean and variance evolution equations accounting the perturbation order two in the system non-linearity and the diffusion coefficient. As a result of this,

$$\begin{aligned}
d\langle\lambda_i\rangle &= (f_i(t,\langle\lambda_t\rangle) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\langle\lambda_t\rangle)\frac{\partial G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_k\rangle} \\
&+ \frac{1}{2}\sum_{p,q} P_{pq}(\frac{\partial^2 f_i(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_q\rangle} + \frac{1}{2}\sum_{k,\gamma}(\frac{\partial G_{k\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle}\frac{\partial^2 G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_q\rangle\partial\langle\lambda_k\rangle} \\
&+ G_{k\gamma}(t,\langle\lambda_t\rangle)\frac{\partial^3 G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_q\rangle\partial\langle\lambda_k\rangle} \\
&+ \frac{\partial^2 G_{k\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_q\rangle}\frac{\partial G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_k\rangle} \\
&+ \frac{\partial G_{k\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_q\rangle}\frac{\partial^2 G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_k\rangle})))dt,
\end{aligned} \quad (14)$$

$$\begin{aligned}
dP_{ij} &= (\sum_p P_{ip}\frac{\partial f_j(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle} + \sum_p P_{jp}\frac{\partial f_i(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle} \\
&+ (GG^T)_{ij}(t,\langle\lambda_t\rangle) + \frac{1}{2}\sum_{p,q} P_{pq}\frac{\partial^2 (GG^T)_{ij}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_q\rangle} \\
&+ \frac{1}{2}\sum_p P_{ip}(\sum_{k,\gamma}(\frac{\partial G_{k\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle}\frac{\partial G_{j\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_k\rangle} \\
&+ G_{k\gamma}(t,\langle\lambda_t\rangle)\frac{\partial^2 G_{j\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_k\rangle})) \\
&+ \frac{1}{2}\sum_p P_{jp}(\sum_{k,\gamma}(\frac{\partial G_{k\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle}\frac{\partial G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_k\rangle} \\
&+ G_{k\gamma}(t,\langle\lambda_t\rangle)\frac{\partial^2 G_{i\gamma}(t,\langle\lambda_t\rangle)}{\partial\langle\lambda_p\rangle\partial\langle\lambda_k\rangle})))dt.
\end{aligned} \quad (15)$$

Remark: Equations (14)-(15) account for the greater partial order attributed to the Stratonovich differential in contrast to the Itô differential. The greater order can be traced back to the mean square convergence of the Stratonovich stochastic integral (Pugachev and Sinitsyn 1987, p.175 and p. 390). Equations (14)-(15) are useful that are exploited to the trajectory estimation of the orbiting satellite under the influence of atmospheric dust perturbation.

Conditional mean and variance evolutions for the satellite dynamics

Conditional mean and variance evolutions for the satellite dynamics in Stratonovich setting can be obtained by considering equations (10), (11), (14), (15). As a result of these, we have

$$d\langle\lambda_1\rangle = \langle\lambda_2\rangle dt, \quad (16)$$

$$d\langle\lambda_2\rangle = ((-b\langle\lambda_2\rangle - \sin\langle\lambda_1\rangle + c\sin 2\langle\lambda_1\rangle)$$
$$+0.5a^2b^2\langle\lambda_2\rangle + 0.5a^2b\sin\langle\lambda_1\rangle$$
$$+0.5P_{11}(\sin\langle\lambda_1\rangle - 4c\sin 2\langle\lambda_1\rangle - 0.5a^2b\sin\langle\lambda_1\rangle))dt, \quad (17)$$

$$dP_{11} = 2P_{12}dt, \quad (18)$$

$$dP_{12} = dP_{21} = (P_{11}(-\cos\langle\lambda_1\rangle + 2c\cos\langle\lambda_1\rangle + 0.5a^2b\cos\langle\lambda_1\rangle)$$
$$+P_{22} - P_{12}(b - 0.5a^2b^2))dt, \quad (19)$$

$$dP_{22} = (2P_{12}(-\cos\langle\lambda_1\rangle + 2c\cos 2\langle\lambda_1\rangle + 0.5a^2b\cos\langle\lambda_1\rangle)$$
$$-2P_{22}(b - 0.5a^2b^2) + a^2b^2\langle\lambda_2\rangle^2 + 2a^2b\langle\lambda_2\rangle\sin\langle\lambda_1\rangle$$
$$+a^2\sin^2\langle\lambda_1\rangle + P_{11}a^2\cos 2\langle\lambda_1\rangle$$
$$-P_{11}a^2b\langle\lambda_2\rangle\sin\langle\lambda_1\rangle + P_{22}a^2b^2 + 2P_{12}a^2b\cos\langle\lambda_1\rangle)dt. \quad (20)$$

3. NUMERICAL SIMULATIONS

For Approximate evolution equations, equations (16)-(20), are hard to evaluate theoretically, since the global properties are replaced with the local. In this paper, the numerical experiments associated with estimation algorithms of the satellite stochastic dynamics in the Itô and Startonovich senses are accomplished by considering two different sets of initial conditions, system parameters. The first set of system parameters and initial conditions is

$$a = 0.3,\ b = 0.6,\ c = 0.3, \langle\lambda_1(0)\rangle = 0.1\text{rad}, \langle\lambda_2(0)\rangle = 0.5\text{rad/sec},$$
$$P_{11}(0) = 0\ \text{rad}^2, P_{22}(0) = 0\ \text{rad}^2/\text{sec}^2, P_{12}(0) = 0\ \text{rad}^2/\text{sec}.$$

For convenience, here we consider that initial variances of the state vector are chosen as zero. This assumption allows studying the mean and variance trajectories under fluctuating atmospheric density explicitly. Non-zero initial variances can be chosen for the numerical testing of the estimation algorithms of the paper as well. The initial conditions and system parameters considered in this paper are associated with equations (16)-(20). The system parameters b, c are damping and oscillatory terms of equation (5) respectively describing the satellite dynamics under the influence of varying atmospheric conditions. The system parameter a is chosen using the criterion that the contribution to the evolution of the state vector $\lambda_t = (\lambda_1, \lambda_2)^T$ stemming from the random part is relatively smaller than the deterministic.

The numerical simulations demonstrated in figures (1)-(4) suggest that the mean and variance trajectories for the pitch angle and the derivative of the pitch angle display the bounded property resulting from estimation algorithms, equations (16)-(20). Furthermore, the mean and variance trajectories for the Stratonovich differential are bounded as well. The variance trajectories of figures (3)-(4) suggest that the difference between the variances, resulting form the Stratonovich and Itô settings. It is difficult to differentiate between the mean trajectories of figure (2) resulting from the Itô and Stratonovich settings. This suggests the contribution to the mean evolution $d\langle\lambda_2\rangle$ coming from the additional term $0.5a^2b^2\lambda_2 + 0.5a^2b\sin\lambda_1$ of the Stratonovich differential is relatively smaller, especially for the first set of system parameters and initial conditions.

The additional correction terms associated with the Stratonovich SDE contribute the additional terms to the mean and variance evolutions leading to the *better* state estimates. Note that the additional correction terms associated with evolution of the conditional mean $\langle\lambda_2\rangle$ are the following: (i) $(0.5(ab)^2\langle\lambda_2\rangle + 0.5a^2b\sin\langle\lambda_1\rangle)$ (ii) $-0.25P_{11}a^2b\sin\langle\lambda_1\rangle$, see equations (15) and (16). Furthermore, the additional correction term with the evolution dP_{22} of conditional variance is $2P_{12}(0.5a^2b\cos\langle\lambda_1\rangle) - 2P_{22}(b - 0.5a^2b^2)$. Thirdly, the term $P_{11}(0.5a^2b\cos\langle\lambda_1\rangle) - P_{12}(b - 0.5a^2b^2)$ of dP_{12} describes the additional correction term stemming from the Stratonovich interpretation. As a result of these correction terms, the estimated state trajectories of the satellite dynamics using the Stratonovich differential will be closer to their actual trajectories.

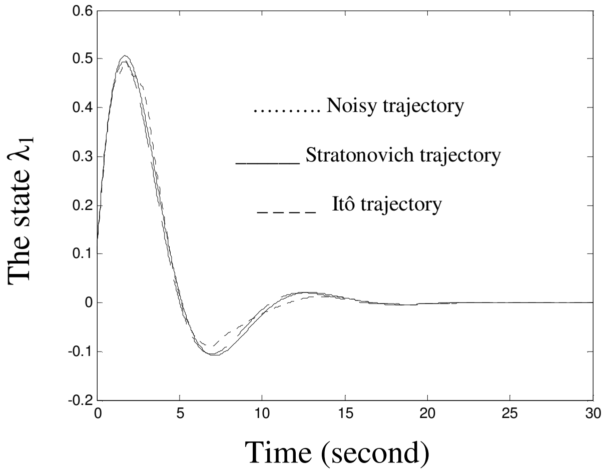

Figure 1: a comparison between estimated trajectories

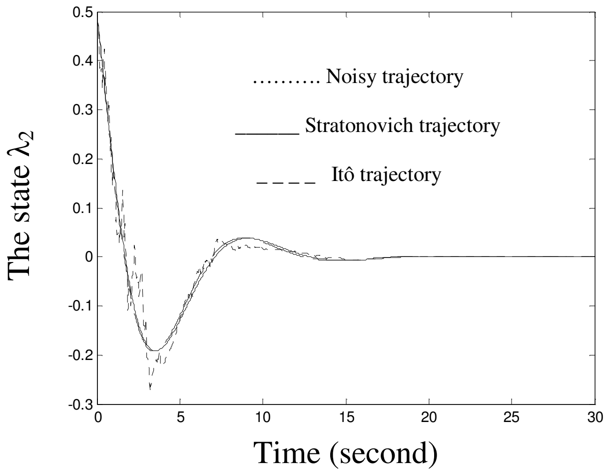

Figure 2: a comparison between estimated trajectories

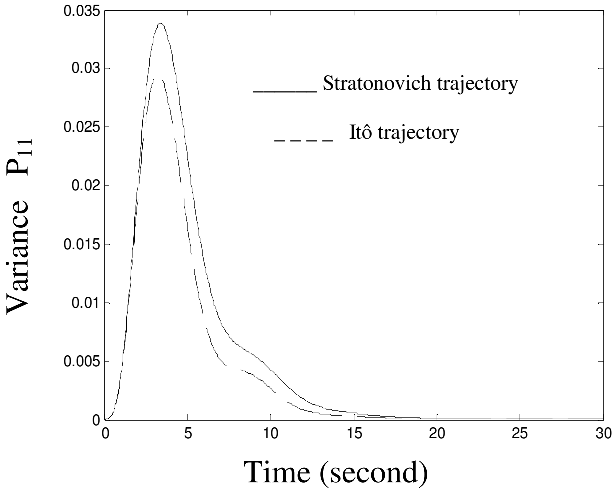

Figure 3: a conditional variance trajectory

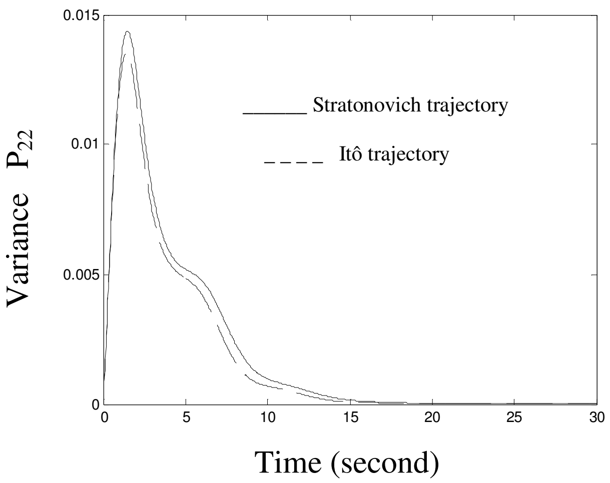

Figure 4: a conditional variance trajectory

4. THREE STOCHASTIC PROBLEMS

In this section, we highlight three stochastic problems. The first problem is about filtering theory of satellite dynamics of this paper. The second and third problems are about the stochastic stability conditions for the satellite dynamics by exploiting the Stratonovich context of the SDE. The stability of dynamical systems becomes quite harder in contrast to classical stability, since its involves the results of stochastic calculus than the ordinary calculus.

(i) In this paper, we developed the estimation algorithm for the satellite motion under the influence of the random atmospheric perturbation without accounting the effect of observations. After accounting observation terms in estimation algorithms, the estimation algorithms become filtering algorithms. Filtered estimates are more accurate in contrast to estimation algorithms unaccounting observations terms. For this reason, stochastic filtering of the satellite dynamics in the Stratonovich setting deserves investigations.

(ii) The stochastic stability theory for the Itô SDE, by constructing the stochastic Lyapunov function, is available in literature (Kushner, 1967). The structure of the differential operator that can be regarded as the Kolmogorov-Fokker-Planck operator is the following,

$$L(.) = \sum_i f_i(t,\lambda_t)\frac{\partial(.)}{\partial\lambda_i} + \frac{1}{2}\sum_{i,j}(GG^T)_{ij}(t,\lambda_t)\frac{\partial^2(.)}{\partial\lambda_i\partial\lambda_j}.$$

The above operator is useful to obtain the derivative of the stochastic Lyapunov function $v(\lambda_t)$. The term $L(v(\lambda_t))$, a derivation of the stochastic Lyapunov function, plays the key role to obtain stability conditions. Secondly, the Itô process is a right continuous strong Markov process as well. On the other hand, the stochastic stability for the Stratonovich SDE involving the concept of the stochastic Lyapunov function is not sufficiently known. The differential operator acting on the stochastic Lyapunov function of the Stratonovich SDE is modified to,

$$L(.) = \sum_i (f_i(t,\lambda_t) + \frac{1}{2}\sum_{k,\gamma} G_{k\gamma}(t,\lambda_t)\frac{\partial G_{i\gamma}(t,\lambda_t)}{\partial\lambda_k})\frac{\partial(.)}{\partial\lambda_i}$$

$$+\frac{1}{2}\sum_{i,j}(GG^T)_{ij}(t,\lambda_t)\frac{\partial^2(.)}{\partial\lambda_i\partial\lambda_j}.$$

However, asymptotic stability in probability is valid for the right continuous strong Markov process. The proof of the Stratonovich process λ_t to be a strong Markov process is cumbersome (Protter, 1990). Thus, deriving stability conditions for the satellite dynamics using the Stratonovich SDE merits investigations. The stochastic stability conditions for the Stratonovich setting would be more stringent and general in contrast to Itô setting.

(iii) Alternatively, the stochastic stability for dynamical systems involving the concept of the Lyapunov exponent received attention in literature. The results on the stochastic stability for the Stratonovich time-varying vector 'bilinear' SDE involving the concept of the Lyapunov exponent are available in literature (Pignol, 1985), see a celebrated book authored by Lin and Cai, (1995) as well. In the stability context, Liu and Liew, (2005) will be also useful. However, the stability results on 'non-linear' Stratonovich SDEs exploiting the concept of the Lyapunov exponent are not sufficiently known. For this reason, deriving stability conditions for the satellite dynamics of this paper, a mathematical problem formalized as a non-linear vector Stratonovich SDE coupled with the concept of the Lyapunov exponent, can be regarded as an open problem.

Contributions to the three stochastic problems of this paper will advance the topics in 'filtering and stochastic stability'.

That will prevent the possibility of inaccurate state estimates and poor stability conditions.

5. CONCLUSION

In this paper, we have developed a perturbed satellite dynamical model by exploiting a Satellite dynamics in the Stratonovich setting that is not available in literature. Notably, it hard to differentiate between state trajectories associated with the satellite dynamics in the Itô and Stratonovich settings, especially under zero initial variances. On the other hand, under non-zero initial variances, the difference between the state trajectories resulting from both settings becomes lager. Thus, the numerical simulation suggests that the Stratonovich setting is a more accurate stochastic interpretation in contrast to the Itô setting. It is further shown that the estimation equations of this paper resulting from the Stratonovich SDE have ability to preserve perturbation effects associated with satellite dynamics, see figures (1)-(8) of the paper.

Another contribution of the paper is to weave equations (14)-(20) using the Stratonovich calculus. Results are useful to the trajectory estimation of the orbiting satellite under the influence of atmospheric dust perturbations.

This paper will provide a direction to address the Itô and Stratonovich dilemma, the choice of a stochastic interpretation, for appealing and non-trivial stochastic problems.

The Stratonovich SDE, equations (10)-(11) of the paper, deserves further investigations that opens up the topics, i.e. Filtering, stochastic stability and control of the satellite dynamics with atmospheric random perturbation. Another contribution of the paper is to highlight three potential stochastic problems by briefly discussing them. Resolving them will lead to advancing the topic.

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