Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Observer-based Robust H_∞ Control for Uncertain Discrete-time T-S Fuzzy Systems

A. BENTALEB * A. BENZAOUIA ** A. EL HAJJAJI ***
A. KARAMA ****

* LAEPT, University Cadi Ayyad, Faculty of Science Semlalia, P.B. 2390, Marrakech, Morocco (e-mail: ahmed.bentaleb@edu.uca.ma).
** LAEPT, University Cadi Ayyad, Faculty of Science Semlalia, P.B. 2390, Marrakech, Morocco (e-mail: benzaouia@uca.ac.ma).
*** UFR of Sciences, MIS University of Picardie Jules Verne, Amiens, France (e-mail: hajjaji@u-picardie.fr).
**** LAEPT, University Cadi Ayyad, Faculty of Science Semlalia, P.B. 2390, Marrakech, Morocco (e-mail: karama@uca.ac.ma).

Abstract: This paper investigates robust observer based H_∞ control problem for uncertain discret-time Takagi-Sugeno fuzzy systems. By using fuzzy Lyapunov functions and some special derivations, Sufficient relaxed conditions for synthesis of a fuzzy observer and a fuzzy controller for T-S fuzzy systems are derived in terms of a set of linear matrix inequalities (LMIs) which can be solved using a single-step procedure. The proposed approach provides more relaxed conditions comparing with the existing techniques in literature which use a quadratic Lyapunov function and the so-called two-step procedure, also ensures better H_∞ control performance. Simulation example is presented to show the effectiveness of the proposed design method.

Keywords: H_∞ Control, Uncertain T-S Fuzzy Models, Fuzzy Observer, Fuzzy Lyapunov Functions, LMIs.

1. INTRODUCTION

In recent years, the Takagi–Sugeno (T–S) fuzzy model has been proved to be a good representation of a wide class of nonlinear dynamic systems. The TS fuzzy dynamics model is a system described by the fuzzy if-then rules which gives local linear representations of nonlinear systems. In (Chang et al., 2011; Benzaouia and El Hajjaji, 2016), the stability analysis and synthesis problems for the T–S fuzzy models are studied by using the LMI technique.

According to the T–S fuzzy model, the concept of PDC (Benzaouia and El Hajjaji, 2016; Benzaouia et al., 2010; Benzaouia and El Hajjaji, 2011) was employed to design the observer-based fuzzy controller (Benzaouia and El Hajjaji, 2016; Lo and Lin, 2004). In the state feedback control theory, the whole information of the state are assumed as known and measurable. But, in many practical nonlinear control systems, some states are often unavailable. For this reason, the output feedback (Benzaouia and El Hajjaji, 2016; Nachidi et al., 2011) and the observer design technique (Lo and Lin, 2004; Benzaouia and El Hajjaji, 2016) are used. Applying the observer design approach, the fuzzy controller can be designed by the PDC technique with estimated states. In general, the design problem of the observer-based fuzzy controller cannot be directly solved by the linear matrix inequalities (LMIs) technique. Therefore, In (Lo and Lin, 2004), conditions guaranteeing the existence of observer-based H_∞ controllers are given in terms of bilinear matrix inequalities (BMIs), which are not convex and NP-hard to solve. Besides, the so-called two-step procedure which appears as a drawback is proposed based on a single-quadratic Lyapunov function approach to both continuous and discrete-time problems with or without uncertainty. For discrete-time fuzzy systems without uncertainties, (Chang et al., 2011) has proposed a new design method of robust observer-based control based on H_∞ norm and on a non-quadratic Lyapunov function. However, their technique uses also two steps for solving the stability conditions. An improvement of the control design method using one step procedure is proposed in (El Haiek et al., 2017).

This work presents a novel approach for the fuzzy observer-based H_∞ control for a class of uncertain discrete-time T–S fuzzy systems. By applying the fuzzy Lyapunov function instead of a single-quadratic Lyapunov function approach together with some special derivations, relaxed stabilization conditions are proposed in terms of a set of LMIs, which gives less conservative results than that in (Lo and Lin, 2004). Furthermore, single-step procedure is used instead of the so-called two-step procedure. Simulation example and a comparison with some existing results are given to illustrate the merits of the proposed method.

The organization of this paper is as follows. Section 2 presents the structure of uncertain fuzzy system and gives some preliminaries. Section 3 contains the main result where new stability conditions for an uncertain fuzzy system via an observer-based robust H_∞ control are given.

In Section 4, simulation example is provided to illustrate the design effectiveness. Finally, the conclusion is given in Section 5.

Notations : $X = X^T < 0$ ($X = X^T \leq 0$) means the matrix X is symmetric and negative definite (symmetric and negative semi-definite). X^T denotes the transpose of X. sym(M) means $M+M^T$. The symbols I and 0 represent the identity and zero matrix with appropriate dimensions, respectively. The symbol '*' represents the symmetric term in a block matrix. The notation Y_h stands for $\sum_{i=1}^{r} h_i(k)Y_i$ and $Y_{h\Delta}$ for $Y_h + \Delta Y_h$.

2. PRELIMINARIES

Consider a discrete-time T–S fuzzy dynamic model with uncertainties, in which the ith fuzzy IF-THEN rule is described as follows:

R^i : if $\xi_1(k)$ is M_{1i} and ... $\xi_p(k)$ is M_{pi} then

$$\begin{cases} x(k+1) = (A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k) + (E_i + \Delta E_i)w(k) \\ z(k) = (C_{1i} + \Delta C_{1i})x(k) + (D_i + \Delta D_i)u(k) + (F_i + \Delta F_i)w(k) \\ y(k) = (C_{2i} + \Delta C_{2i})x(k) + (R_i + \Delta R_i)w(k) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $z(t) \in \mathbb{R}^q$ is the controlled output variable, $w(t) \in \mathbb{R}^v$ is the disturbance variable, $u(t) \in \mathbb{R}^m$ is the input variable, $y(t) \in \mathbb{R}^c$ is the output variable. Matrices $A_i \in \mathbb{R}^{n\times n}$ $B_i \in \mathbb{R}^{n\times m}$, $C_{1i} \in \mathbb{R}^{q\times n}$, $D_i \in \mathbb{R}^{q\times m}$, $C_{2i} \in \mathbb{R}^{c\times n}$, $E_i \in \mathbb{R}^{n\times v}$, $F_i \in \mathbb{R}^{q\times v}$, $R_i \in \mathbb{R}^{c\times v}$ are known real constant matrices, M_{di}, $i = 1, 2, ..., r, d = 1, 2, ..., p$ are the fuzzy sets, r is the number of fuzzy rules, and ξ_d, d=1,2,...,p are premise variables. We assume that the premise variables do not depend on the state variables estimated by a fuzzy observer and control input vector.

ΔA_i, ΔB_i, ΔE_i, ΔC_{1i}, ΔD_i, ΔF_i, ΔC_{2i}, ΔE_i represent the time-varying uncertain matrices of appropriate dimensions, and can describe the modeling errors of the nonlinear system or the identification errors between an original nonlinear system and its local linear representation. We adopt the following form of uncertainties described in (Lo and Lin, 2004) :

$$\begin{bmatrix} \Delta A_i & \Delta E_i & \Delta B_i \\ \Delta C_{1i} & \Delta F_i & \Delta D_i \\ \Delta C_{2i} & \Delta R_i & \end{bmatrix} = \begin{bmatrix} M_{1i} \\ M_{2i} \\ M_{3i} \end{bmatrix} \Delta(k) \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \tag{2}$$

Where the uncertain matrix satisfies $\Delta(k)^T\Delta(k) \leq I$ and $M_{ki}, N_k, k = 1, 2, 3$ are known real matrices of appropriate dimensions.

The T–S fuzzy model (1) is inferred as follows:

$$\begin{cases} x(k+1) = A_{h\Delta}x(k) + B_{h\Delta}u(k) + E_{h\Delta}w(k) \\ z(k) = C_{1h\Delta}x(k) + D_{h\Delta}u(k) + F_{h\Delta}w(k) \\ y(k) = C_{2h\Delta}x(k) + R_{h\Delta}w(k) \end{cases} \tag{3}$$

The normalized membership functions are given by : $h_i(\xi(k)) = \dfrac{w_i(\xi(k))}{\sum_{i=1}^{r} w_i(\xi(k))}$, $w_i(s(k)) = \prod_{d=1}^{p} M_{di}(\xi_d(k))$

where $M_{di}(\xi_d(k))$ is the grade of membership of $\xi_d(k)$ in M_{di}.

By definition, we have : $0 \leq h_i(s(k)) \leq 1$, $\sum_{i=1}^{r} h_i(s(k)) = 1$, $\forall i \in \{1, ..., r\}$

The following fuzzy observer is proposed to deal with the state estimation of system (3) :

$$\begin{cases} \hat{x}(k+1) = A_h\hat{x}(k) + B_hu(k) + L_h(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C_{2h}\hat{x}k) \end{cases} \tag{4}$$

where $\hat{x}(k)$ and $\hat{y}(k)$ are the estimated state and estimated output, respectively. $L_h = \sum_{i=1}^{r} h_i(\xi(k))L_i$, $L_i \in \mathbb{R}^{n\times c}$, $i = 1, ..., r$ are the observer gains .

The following fuzzy controller which based on the PDC concept is employed :

$$u(k) = -K_h\hat{x}(k) \tag{5}$$

where $K_h = \sum_{i=1}^{r} h_i(\xi(k))K_i$, $K_i \in \mathbb{R}^{m\times n}, i = 1, ..., r$ are the controller gains to be determined.

By defining the estimation error as :

$$e(k) = x(k) - \hat{x}(k) \tag{6}$$

And the the augmented state as : $\tilde{x}(k) = \begin{bmatrix} x(k \\ e(k)) \end{bmatrix}$

The closed-loop fuzzy system, comprising of plant (3), observer (4) and PDC controller (5), becomes :

$$\begin{bmatrix} \tilde{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{hh} & B_{hh} \\ C_{hh} & D_{hh} \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \tag{7}$$

$$= \left(\begin{bmatrix} A_{hh0} & B_{hh0} \\ C_{hh0} & D_{hh0} \end{bmatrix} + \begin{bmatrix} \Delta A_{hh} & \Delta B_{hh} \\ \Delta C_{hh} & \Delta D_{hh} \end{bmatrix} \right) * \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix}$$

Where

$$A_{hh0} = \begin{bmatrix} A_h - B_hK_h & B_hK_h \\ 0 & A_h - L_hC_{2h} \end{bmatrix}, B_{hh0} = \begin{bmatrix} E_h \\ E_h - L_hR_h \end{bmatrix}$$

$$C_{hh0} = \begin{bmatrix} C_{1h} - D_hK_h & D_hK_h \end{bmatrix}, \qquad D_{hh0} = F_h$$

$$\Delta A_{hh} = \begin{bmatrix} \Delta A_h - \Delta B_hK_h & \Delta B_hK_h \\ \Delta A_h - \Delta B_hK_h - L_hC_{h2} & \Delta B_hK_h \end{bmatrix}$$

$$\Delta B_{hh} = \begin{bmatrix} \Delta E_h \\ \Delta E_h - L_h\Delta R_h \end{bmatrix}, \qquad \Delta D_{hh} = \Delta F_h$$

$$\Delta C_{hh} = \begin{bmatrix} \Delta C_h - \Delta D_hK_h & \Delta D_hK_h \end{bmatrix}$$

For the formulation of the main result, we recall the following definition and lemmas.

Definition 1. (Gao and Wang, 2005) The closed-loop fuzzy system (7) is said to be asymptotically stable with an H_∞ performance $\gamma > 0$ if it is asymptotically stable with $w(k) \equiv 0$, and under zero initial condition, satisfies the following inequality:

$$\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) \tag{8}$$

Lemma 2. (Chang et al., 2015) For matrices T , Q, U, and W with appropriate dimensions and scalar β. Inequality

$$T + QW + W^TQ^T < 0 \tag{9}$$

is fulfilled if the following condition holds:

$$\begin{bmatrix} T & * \\ \beta Q^T + UW & -\beta U - \beta U^T \end{bmatrix} < 0 \tag{10}$$

Lemma 3. (Petersen, 1987) Let X, Y and $\Delta(k)$ be real matrices with appropriate dimensions and $\Delta^T(k)\Delta(k) \leq I$. Then, for any scalar $\epsilon > 0$

$$X\Delta(k)Y + Y^T\Delta^T(k)X^T \leq \epsilon^{-1}XX^T + \epsilon Y^T Y \tag{11}$$

Lemma 4. (Tuan et al., 2001) Suppose $\Phi_{ijl}, i, j, l = 1, ..., r$, are symmetric matrices. Inequality

$$\sum_{l=1}^{r}\sum_{i=1}^{r}\sum_{j=1}^{r} h_l(k+1)h_i(k)h_j(k)\Phi_{ijl} < 0 \tag{12}$$

is fulfilled if the following conditions hold

$$\Phi_{iil} < 0, \quad i, l = 1, ..., r \tag{13}$$

$$\frac{1}{r-1}\Phi_{iil} + \frac{1}{2}(\Phi_{ijl} + \Phi_{jil}) < 0, \;\; i, j, l = 1, ..., r \quad i \neq j \tag{14}$$

3. MAIN RESULTS

The control objective of this paper is to develop a new conditions in terms of strict LMIs which can be solved in one step to determine the fuzzy controller and observer gains (K_i, L_i) such that closed-loop augmented system (7) is stabilizable by based-observer controller (5) in the presence of bounded disturbances.

Lemma 5. Closed-loop fuzzy system (7) is asymptotically stable with an H_∞ performance $\gamma > 0$ if there exist positive scalars $\varepsilon_{1ijl}, \varepsilon_{2ijl}$, symmetric matrices $P_{1i}(P_{2i})$, $P_{1l}(P_{2l})$, matrices G_1, G_2, G_3, Y_i, and K_j, for $i, j, l = 1, 2, ..., r$, such that the following conditions are satisfied

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(k)h_j(k)h_l(k+1)\Omega_{ijl} < 0 \tag{15}$$

Where

$$\Omega_{ijl} = \begin{bmatrix} -P_{1i} & * & * & * & * & * & * & * & * & * \\ 0 & -P_{2i} & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ \Omega_{41} & G_1B_iK_j & G_1E_i & \Omega_{44} & * & * & * & * & * & * \\ 0 & \Omega_{52} & \Omega_{53} & 0 & \Omega_{55} & * & * & * & * & * \\ \Omega_{61} & G_3D_iK_j & G_3F_i & 0 & 0 & -I & * & * & * & * \\ 0 & 0 & 0 & M_{1i}^T & M_{1i}^T & M_{2i}^T & -\varepsilon_{1ijl} & * & * & * \\ 0 & 0 & 0 & 0 & -M_{3i}^T L_j^T & 0 & 0 & -\varepsilon_{2ijl} & * & * \\ \Omega_{91} & \varepsilon_{1ijl}N_3K_j & \varepsilon_{1ijl}N_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ijl} & * \\ \varepsilon_{2ijl}N_1 & 0 & \varepsilon_{2ijl}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ijl} \end{bmatrix} < 0 \tag{16}$$

$$\Omega_{41} = G_1A_i - G_1B_iK_j \qquad \Omega_{52} = G_2A_i - Y_iC_{2j}$$

$$\Omega_{61} = G_3C_{1i} - G_3D_iK_j \qquad \Omega_{91} = \varepsilon_{1ijl}N_1 - \varepsilon_{1ijl}N_3K_j$$

$$\Omega_{53} = G_2E_i - Y_iR_j \qquad \Omega_{55} = -G_2 - G_2^T + P_{2l}$$

$$\Omega_{44} = -G_1 - G_1^T + P_{1l} \qquad Li = G_2^{-1}Y_i$$

Proof.

From (Lo and Lin, 2004; Chang et al., 2011), the closed-loop fuzzy system is asymptotically stable via observer-based control with the required H_∞ constraint if the following conditions hold :

$$\underbrace{\begin{bmatrix} -P_h & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{hh0} & B_{hh0} & P_h^{-1} & * \\ C_{hh0} & D_{h0} & 0 & -I \end{bmatrix}}_{J_{hhh}} + \underbrace{\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ \Delta A_{hh} & \Delta B_{hh} & 0 & * \\ \Delta C_{hh} & \Delta D_{h0} & 0 & 0 \end{bmatrix}}_{\Delta J_{hh}} < 0 \tag{17}$$

From (2), ΔJ_{hh} can be given by :

$$\Delta J_{hh} = sym(S_1\Delta(t)T_1) + sym(S_2\Delta(t)T_2) \tag{18}$$

Where :

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & M_{1h}^T & M_{1h}^T & M_{2h}^T \end{bmatrix}^T$$
$$T_1 = \begin{bmatrix} N_1 - N_3K_h & N_3K_h & N_2 & 0 & 0 & 0 \end{bmatrix}$$
$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & (-L_hM_{3h})^T & 0 \end{bmatrix}^T$$
$$T_2 = \begin{bmatrix} N_1 & 0 & N_2 & 0 & 0 & 0 \end{bmatrix}$$

The equation (17) can be written as follows :

$$\underbrace{J_{hhh} + \underbrace{\underbrace{\begin{bmatrix} S_1 & S_2 \end{bmatrix}}_{S} \begin{bmatrix} \Delta(t) & 0 \\ 0 & \Delta(t) \end{bmatrix} \underbrace{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}_{T}}_{L} + L^T}_{Z} < 0 \tag{19}$$

By applying Lemma (3), it comes that:

$$Z < J_{hhh} + \varepsilon_{hh}^{-1}SS^T + \varepsilon_{hh}T^TT \tag{20}$$
$$< \underbrace{J_{hhh} + (-)\begin{bmatrix} S & T^T \end{bmatrix} \begin{bmatrix} -\varepsilon_{hh} & 0 \\ 0 & -\varepsilon_{hh}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} S^T \\ T \end{bmatrix}}_{X}$$

Where : $\varepsilon_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(k)h_j(k)\begin{bmatrix} \varepsilon_{1ij} & * \\ 0 & \varepsilon_{2ij} \end{bmatrix}$ Applying the Schur complement, X is equivalent to:

$$\begin{bmatrix} J_{hhh} & * & * \\ S^T & -\varepsilon_{hhh} & * \\ T & 0 & -\varepsilon_{hhh}^{-1} \end{bmatrix} < 0 \tag{21}$$

Substituting each term of (21), we have the following conditions :

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(k)h_j(k)h_l(k+1)\Gamma_{ijl} \tag{22}$$

Where

$$\Gamma_{ijl} = \begin{bmatrix} -P_{1i} & * & * & * & * & * & * & * & * & * \\ 0 & -P_{2i} & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ A_i - B_iK_j & B_iK_j & E_i & -P_{1l}^{-1} & * & * & * & * & * & * \\ 0 & A_i - L_iC_{2j} & E_i - L_iR_j & 0 & P_{2l}^{-1} & * & * & * & * & * \\ C_{1i} - D_iK_j & D_iK_j & F_i & 0 & 0 & -I & * & * & * & * \\ 0 & 0 & 0 & M_{1i}^T & M_{1i}^T & M_{2i}^T & -\varepsilon_{1ijl} & * & * & * \\ 0 & 0 & 0 & 0 & -M_{3i}^T L_j^T & 0 & 0 & -\varepsilon_{2ijl} & * & * \\ N_1 - N_3K_j & N_3K_j & N_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ijl}^{-1} & * \\ N_1 & 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ijl}^{-1} \end{bmatrix} < 0 \quad (23)$$

It is noted that

$$(-V - Q)Q^{-1}(-V - Q)^T \ \leq \ 0 \ \ Q > 0,$$

implies that

$$-VQ^{-1}V^T \ \leq \ -V - V^T + Q$$

Thus, Pre-and post-multiplying (22) by

$diag\{I, I, I, G_1, G_2, G_3, I, I, \varepsilon_{1ijl}, \varepsilon_{2ijl}\}$ and its transpose, respectively, and using the previous inequality, we can see that $\Gamma_{ijl} \leq \Omega_{ijl}$, It follows that if (15) holds, then (22) is satisfied. This completes the proof.

Inequalities Ω_{ijl} are expressed by a set of bilinear matrix inequalities (BMIs) which are very hard to solve numerically due to the existence of the bilinear (i.e., product) terms $G_1B_iK_j$, $G_3D_iK_j$ and $\varepsilon_{1ijl}N_3K_j$. In the literature, the so-called two-step design approaches is used to handle these bilinearities. In this paper , a new H_∞ performance synthesis criterion is presented in the following theorem, where the obtained conditions are given by a set of LMIs which can be solved using a single-step procedure.

Theorem 1. Closed-loop fuzzy system (7) is asymptotically stable with an H_∞ performance $\gamma > 0$ if there exist a known scalar β, positive scalars $\varepsilon_{1ijl}, \varepsilon_{2ijl}$, symmetric matrices $P_{1i}(P_{2i})$, $P_{1l}(P_{2l})$, matrices G_1, G_2, G_3, H_1, H_2, U, and Y_i, for $i, j, l = 1, 2, ..., r$, such that the following conditions are satisfied :

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(k)h_j(k)h_l(k+1)\Phi_{ijl} < 0 \quad (24)$$

Where

$$\Phi_{ijl} = \begin{bmatrix} -P_{1i} & * & * & * & * & * & * & * & * & * & * \\ 0 & -P_{2i} & * & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * & * \\ \Phi_{41ij} & B_iH_j & G_1E_i & \Phi_{44l} & * & * & * & * & * & * & * \\ 0 & \Phi_{52ij} & \Phi_{53ij} & 0 & \Phi_{55l} & * & * & * & * & * & * \\ \Phi_{61ij} & D_iH_j & G_3F_i & 0 & 0 & \Phi_{66} & * & * & * & * & * \\ 0 & 0 & 0 & \Phi_{74i} & \Phi_{75i} & \Phi_{76i} & -\varepsilon_{1ijl}I & * & * & * & * \\ 0 & 0 & 0 & 0 & \Phi_{85i} & 0 & 0 & -\varepsilon_{2ijl}I & * & * & * \\ \Phi_{91j} & N_3H_j & \varepsilon_{1ijl}N_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ijl}I & * & * \\ \varepsilon_{2ijl}N_1 & 0 & \varepsilon_{2ijl}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ijl}I & * \\ -H_j & H_j & 0 & \Phi_{114i} & 0 & \Phi_{116i} & 0 & 0 & \Phi_{119} & 0 & \Phi_{1111} \end{bmatrix}$$

$\Phi_{41ij} = G_1A_i - B_iH_j$ $\quad \Phi_{61ij} = G_3C_{1i} - D_iH_j$

$\Phi_{91j} = \varepsilon_{1ijl}N_1 - N_3H_j$ $\quad \Phi_{52ij} = G_2A_i - Y_jC_{2i}$

$\Phi_{53ij} = G_2E_i - Y_jR_i$ $\quad \Phi_{44l} = -G_1 - G_1^T + P_{1l}$

$\Phi_{74i} = M_{1i}^T G_1^T$ $\quad \Phi_{114i} = \beta(G_1B_i - B_iU)^T$

$\Phi_{55l} = -G_2 - G_2^T + P_{2l}$ $\quad \Phi_{75i} = M_{1i}^T G_2^T$

$\Phi_{85i} = M_{3i}^T Y_j^T$ $\quad \Phi_{66} = -G_3 - G_3^T + I$

$\Phi_{76i} = M_{2i}^T G_3^T$ $\quad \Phi_{116i} = \beta(G_3D_i - D_iU)^T$

$\Phi_{119} = \beta(\varepsilon_{1ijl}N_3 - N_3U)^T$ $\quad \Phi_{1111} = -\beta U - \beta U^T$

the controller and observer gains are given by $K_j = U^{-1}H_j$ *and* $L_j = G_2^{-1}Y_j$, respectively.

Proof. Suppose that inequality (24) holds. The feasible solution of this inequality satisfies $-\beta U^T - \beta U^T < 0$, which implies that the matrix U is non-singular.

By Lemma (2) with :

$$Q = \begin{bmatrix} 0 & 0 & 0 & Q_{14}^T & 0 & Q_{16}^T & 0 & 0 & Q_{19}^T & 0 \end{bmatrix}^T$$

$$W = U^{-1}\begin{bmatrix} -H_j & H_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} -P_{1i} & * & * & * & * & * & * & * & * & * \\ 0 & -P_{2i} & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ \Phi_{41ij} & B_iH_j & G_1E_i & \Phi_{44l} & * & * & * & * & * & * \\ 0 & \Phi_{52ij} & \Phi_{53ij} & 0 & \Phi_{55l} & * & * & * & * & * \\ \Phi_{61ij} & D_iH_j & G_3F_i & 0 & 0 & \Phi_{66} & * & * & * & * \\ 0 & 0 & 0 & \Phi_{74i} & \Phi_{75i} & \Phi_{76i} & -\varepsilon_{1ijl}I & * & * & * \\ 0 & 0 & 0 & 0 & \Phi_{85i} & 0 & 0 & -\varepsilon_{2ijl}I & * & * \\ \Phi_{91j} & N_3H_j & \varepsilon_{1ijl}N_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ijl}I & * \\ \varepsilon_{2ijl}N_1 & 0 & \varepsilon_{2ijl}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ijl}I \end{bmatrix}$$

Where :

$Q_{14} = G_1B_i - B_iU, \quad Q_{16} = G_3D_i - D_iU$

$Q_{19} = \varepsilon_{1ijl}N_3 - N_3U$

the inequality in (24) leads to:

$$T = \begin{bmatrix} -P_{1i} & * & * & * & * & * & * & * & * & * \\ 0 & -P_{2i} & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ \Phi_{41ij} & B_iH_j & G_1E_i & \Phi_{44l} & * & * & * & * & * & * \\ 0 & \Phi_{52ij} & \Phi_{53ij} & 0 & \Phi_{55l} & * & * & * & * & * \\ \Phi_{61ij} & D_iH_j & G_3F_i & 0 & 0 & \Phi_{66} & * & * & * & * \\ 0 & 0 & 0 & \Phi_{74i} & \Phi_{75i} & \Phi_{76i} & -\varepsilon_{1ijl}I & * & * & * \\ 0 & 0 & 0 & 0 & \Phi_{85i} & 0 & 0 & -\varepsilon_{2ijl}I & * & * \\ \Phi_{91j} & N_3H_j & \varepsilon_{1ijl}N_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ijl}I & * \\ \varepsilon_{2ijl}N_1 & 0 & \varepsilon_{2ijl}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ijl}I \end{bmatrix}$$

$$+sym\left\{\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1B_i - B_iU \\ 0 \\ G_3D_i - D_iU \\ 0 \\ 0 \\ \varepsilon_{1ijl}N_3 - N_3U \\ 0 \end{bmatrix} U^{-1}\begin{bmatrix} -H_j & H_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{R}\right\} \tag{25}$$

By defining $K_j = U^{-1}H_j$, R can be written as follows:

$$\begin{bmatrix} 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ -G_1B_iK_j + B_iH_j & G_1B_iK_j - B_iH_j & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ -G_3D_iK_j + D_iH_j & G_3D_iK_j - D_iH_j & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ -\varepsilon_{1ijl}N_3K_j + N_3H_j & \varepsilon_{1ijl}N_3K_j - N_3H_j & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix} \tag{26}$$

from (25) and (26) we can see that inequality (15) in Lemma 5 holds.

Theorem 2. Closed-loop fuzzy system (7) is asymptotically stable with an H_∞ performance $\gamma > 0$ if there exist a known scalar β, positive scalars $\varepsilon_{1ijl}, \varepsilon_{2ijl}$, symmetric matrices $P_{1i}(P_{2i})$, $P_{1l}(P_{2l})$, matrices G_1, G_2, G_3, H_1, H_2, U and Y_i, for $i, j, l = 1, 2, ..., r$, such that the following conditions are satisfied :

$$\Phi_{iil} < 0, \qquad i, j, l = 1, 2, ..., r \tag{27}$$

$$\frac{1}{r-1}\Phi_{iil} + \frac{1}{2}(\Phi_{ijl} + \Phi_{jil}) \qquad i, j, l = 1, 2, ..., r,\ i \neq j \tag{28}$$

the controller and observer gains are given by $K_j = U^{-1}H_j$ *and* $L_j = G_2^{-1}Y_j$, respectively.

Proof. The proof follows directly from Theorem 1 by applying Lemma 4

Remark 1. Theorem 2 presents a new condition for designing observer-based H_∞ controllers for uncertain discrete-time T–S fuzzy systems which is of LMIs and can be effectively solved via LMI Control Toolbox. In contrast with the so-called two step approach (Lo and Lin, 2004), a single approach based on a fuzzy lyapunov function and a slack variables have been used which may help in reducing of the conservatism in the derived results.

Remark 2. It is noted that when $\Delta = 0$, the result of Theorem.2 reduces to that of Theorem 3.3 in (El Haiek et al., 2017).

4. NUMERICAL EXAMPLE

To illustrate the obtained results, consider the following numerical T-S fuzzy system (Chang et al., 2011) with uncertainties composed of two subsystems:

R^1 : if $\xi_1(k)$ is M_{11} then

$$\begin{cases} x(k+1) &= (A_1 + M_1\Delta N_1)x(k) + B_1u(k) + E_1w(k) \\ z(k) &= C_{11}x(k) + D_1u(k) + F_1w(k) \\ y(k) &= (C_{21} + M_3\Delta N_1)x(k) + R_1w(k) \end{cases} \tag{29}$$

R^2 : if $\xi_1(k)$ is M_{12} then

$$\begin{cases} x(k+1) &= A_2x(k) + B_2u(k) + E_2w(k) \\ z(k) &= C_{12}x(k) + D_2u(k) + F_2w(k) \\ y(k) &= C_{22}x(k) + R_2w(k) \end{cases} \tag{30}$$

Where :

$$A_1 = \begin{bmatrix} 1 & -\alpha \\ -1 & -0.5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & \alpha \\ -1 & -0.5 \end{bmatrix} \quad B_1 = \begin{bmatrix} 5+\alpha \\ 2\alpha \end{bmatrix} \quad D_1 = 0.5$$

$$B_2 = \begin{bmatrix} 5-\alpha \\ -2\alpha \end{bmatrix} \quad E_1 = \begin{bmatrix} -0.3 \\ 0.1 \end{bmatrix} \quad E_2 = \begin{bmatrix} -0.3 \\ 0.1 \end{bmatrix} \quad D_2 = 0.5$$

$$C_{11} = \begin{bmatrix} -0.1 \\ 0.05 \end{bmatrix}^T \quad C_{12} = \begin{bmatrix} -0.1 \\ -0.05 \end{bmatrix}^T \quad C_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \quad C_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$$

$$R_1 = -0.1 \quad R_2 = 0.1 \quad F_1 = 0.4 \quad F_2 = 0.4$$

$$M_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T \quad N_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix} \quad \|\Delta\| \leq I$$

$$\begin{cases} h_1(k) = \dfrac{x_1(k) + \alpha}{2\alpha} \\ h_2(k) = 1 - h_1(k) \end{cases} \quad \begin{cases} w(k) = 0.01\sin(k) \\ x_1(k) \in \begin{bmatrix} -\alpha & \alpha \end{bmatrix} \end{cases}$$

We apply Theorem 4 (Lo and Lin, 2004) and Theorem 2 to this fuzzy model. By Theorem 4 (Lo and Lin, 2004), a solution is obtainable when $\alpha \leq 1.488$, while by Theorem 2, a solution is obtainable when $\alpha \leq 1.503$. It implies that Theorem 2 is more relaxed than Theorem 4 (Lo and Lin, 2004) for this example. For example, by MATLAB LMI toolbox, choosing $\alpha = 1.488$ (A_1 is an unstable sub-system matrix and A_2 is a stable sub-system matrix.), the minimum H_∞ attenuation level calculated by Theorem 4 (Lo and Lin, 2004) is $\gamma_{min} = 15.3020$ whereas by Theorem 2 is $\gamma_{min} = 4.1275$. It comes that Theorem 2 ensures better H_∞ performance than Theorem 4 (Lo and Lin, 2004) for this example. In other hand, when $\alpha = 1.503$ (the two sub-system matrices are unstable), we cannot find a feasible solution for Theorem 4 (Lo and Lin, 2004), but for Theorem 2 with $\beta = 1.31$, the closed-loop system is asymptotically stable with H_∞ performance $\gamma_{min} = 15.7276$ as we can

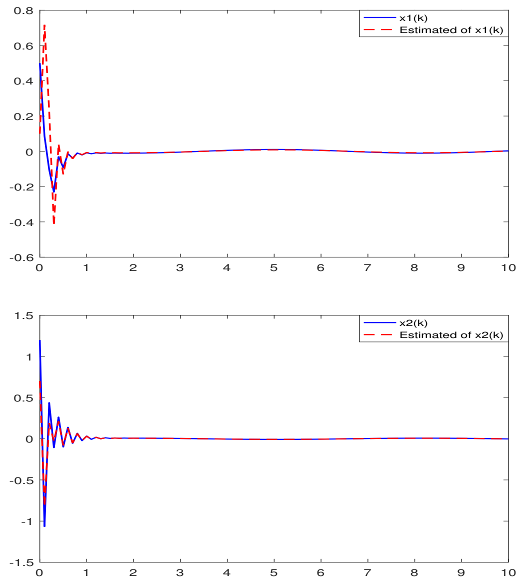

Fig. 1. State trajectories of the closed-loop model.

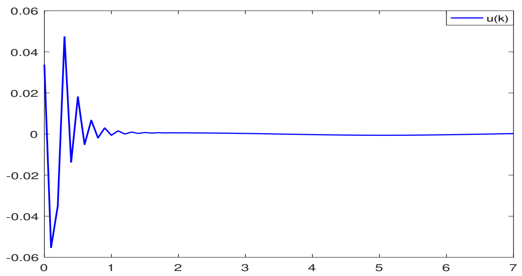

Fig. 2. System control signal u(k).

see in Figures 1, 2 and 3. Figure 1 shows the evolution of the closed-loop states and the estimated states. It is seen that the estimated states (dashed lines) converge to the true states (solid lines) starting from the initial conditions $x(0) = [0.5\ 1.2]^T$, $\hat{x}(0) = [0.1 \quad 0.7]^T$ in the presence of the disturbance signal $w(k)$. Control signal $u(k)$ is shown in Figure 2. The obtained matrices are :

$$K_1 = \begin{bmatrix} 0.0758 & -0.2619 \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0.1295 & 0.3382 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1.0996 & -0.8776 \end{bmatrix}^T \qquad L_2 = \begin{bmatrix} 1.5000 & -1.0132 \end{bmatrix}^T$$

$$P_{11} = \begin{bmatrix} 6.9504 & 0.6207 \\ 0.6207 & 2.7353 \end{bmatrix} \qquad P_{12} = \begin{bmatrix} 6.7987 & 0.8037 \\ 0.8037 & 3.0982 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 10.2958 & -0.1524 \\ -0.1524 & 153.5840 \end{bmatrix} \qquad P_{22} = \begin{bmatrix} 11.1647 & 5.7938 \\ 5.7938 & 137.3817 \end{bmatrix}$$

From Figure 3, we can see that the H_∞ performance $\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k)$ is achieved.

5. CONCLUSION

In this paper, a stability analysis and design of uncertain discrete time T-S fuzzy system via an observer-based controller satisfying the H_∞ performance requirement has been investigated. Controller and observer gains are obtained by solving a set of strict LMIs using single-step

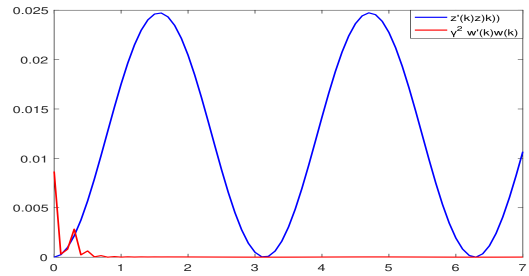

Fig. 3. H_∞ performance.

method. Less conservative results have been obtained by considering a fuzzy Lyapunov function and slack variables. Numerical example has been given to illustrate the effectiveness of the proposed method. Extension of the proposed approach to time-delay TS fuzzy systems can be the focus of our future work.

REFERENCES

Benzaouia, A. and El Hajjaji, A. (2011). Delay-dependent stabilization conditions of controlled positive ts fuzzy systems with time varying delay. *International Journal of Innovative Computing, Information and Control*, 7(4), 1533–1547.

Benzaouia, A. and El Hajjaji, A. (2016). *Advanced Takagi-Sugeno Fuzzy Systems*. Springer.

Benzaouia, A., Hmamed, A., and EL Hajjaji, A. (2010). Stabilization of controlled positive discrete-time t-s fuzzy systems by state feedback control. *International Journal of Adaptive Control and Signal Processing*, 24(12), 1091–1106.

Chang, X., Yang, G., and Wang, H. (2011). Observer-based h_∞-control for discrete-time t-s fuzzy systems. *International Journal of Systems Science*, 42(10), 1801–1809.

Chang, X.H., Zhang, L., and Park, J.H. (2015). Robust static output feedback h_∞ control for uncertain fuzzy systems. *Fuzzy Sets and Systems*, 273, 87–104.

El Haiek, B., Hmamed, A., El Hajjaji, A., and Tissir, E.H. (2017). Improved results on observer-based control for discrete-time fuzzy systems. *International Journal of Systems Science*, 48(12), 2544–2553.

Gao, H. and Wang, Z.W.C. (2005). Improved h_∞ control of discrete-time fuzzy systems: a cone complementarity linearization approach. *Information sciences*, 175(1-2), 57–77.

Lo, J. and Lin, M. (2004). Observer-based robust h_∞ control for fuzzy systems using two step procedure. *IEEE Transactions on Fuzzy Systems*, 12(3), 350–359.

Nachidi, M., Tadeo, F., Benzaouia, A., and Ait Rami, M. (2011). Static output-feedback for takagi–sugeno systems with delays. *International Journal of Adaptive Control and Signal Processing*, 25(4), 295–312.

Petersen, I. (1987). A stabilization algorithm for a class of uncertain linear systems. *Systems & Control Letters*, 8(4), 351–357.

Tuan, H., Apkarian, P., Narikiyo, T., and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on fuzzy systems*, 9(2), 324–332.