

Event-triggered Finite-time Consensus under Directed Graphs^{*}

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Abstract: This paper focuses on deal with the finite-time consensus with event-triggered control strategy for multi-agent systems (MASs). An event-triggered protocol for finite-time consensus is designed using relative measurements. The coordination measurement error is utilized in the triggering condition design for the purpose of removing the prerequisite of topology graph knowledge. Under strongly connected graph assumptions, by utilizing the proposed consensus protocol, all agents can complete consensus and Zeno behaviour will not happen in a settling time. Next, by decomposing the Laplacian matrix in Frobenius norm form, the results are extended to the more general graphs containing a directed spanning tree. At last, a numerical example demonstrates the validity of the algorithm results.

Keywords: Multi-agent system, finite-time consensus, event-triggered, directed graphs

1. INTRODUCTION

Last few decades have witnessed the much progress in the research of multi-agent systems (MASs) Qin et al. (2017); Olfati-Saber et al. (2007). The essential concern in the filed of MASs is consensus Tang et al. (2015); Wu et al. (2016), which aims to propose a protocol to drive all the agents to reach an agreement. Owing to the widely applications, this topic has been investigated in various research domains such as attitude synchronization Thunberg et al. (2014), formation control Dong et al. (2015), and distributed optimization Nedić and Olshevsky (2015).

In the most of real applications of consensus control for MASs, the computation, energy and communication resources are quite limited. For example, in the formation control of quadrotors, each quadrotor is battery-driven with a very limited power. In addition, since the communication channels of wireless networks is limited, the network congest will easily occur with a large number of quadrotors. Due to these facts, it is meaningful to consider how to use the resource efficiently in MASs. In the past few years, a novel event-triggered mechanism is put forward to deal with this issue Dimarogonas et al. (2012); Zhu et al. (2014); Zhang et al. (2019). In the event-triggered mechanism, each agent updates their control protocol at triggering instants, which implies that they don't need to

continuously interact with their neighbours. The triggering moments of each agent are defined through an event-triggered condition (ETC), which should be appropriately designed.

The pioneering result of event-triggered control for MASs is reported in Dimarogonas et al. (2012). The centralized and distributed event-triggered consensus protocols for MASs are both discussed. In Zhu et al. (2014), each agent samples their neighbors' state information and update the protocol at its own triggering instants which can further reduce the update times of controllers. To avoid the continuous communication in the event-detection, the self-triggered mechanism is taken into account in Tang et al. (2016). The latest progress in event-triggered consensus control is survey by Nowzari et al. (2019). Note that in some research Dimarogonas et al. (2012); Zhu et al. (2014), the global information of the topology network is required to be known in event-triggered consensus protocols and event-triggered conditions (ETCs), which makes these protocols are not fully distributed. Recently, an adaptive event-based consensus protocol is put forward to overcome this drawback Cheng and Li (2019); Li et al. (2019). However, in Cheng and Li (2019), the assumption of undirected graphs is still needed.

It should be mentioned that the existing research in event-triggered consensus of MASs above are to achieve the asymptotic consensus. Note that in some real applications, the consensus convergence performance is quite meaningful for MASs. In addition, event-triggered control only takes a control action at some triggering instants which may decrease the convergence rate. Thus, it is worth proposing a finite-time consensus protocol under the event-triggered mechanism to guarantee convergence performance and low resource consumption both.

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Motivated by these discussion, we focus on event-triggered control of finite-time consensus under directed graphs for MASs in this paper. Inspired by the early works on the finite-time consensus Bhat and Bernstein (2000), a finite-time protocol under event-triggered mechanisms is studied for MASs under directed graphs. Furthermore, considering about the fully distributed protocol, the event-triggered finite-time protocol should not use the graph knowledge. Since the protocol and ETC are both related to graphs, designing an event-triggered protocol without utilizing graph knowledge under directed graphs is a great challenge. The merits in this paper are outlined below, 1) An event-triggered protocol is proposed for the finite-time consensus of MASs. It can be revealed that consensus is achieved by using the proposed protocol in a settling time. The communication consumptions and control protocol updates will be reduced compared with continuous consensus protocols. 2) Compare with Zhu et al. (2014); Cheng and Li (2019), the consensus protocol and ETC designed in this work are fully distributed. Furthermore, the results is improved under the directed graphs with spanning trees.

Notation: \mathbb{N} , \mathbb{N}_+ denote a non-negative integer and a positive integer, respectively. \mathbb{R}^N and $\mathbb{R}^{N \times N}$ stand for vector space and matrix space with N dimension and $N \times N$ dimension. Define a vector $x \in \mathbb{R}^N$, $|x| = [|x^{(1)}|, \dots, |x^{(N)}|]^T$ and $x^\sigma = [(x^{(1)})^\sigma, \dots, (x^{(N)})^\sigma]^T$ denote absolute value and power of the vectors, where $x^{(v)}$ typifies the v th element of the vector. Denote $\text{sig}(x)^\gamma = [|x^{(1)}|(x^{(1)})^\gamma, \dots, |x^{(N)}|(x^{(N)})^\gamma]$. $\|x\|$ denotes the Euclidean norm of x and \otimes represents Kronecker product.

2. MATHEMATICAL KNOWLEDGE AND PROBLEM STATEMENT

2.1 Graph

Denote $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ a directed graph, in which $\mathcal{N} = \{1, \dots, N\}$ represents the node set and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ represents the edge set. If agent j can send messages to agent i , then (i, j) is in the \mathcal{E} and agent j is known as a neighbor of agent i . Let \mathcal{N}_i denote the all the agent i 's neighbors set. One directed link between agent i and agent j in the graph is defined as a sequence of connections such as $(j, l_1), (l_1, l_2), \dots, (l_k, i), k \in \mathbb{N}_+$. One directed graph is a tree means that the graph contains a spanning tree means that there is a root node which has directed links to other nodes. An adjacency matrix of a directed graph is formed as $a_{ij} > 0$ if agent j is the neighbors of agent i and $a_{ij} = 0$ otherwise. In addition, it assumes that $a_{ii} = 0$ here which means that the self-loop is excluded. A Laplacian matrix of a directed graph is formed as $l_{ii} = \sum_{j=1}^N a_{ij}$, for $i \neq j$, $l_{ij} = -a_{ij}$ where $i, j = 1, \dots, N$.

2.2 Problem statement

Define a MAS comprising N agents under directed graphs. The agents are modelled by $i = 1, \dots, N$,

$$\dot{x}_i(t) = u_i(t), \quad (1)$$

where $x_i \in \mathbb{R}^n$ denotes the state variable, $u_i \in \mathbb{R}^n$ denotes the input for each agent.

Definition 1. Given a MAS comprising N agents, then finite-time consensus is accomplished if $\forall i, j = 1, \dots, N$,

$$\lim_{t \rightarrow T} (x_i(t) - x_j(t)) = 0, \quad (2)$$

where T is a positive settling time constant.

Since event-triggered control is introduced, the control input is reformed as $i = 1, \dots, N$,

$$u_i(t) = u_i(t_k^i), t \in [t_k^i, t_{k+1}^i), \quad (3)$$

where $t_k^i, k \in \mathbb{N}_+, i = 1, \dots, N$ denote event-triggered moments of each agent.

The primary task in the next subsection is to propose a event-triggered algorithm that can complete the finite-time consensus in Definition 1.

Before introducing the main results, some useful lemmas should be presented firstly.

Lemma 1. (Bhat and Bernstein (2000)) Given a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(x) > 0$ where $x \neq 0$, $V(0) = 0$ and

$$D^+V(x) \leq -\varepsilon V(x)^\beta, x \in \mathbb{R}^n \setminus \{0\}, \quad (4)$$

where $0 < \beta < 1$ and $\varepsilon > 0$, then $x = 0$ is a finite-time stability point and there exists a settling time $T = \frac{1}{\varepsilon(1-\beta)} V(0)^{1-\beta}$.

Lemma 2. (Zhang et al. (2012)) Given a strongly connected graph \mathcal{G} containing N nodes and the associated Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N \times N}$. Then there exists a positive vector $w \in \mathbb{R}^N$ such that $w^T \mathcal{L} = 0$. Furthermore, let $W = \text{diag}\{w_i\}$, $i = 1, \dots, N$, then $W\mathcal{L} + \mathcal{L}^T W \geq 0$.

3. EVENT-TRIGGERED FINITE-TIME CONSENSUS UNDER DIRECTED GRAPHS

In this part, an event-triggered protocol for finite-time consensus is designed for (1). By means of Lyapunov methods, consensus will be accomplished in finite time under assumptions of strongly connected graphs. Next, with the help of decomposing the Laplacian matrix into a Frobenius norm form, the event-triggered finite-time consensus is considered under a tree topology.

3.1 Strongly connected graphs case

Let $x_{ij}(t) = x_i(t) - x_j(t), j \in \mathcal{N}_i$ represent the relative measurement between i th agent and its neighbours. Next, a finite-time consensus protocol based on event-triggered mechanism is constructed by

$$\dot{x}_i(t) = -\beta_i \text{sig} \left(\sum_{j=1}^N a_{ij} (x_{ij}(t_k^i)) \right)^\sigma, \quad (5)$$

where $0 < \sigma < 1$ and $\beta_i > 0$. According to the event-triggered protocols above, each agent will sample the relative information between neighbors at its own triggering instants t_k^i . Let $\tilde{x}_i(t_k^i) = \sum_{j=1}^N a_{ij} (x_{ij}(t_k^i))$ and define the measurement error as,

$$e_i(t) = \text{sig}(\tilde{x}_i(t_k^i))^\sigma - \text{sig}(\tilde{x}_i(t))^\sigma, t \in [t_k^i, t_{k+1}^i). \quad (6)$$

Then, the event-triggered instants of each agent are determined as,

$$t_{k+1}^i = \inf \left\{ t > t_k^i : \beta_i \|e_i(t)\|^2 > k_3 \left(\delta_i \|\tilde{x}_i^\sigma(t)\|^2 + \eta_i^r(t) \right) \right\}, \quad (7)$$

where r is a positive constant such that $0 < r < 1$. β_i , δ_i are positive parameters that will be defined later. η_i is a dynamic variable which is governed by the following equation,

$$\dot{\eta}_i(t) = -k_1 \eta_i^r(t) + k_2 \left(\delta_i \|\tilde{x}_i^\sigma(t)\|^2 - \frac{\beta_i}{k_3} \|e_i(t)\|^2 \right), \quad (8)$$

where $\eta_i(0) > 0$, $k_1 > 0$ and $k_2 > 0$ which will be defined later.

Lemma 3. Given a dynamic variable η_i by (8), under the ETC (7), it can be shown that $\eta_i(t) \geq (-w_1 t + w_2)^{\frac{1}{1-r}}$, where $w_1 = (k_1 + k_2)(1 - r)$, $w_2 = \eta_i(0)^{(1-r)}$.

Proof. Combining with the ETC (7) and (8), one has

$$\dot{\eta}_i(t) \geq -(k_1 + k_2) \eta_i^r(t). \quad (9)$$

Supposed that $y(t)$ is a solution of the differential equation $\dot{y}(t) = -(k_1 + k_2)y^r(t)$. Multiplying $y^{-r}(t)$ on the both sides of the equation, we have the following equivalent equation

$$\frac{d(y^{1-r})}{dt} = -(k_1 + k_2)(1 - r). \quad (10)$$

The differential equation above can be solved by separating variables, which means that $y^{1-r}(t) = -(k_1 + k_2)(1 - r)t + y(0)^{1-r}$. By using comparison lemma, one can obtain that $\eta_i(t) \geq y(t) = [-(k_1 + k_2)(1 - r)t + y(0)^{1-r}]^{\frac{1}{1-r}}$. Hence, the proof is completed.

Remark 1. Inspired by Girard (2015), a dynamic internal variable η_i is introduced in the ETC (7). In the event-triggered mechanism, the important task is to exclude the Zeno behavior. Note that this issue has not been well solved for the static triggering condition Yi et al. (2019). For the purpose of excluding the Zeno behavior explicitly, a dynamic event-triggering condition is introduced in this paper.

Now, the main results can be given below,

Theorem 1. Given a MAS (1) with the strongly connected graph, under event-triggered protocol (5), then the finite-time consensus is accomplished and the Zeno behavior is avoided both in a settling time.

Proof. Given a candidate function as,

$$V(t) = \sum_{p=1}^n \sum_{i=1}^N \frac{w_i}{\lambda(1 + \sigma)} \left| \tilde{x}_i^{(p)}(t) \right|^{1+\sigma} + \sum_{i=1}^N \eta_i(t), \quad (11)$$

where w_i is defined in Lemma 2 above. Since the discontinuity of the right hand side of (5), the Dini derivative of V is calculated below,

$$\begin{aligned} D^+V(t) &= \sum_{p=1}^n \sum_{i=1}^N -\frac{w_i}{\lambda} \text{sig} \left(\tilde{x}_i^{(p)}(t) \right)^\sigma \sum_{j=1}^N \mathcal{L}_{ij} \\ &\quad \times \left[\beta_j \text{sig} \left(\tilde{x}_j^{(p)}(t) \right)^\sigma + \beta_j e_j^{(p)}(t) \right] + \sum_{i=1}^N \dot{\eta}_i(t) \\ &= \sum_{p=1}^n \sum_{i=1}^N -\frac{w_i}{\lambda} \text{sig} \left(\tilde{x}_i^{(p)}(t) \right)^\sigma \sum_{j=1}^N \mathcal{L}_{ij} \\ &\quad \times \left[\beta_j \text{sig} \left(\tilde{x}_j^{(p)}(t) \right)^\sigma \right] - \sum_{p=1}^n \sum_{i=1}^N \frac{w_i}{\lambda} \text{sig} \left(\tilde{x}_i^{(p)}(t) \right)^\sigma \\ &\quad \times \sum_{j=1}^N \mathcal{L}_{ij} \left[\beta_j e_j^{(p)}(t) \right] + \sum_{i=1}^N \dot{\eta}_i(t). \end{aligned} \quad (12)$$

In order to make the proof concise, we define $\hat{x}_i = \left[(\tilde{x}_i^{(1)})^\sigma, (\tilde{x}_i^{(2)})^\sigma, \dots, (\tilde{x}_i^{(n)})^\sigma \right]^T$ and $\hat{x} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N]$. Then follows from (12), we have

$$\begin{aligned} D^+V(t) &= -\frac{1}{2\lambda} \hat{x}^T(t) \left[B(G\mathcal{L} + \mathcal{L}^T G) \otimes I_n \right] \hat{x}(t) \\ &\quad - \frac{1}{2\lambda} \hat{x}^T(t) \left[B(G\mathcal{L} + \mathcal{L}^T G) \otimes I_n \right] e(t) \\ &\quad + \sum_{i=1}^N k_2 \left(\delta_i \|\hat{x}_i(t)\|^2 - \beta_i \|e_i(t)\|^2 \right) \\ &\quad - \sum_{i=1}^N k_1 \eta_i^r(t). \end{aligned} \quad (13)$$

According to Lemma 2 and by virtue of the facts that $WL + L^T W \geq \lambda I_N$, we have

$$\begin{aligned} D^+V(t) &\leq -\frac{1}{2} \hat{x}(t)^T (B \otimes I_n) \hat{x}(t) - \frac{1}{2} \hat{x}(t)^T (B \otimes I_n) e(t) \\ &\quad + \sum_{i=1}^N k_2 \left(\delta_i \|\hat{x}_i(t)\|^2 - \beta_i \|e_i(t)\|^2 \right) - \sum_{i=1}^N k_1 \eta_i^r(t) \\ &\leq -\frac{1}{2} \sum_{i=1}^N \beta_i \|\hat{x}_i(t)\|^2 + \frac{\gamma}{4} \sum_{i=1}^N \beta_i \|\hat{x}_i(t)\|^2 \\ &\quad + \frac{1}{4\gamma} \sum_{i=1}^N \beta_i \|e_i(t)\|^2 + \sum_{i=1}^N k_2 \left(\delta_i \|\hat{x}_i(t)\|^2 \right. \\ &\quad \left. - \beta_i \|e_i(t)\|^2 \right) - \sum_{i=1}^N k_1 \eta_i^r(t). \end{aligned} \quad (14)$$

Using the ETC (7) and choosing $\gamma = \frac{1}{2}$, we have

$$\begin{aligned} D^+V(t) &\leq -\frac{1}{4} \sum_{i=1}^N \left[\beta_i - k_2 \delta_i - (1 - k_2) k_3 \delta_i \right] \|\hat{x}_i(t)\|^2 \\ &\quad - \sum_{i=1}^N \left[k_3(1 - k_2) + k_1 \right] \eta_i^r(t) \\ &= -C_3 \sum_{p=1}^n \sum_{i=1}^N \left[\frac{\beta_i}{\lambda(1 + \sigma)} \left| \tilde{x}_i^{(p)}(t) \right|^{1+\sigma} \right]^{\frac{2\sigma}{1+\sigma}} \\ &\quad - C_2 \sum_{i=1}^N \eta_i^r(t), \end{aligned} \quad (15)$$

where $C_1 = \frac{\beta_{\min} - [k_2 + (1-k_2)k_3]\delta_{\max}}{4}$, $C_2 = k_3(1-k_2) + k_1$, $C_3 = C_1 \frac{\lambda(1+\sigma)}{w_{\max}}$. Choosing $r = \frac{2\sigma}{1+\sigma}$, $k_3(1-k_2) + k_1 > 0$ and $\beta_{\min} - [k_2 + (1-k_2)k_3]\delta_{\max} > 0$, from (15), we can derive that $D^+V \leq -KV \frac{2\sigma}{1+\sigma}$, where $K = \min\{C_2, C_3\}$.

By using Lemma 3, it shows that $V = 0$ after a settling time $T = \frac{(1+\sigma)V(0) \frac{1+\sigma}{1-\sigma}}{K(1-\sigma)}$. Thus, one has $\tilde{x}(t) = Lx = 0$. Since the $\text{Null}(L) = aI_N$ and a is a constant, then one can derive that the consensus is accomplished in a settling time.

The following part is to show that Zeno-behavior will not happen before the settling time T_2 . Note that $\|f - g\|^2 \leq 2\|f\|^2 + 2\|g\|^2, \forall f, g \in \mathbb{R}^n$, we have $\beta_i \|e_i(t)\|^2 \leq 2\beta_i (\|\hat{x}_i(t_k^i)\|^2 + \|\hat{x}_i(t)\|^2)$. Using Lemma 3 and the ETC (7), we can derive a sufficient condition for (7),

$$2\beta_i \|\hat{x}_i(t_k^i)\|^2 + (2\beta_i - \delta_i) \|\hat{x}_i(t)\|^2 \leq (-w_1 t + w_2) \frac{r}{1-r}. \quad (16)$$

Based on the proof above, we know that $\|\hat{x}_i\|$ is bounded before the settling time T . Thus, there exists a triggering instant t_k^i such that $(4\beta_i - \delta_i) \|\hat{x}_i(t_k^i)\|^2 \geq (-w_1 t_k^i + w_2) \frac{r}{1-r}$. Let $\mathcal{M}(t_k^i) = (4\beta_i - \delta_i) \|\hat{x}_i(t_k^i)\|^2$. Since $\|\hat{x}_i\|$ is decreasing, there must exist another triggering instants t_{k+1}^i such that $\mathcal{M}(t_{k+1}^i) < \mathcal{M}(t_k^i)$. Since the next event will trigger when $\mathcal{M}(t_{k+1}^i) \geq (-w_1 t_{k+1}^i + w_2) \frac{r}{1-r}$, we can conclude that

$$\mathcal{M}(t_k^i) - \mathcal{M}(t_{k+1}^i) \leq \mu(t_k^i)(t_{k+1}^i - t_k^i), \quad (17)$$

where $\mu(t_k^i)$ denotes the derivative of the function in the right side of (16) at triggering instant t_k^i . Then, we can obtain that $(t_{k+1}^i - t_k^i) > \frac{\mathcal{M}(t_k^i) - \mathcal{M}(t_{k+1}^i)}{\mu(t_k^i)}$, which means that the Zeno behavior can not happen before the settling time T . \square

Remark 2. In this subsection, a finite-time consensus protocol is investigated under the event-triggered mechanism, which has two aspects of benefits. Firstly, the protocol and ETC only contain the relative measurement information for each agent. Note that the relative information is more easily obtained than the absolute information in some situations such as vision based multi-robot systems. Each robot can measure the relative bearing information by vision sensors. Secondly, by using the coordination measurement error which is different from the traditional measurement error, the Laplacian matrix information is not included in the protocol and ETC, which means that the protocol we designed is fully distributed.

Remark 3. Note that the continuous relative information are utilized in the ETC (7) which requires each agent continuously monitors the relative information with their neighbors. This problem can be solved by predicting the current neighbors' states. In fact, from (5) we know that each agent dynamics is piece-wise constant in t , thus each agent can calculate their neighbors' current states based on the sampling information of their neighbors.

Remark 4. The finite-time event-triggered consensus protocol is proposed in Dong and Xian (2017) recently. In order to handle with directed graph cases, the signed control law is used in the consensus protocol. However, it should be noted that the signed control law is a discontinuous signal, which may result in chattering phenomenon. In

addition, the parameter in protocol is dependent on the global information of the graph. Compared with the result in Dong and Xian (2017), the consensus protocol proposed in this paper under directed graphs is continuous and does not use any global information of the graph.

3.2 Directed graphs cases

This part, we strengthen the result in above subsection to the directed graphs which contains a directed spanning tree. Noted that the Laplacian matrix of a strongly directed graph is an irreducible matrix. For a Laplacian matrix of a tree topology, we can decompose it into a Frobenius norm form as follows,

$$P^T \mathcal{L} P = \left[\begin{array}{ccc|c} \tilde{\mathcal{L}}_{11} & \tilde{\mathcal{L}}_{12} & \cdots & \tilde{\mathcal{L}}_{1m} \\ 0 & \tilde{\mathcal{L}}_{22} & \cdots & \tilde{\mathcal{L}}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathcal{L}_{mm} \end{array} \right] = \left[\begin{array}{cc} \mathcal{L}_p & \mathcal{L}_{pq} \\ 0 & \mathcal{L}_q \end{array} \right] \quad (18)$$

where P is a permutation matrix and the diagonal block matrices $\tilde{\mathcal{L}}_{ii}, i = 1, \dots, m$ are irreducible matrices. Therefore, the Frobenius norm form of the Laplacian matrices associated with directed graphs containing a directed spanning tree corresponds to the decomposing the graph into strongly connected components. For the matrix \mathcal{L}_q , according to Lemma 2, there is a positive vector h_q satisfying that $h_q^T \tilde{\mathcal{L}}_q = 0$. Furthermore, let $H_q = \text{diag}\{h_q\}$, one has $H_q \tilde{\mathcal{L}}_q + \tilde{\mathcal{L}}_q^T H_q \geq \lambda_q I_N$, where λ_q is the second smallest eigenvalue of the matrix $H_q \tilde{\mathcal{L}}_q + \tilde{\mathcal{L}}_q^T H_q$. Also note that the matrix $\tilde{\mathcal{L}}_q$ has a single zero eigenvalue. Thus, we can deduce that eigenvalues of matrix $\tilde{\mathcal{L}}_p$ are positive.

Following from the Frobenius norm form (18), we define

$$\left[\begin{array}{c} \tilde{x}_p \\ \tilde{x}_q \end{array} \right] = \left[\begin{array}{cc} \mathcal{L}_p & \mathcal{L}_{pq} \\ 0 & \mathcal{L}_q \end{array} \right] \otimes I_n \left[\begin{array}{c} x_p \\ x_q \end{array} \right] \left. \vphantom{\left[\begin{array}{c} \tilde{x}_p \\ \tilde{x}_q \end{array} \right]} \right\} \begin{array}{l} N - N_q \text{ rows} \\ N_q \text{ rows} \end{array} \quad (19)$$

where $x = [x_q, x_p]^T, x_q \in \mathbb{R}^{nN_q}$ and $x_p \in \mathbb{R}^{n(N-N_q)}$. Similar from above, let $e = [e_p, e_q]^T$ and $\eta = [\eta_p, \eta_q]^T$. The main results in this subsection are shown below,

Theorem 2. Given a MAS (1) under a tree topology, using the event-triggered protocol (5), then the finite-time consensus is accomplished and Zeno behavior is avoided in a settling time.

Proof. Firstly, we consider the following Lyapunov candidate function,

$$V_q(t) = \sum_{\kappa=1}^n \sum_{i=N-N_q+1}^N \frac{h_i}{\lambda_q(1+\sigma)} \left| \tilde{x}_i^{(\kappa)}(t) \right|^{1+\sigma} + \sum_{i=N-N_q+1}^N \eta_i(t). \quad (20)$$

where λ_q is the smallest eigenvalue of the matrix \mathcal{L}_q . After using similar techniques in the proof of Theorem 1, it can be shown that $D^+V_q \leq -K_q V_q \frac{2\sigma}{1+\sigma}$, where $K_q = \min\{C_4, C_5\}$, $C_4 = \frac{\lambda_q(1+\sigma)}{h_{\max}} \times \frac{\beta_{\min} - [k_2 + (1-k_2)k_3]\delta_{\max}}{4}$, $C_5 = k_3(1-k_2) + k_1$. Therefore, we can obtain that

$\tilde{x}_q = 0$ after the settling time $T_1 = \frac{(1+\sigma)V_q(0)^{\frac{1+\sigma}{1-\sigma}}}{K_q(1-\sigma)}$. Next, considering the Lyapunov candidate function,

$$V_p(t) = \sum_{\kappa=1}^n \sum_{i=1}^{N-N_q} \frac{1}{\lambda_p(1+\sigma)} |\tilde{x}_i^{(\kappa)}(t)|^{1+\sigma} + \sum_{i=1}^{N-N_q} \eta_i(t). \quad (21)$$

Taking the Dini derivative of the V_p follows that,

$$\begin{aligned} D^+V_p(t) &= \sum_{\kappa=1}^n \sum_{i=1}^{N-N_q} -\frac{1}{\lambda_p} \text{sig}\left(\tilde{x}_i^{(\kappa)}(t)\right)^\sigma \\ &\quad \times \left[\sum_{j=1}^{N-N_q} \mathcal{L}_{ij} \left[\beta_j \text{sig}\left(\tilde{x}_j^{(\kappa)}(t_k^i)\right)^\sigma \right] \right. \\ &\quad \left. + \sum_{j=N-N_q+1}^N \mathcal{L}_{ij} \left[\beta_j \text{sig}\left(\tilde{x}_j^{(\kappa)}(t_k^i)\right)^\sigma \right] \right] \\ &\quad + \sum_{i=1}^{N-N_q} \dot{\eta}_i(t). \end{aligned} \quad (22)$$

In line with the analysis above, there is a settling time T_1 such that $\tilde{x}_q(\tilde{t}) = 0, \tilde{t} > T_1$. Hereafter,

$$\begin{aligned} D^+V_p(\tilde{t}) &= \sum_{\kappa=1}^n \sum_{i=1}^{N-N_q} -\frac{1}{\lambda_p} \text{sig}\left(\tilde{x}_i^{(\kappa)}(\tilde{t})\right)^\sigma \\ &\quad \times \sum_{j=1}^{N-N_q} \mathcal{L}_{ij} \left[\beta_j \text{sig}\left(\tilde{x}_j^{(\kappa)}(\tilde{t})\right)^\sigma + \beta_j e_j^{(\kappa)}(\tilde{t}) \right] \\ &\quad + \sum_{i=1}^{N-N_q} \dot{\eta}_i(\tilde{t}) \\ &= -\frac{1}{\lambda_p} \hat{x}_p(\tilde{t})^T (B_p \mathcal{L}_p \otimes I_n) \hat{x}_p(\tilde{t}) \\ &\quad - \frac{1}{\lambda_p} \hat{x}_p(\tilde{t})^T (B_p \mathcal{L}_p \otimes I_n) e_p(\tilde{t}) + \sum_{i=1}^{N-N_q} \dot{\eta}_i(\tilde{t}) \\ &\leq -\frac{1}{2} \sum_{i=1}^{N-N_q} \beta_i \|\hat{x}_i(\tilde{t})\|^2 + \frac{1}{2} \sum_{i=1}^{N-N_q} \beta_i \|e_i(\tilde{t})\|^2 \\ &\quad + \sum_{i=1}^{N-N_q} k_2 (\delta_i \|\hat{x}_i(\tilde{t})\|^2 - \beta_i \|e_i(\tilde{t})\|^2) \\ &\quad - \sum_{i=1}^{N-N_q} k_1 \eta_i^r(\tilde{t}). \end{aligned} \quad (23)$$

Then combining with the ETC (7), it follows that,

$$\begin{aligned} D^+V_p(\tilde{t}) &\leq -\frac{1}{2} \sum_{i=1}^{N-N_q} \left[\beta_i - k_2 \delta_i - (1-k_2)k_3 \delta_i \right] \|\hat{x}_i(\tilde{t})\|^2 \\ &\quad - \sum_{i=1}^{N-N_q} \left[k_3(1-k_2) + k_1 \right] \eta_i^r(\tilde{t}) \\ &\leq -C_6 \sum_{p=1}^n \sum_{i=1}^{N-N_q} \left[\frac{1}{\lambda_p(1+\sigma)} |\tilde{x}_i^p(\tilde{t})|^{1+\sigma} \right]^{\frac{2\sigma}{1+\sigma}} \\ &\quad - C_7 \sum_{i=1}^{N-N_q} \eta_i^r(\tilde{t}), \end{aligned} \quad (24)$$

where $C_6 = \lambda_p(1+\sigma) \times \frac{\beta_{\min} - [k_2 + (1-k_2)k_3]\delta_{\max}}{4}$, $C_7 = k_3(1-k_2) + k_1$. Choosing $r = \frac{2\sigma}{1+\sigma}$, then we can obtain that $D^+V_p(\tilde{t}) \leq -K_p V_p(\tilde{t})^{\frac{2\sigma}{1+\sigma}}, \tilde{t} > T_1$, where $K_p = \min\{C_6, C_7\}$. Hereafter, we can conclude that $V_p(t) \rightarrow 0$ after a settling time $T_2 = \frac{(1+\sigma)V_p(T_1)^{\frac{1+\sigma}{1-\sigma}}}{K_p(1-\sigma)}$, which means that $\tilde{x}(t) = 0, t > T_2$. The exclusion of the Zeno behavior can be obtained by using the similar procedure in the proof of Theorem 1.

Remark 5. At this part, the result of event-triggered finite-time consensus protocol (5) and ETC (7) is extended to the more general graph cases which has a directed spanning tree. In a recent research Hu et al. (2019), an event-triggered protocol for finite-time consensus is proposed for MASs under undirected graphs and the agents' states are restricted to one dimension. In this paper, the directed graph with a tree topology is considered and each agent state is a N dimensional vector.

4. SIMULATION

In this part, an illustrative example is shown to verify the validity of event-triggered protocols in Theorem 1 and Theorem 2. Given a MAS involving seven agents under directed graphs with a tree topology. In addition, we present the Laplacian matrix associated with the graph below,

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}. \quad (25)$$

The initial value of each agent is chosen as $x_i(0) = [0.1 * i, 0.5 * i]^T, i = 1, \dots, N$. Without losing the generality, the parameters in the event-triggered protocol (5) and ETC (7) are chosen as $\sigma = 0.8, \beta = 1, \delta = 0.5, k_1 = 0.5, k_2 = 0.125, k_3 = 1$. Simulation conclusions are demonstrated in Figs. 1-3. As illustrated in Fig. 1 and Fig. 2, the first and the second component of states for each agent reach consensus in finite-time, respectively. In Fig. 3, the triggering instants of each agent are illustrated, which shows the advantages compared with the continuous protocols.

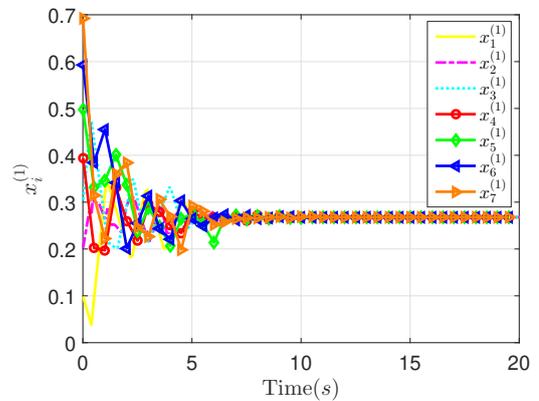


Fig. 1. The first coordinates of each agent.

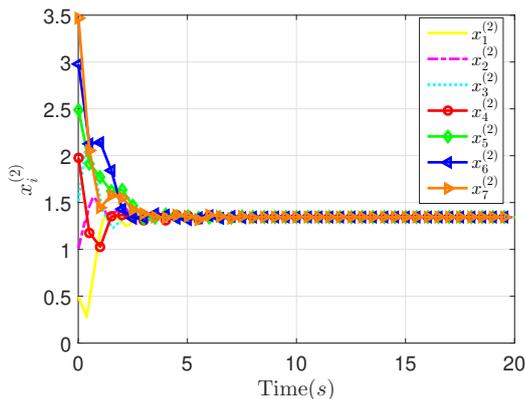


Fig. 2. The second coordinates of each agent.

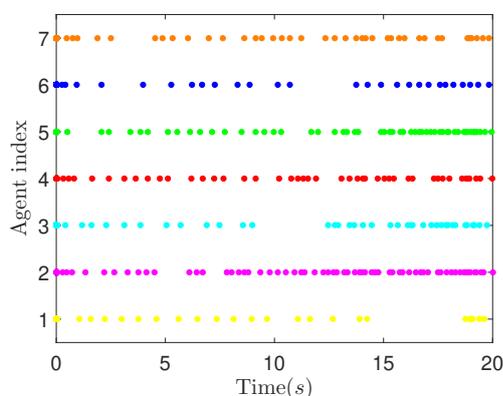


Fig. 3. The event-triggered instants for each agent.

5. CONCLUSION

This paper considers the event-triggered finite-time consensus of MASs under directed graphs. Firstly, an event-triggered protocol of finite-time consensus is proposed for MASs under the strongly connected graph. Each agent only need to measure the relative information with neighbors to update the control protocols and decide triggering or not. In addition, a dynamic ETC is considered to obtain the larger inter-event intervals. Secondly, the more general directed graph with a tree topology is considered. Furthermore, the event-triggered protocol and ETC are fully distributed without the usage of the global information. Future works will concentrate on the event-triggered consensus with switching topologies.

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