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# Topology Identification of Fractional Complex Networks with An Auxiliary Network

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**Abstract:** In this paper, topology identification of fractional complex networks is investigated. First of all, an important result is obtained, which reveals some special relations between the primitive function and its fractional derivative. Then an auxiliary network consisting of isolated nodes and a regulation mechanism are designed in order that there is no need to check the linear independence condition ($LIC$) and the identification failure caused by network synchronization can be avoided. By applying inequality techniques, the realizability of topology identification for fractional systems is proved. And then two algorithms are given to identify the unknown parameters in the original network. In order to have more realistic significance, the accuracy function, which is an upper bound of the estimation error between the estimated results and the corresponding unknown parameters, is considered to evaluate the validity of our estimation algorithms. Furthermore, an example is provided to demonstrate the effectiveness of the main results.

*Keywords:* Fractional calculus, Complex networks, Topology identification, Synchronization, Adaptive control.

## 1. INTRODUCTION

Complex networks Strogatz (2001), which consist of a great number of nodes interconnected by edges, are ubiquitous in various fields of the real world, such as power grids Albert et al. (2004), telecommunication networks Schintler et al. (2005), the Internet Maslov et al. (2004) and so on. Over the past few decades, research on complex networks has been done and fruitful results have been obtained, such as synchronization control Yi et al. (2017),Xu et al. (2018), structure optimization Zhou et al. (2006) and so on. Note that the dynamical systems with known parameters have been investigated in the aforementioned papers. But, in most of practical situations, the topology of a large-scale network is usually not accessible. Thus an important topic called topology identification or parameter estimation has emerged and it has attracted more and more attention Xu et al. (2017),Wu et al. (2016).

One of the main methods used in the field of topology identification for unknown network is the so-called synchronization-based method Yu et al. (2006),Zhou et al. (2007),Wu. (2008),Liu et al. (2009) and a key assumption is needed in these existing papers that all coupling terms of the unknown network are linearly independent of the synchronization manifold between the drive (unknown) network and a designed response network, which is called the linear independence condition ($LIC$). However, there are two shortcomings associated with the $LIC$ that should be noted. Firstly, there is still no effective method to verify the $LIC$ so far. Secondly, the topology identification is likely to fail if the $LIC$ does not hold. For example, a conclusion has been obtained in Chen et al. (2009) that, if the drive network is in a synchronous situation, the $LIC$ will not be satisfied and the topology identification process based on Yu et al. (2006),Zhou et al. (2007),Wu. (2008),Liu et al. (2009) will fail. Therefore, it is necessary to explore some new methods on topology identification, which can overcome the disadvantages. Note that some efforts have been devoted in Zhu et al. (2019) but it is still a challenging problem.

It should be pointed out that all the mathematical models considered in this paper are fractional systems. As one of the important branches of mathematics, fractional calculus Oldham et al. (1974) was born in 1695 almost simultaneously with classical integer-order integration and differentiation. In recent years, people have gradually realized that fractional systems can provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. And the theories of fractional calculus have been applied to a wide range of areas successfully, which include mathematical biology Ahmed et al. (2007), finance Laskin. (2000), engineering Zhang et al. (2017),Ren et al. (2016) and so on. Therefore to investigate fractional complex networks will enrich and perfect the nonlinear theories and applications.

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It is noteworthy that some conclusions may not hold in the sense of fractional calculus while they are holding in the sense of integer-order calculus. For example, *Lemma* 1 in Zhu et al. (2018) (or *Lemma* 2.2 in Zhao et al. (2020)), which plays a key role in the process of topology identification, has not been proved in the sense of fractional calculus. Based on the difference between fractional and integer-order calculus and the lack of some important conclusions, two problems should be highlighted. 1) By using control theories, such as synchronization-based method, can the topology identification of fractional systems be realized? 2) How to design an effective strategy to identify unknown parameters? And can an appropriate mechanism be given to evaluate the validity of the estimation strategy?

According to the above discussion, the main contributions of this manuscript are summarized as follows. An important result is proposed and proved, which can reveal some special relations between the primitive function and its fractional derivative. And an auxiliary network consisting of isolated nodes is constructed and a regulation protocol is designed in order that there is no need to check the $LIC$ and the identification failure caused by network synchronization can be avoided. And then, by using the synchronization-based method and inequality techniques, the realizability of topology identification for fractional systems is proved. Then two algorithms are proposed to estimate the unknown parameters in the original network. And the accuracy function is considered to denote the accuracy of estimated results and this function can also be used to evaluate the effectiveness of the algorithms.

The rest of this paper is organized as follows. Section 2 gives some mathematical preliminaries. The main results about topology identification of fractional complex networks are presented in section 3. Next, a numerical simulation in section 4 is provided to demonstrate the effectiveness and correctness of the theoretical results. Finally, this manuscript is concluded in section 5.

*Notations*: Let $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, $R^n$ be the $n$-dimensional Euclidean space and $R^{n\times m}$ be the space of $n \times m$ real matrices. $Z_+$ represents the collection of all positive integers. $I_n$ denotes $n \times n$ real identity matrix. $A^T$ means the transpose of the matrix $A$. $B^{-1}$ means the inverse of the matrix $B$. $\parallel \beta \parallel$ denotes the Euclidean norm of a vector $\beta$, which is defined by $\parallel \beta \parallel = \sqrt{\sum_{i=1}^n \beta_i^2}$ where $\beta = (\beta_1, \cdots, \beta_n)^T$. $\parallel \Psi \parallel$ stands for the Euclidean norm of a matrix $\Psi$, which is defined by $\parallel \Psi \parallel = \sqrt{\lambda_{max}(\Psi^T\Psi)}$ where $\lambda_{max}(\cdot)$ represents the maximum eigenvalue of the corresponding matrix. $Rank(A)$ denotes the rank of matrix $A$. $\mid f(t) \mid$ denotes the absolute value of a function $f(t) \in R$. Let ${}_{t_0}^{C}D_{\gamma}^{\alpha}x(t) = {}_{t_0}^{C}D_t^{\alpha}x(t) \mid_{t=\gamma}$.

## 2. PRELIMINARIES

### *2.1 Fractional calculus*

In this subsection, two common definitions of fractional calculus are introduced and several necessary lemmas are presented.

*Definition 1.* (Kilbas et al. (2006)) Caputo fractional derivative with order $\alpha$ for a function $x : R^+ \to R$ is defined as

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} \frac{d^m x(\tau)}{d\tau^m} d\tau, $$

where $0 \le m-1 < \alpha < m$, $m \in Z_+$ and $\Gamma(\cdot)$ is Gamma function.

*Definition 2.* (Kilbas et al. (2006)) Riemann-Liouville fractional integral with order $\alpha$ for a function $f : R^+ \to R$ is defined by

$$ {}_{t_0}^{R}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

where $\alpha > 0$ and $\Gamma(\cdot)$ is Gamma function.

*Lemma 3.* (Kilbas et al. (2006)) For any constants $k_1$ and $k_2$ , the linearity of Caputo fractional derivative is described by

$$ {}_{t_0}^{C}D_t^{\alpha}(k_1 g(t) + k_2 h(t)) = k_1 {}_{t_0}^{C}D_t^{\alpha} g(t) + k_2 {}_{t_0}^{C}D_t^{\alpha} h(t). $$

*Lemma 4.* (Kilbas et al. (2006)) If $0 < \alpha < 1$ and $x(t) : R \to R$ is a differentiable function, then the following equality holds,

$$ {}_{t_0}^{R}I_t^{\alpha} {}_{t_0}^{C}D_t^{\alpha} x(t) = x(t) - x(t_0), $$

where $t \ge t_0$.

*Lemma 5.* (Duarte-Mermoud et al. (2015)) Let $v(t) : R \to R^n$ be a vector of differentiable function. Then, for any time instant $t \ge t_0$, the following inequality holds,

$$ {}_{t_0}^{C}D_t^{\alpha}(v^T(t)Pv(t)) \le 2v^T(t)(P {}_{t_0}^{C}D_t^{\alpha} v(t)), $$

where $P \in R^{n\times n}$ is a positive definite matrix and $0 < \alpha < 1$.

*Lemma 6.* (Kilbas et al. (2006)) If $\alpha > 0$ and $\beta > 0$, then

$$ {}_{t_0}^{R}I_t^{\alpha}(t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-t_0)^{\alpha+\beta-1}, $$

where $\Gamma(\cdot)$ is Gamma function.

### *2.2 Problem formulation*

A fractional complex network consisting of $N$ nodes is considered as follows,

$$ {}_{t_0}^{C}D_t^{\alpha} x_i(t) = f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} G x_j(t-\tau(t)), \quad 1 \le i \le N, \ i \in Z_+, \tag{1} $$

where $0 < \alpha < 1$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ is the state vector of the $i$th node at time $t$, $f_i : R^n \to R^n$ is a continuously differentiable vector function, $\tau(t) \ge 0$ denotes time-varying delay at time $t$, $G \ne 0 \in R^{n\times n}$ is the inner coupling matrix and $A = (a_{ij})_{N\times N}$ is the unknown weight configuration matrix. If there exists a link from node $i$ to node $j$ $(i \ne j)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$ $(i \ne j)$; $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$. Note that the configuration matrix $A$ does not have to be symmetric or irreducible, which means that network (1) can be directed or undirected, connected or unconnected.

In this paper, the topological structure of network (1) will be identified. In order to estimate the configuration matrix

$A$, an auxiliary network composed of $N$ isolated nodes is constructed as follows,

$$y_i(t) = v_i g(t),\ 1 \le i \le N,\ i \in Z_+, \tag{2}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t))^T \in R^n$ is the state vector of the $i$th node, $v_i \in R^n$ is a constant vector and $g(t) : R \to R$ is a continuously differentiable function. Furthermore, a regulation mechanism is added to network (1) and the regulated network can be described by

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} x_i(t) = f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} G\ x_j(t - \tau(t)) + u_i(t), \\ 1 \le i \le N,\ i \in Z_+, \end{aligned} \tag{3}$$

where $u_i(t)$ is the regulation protocol which will be designed later.

Then three necessary assumptions are given as follows.

*Assumption 7.* Assume that $\tau(t)$ is bounded and differentiable. And $\dot{\tau}(t)$ is also bounded. Namely, there exist two positive constants $\phi_1, \phi_2$ such that $\tau(t) \le \phi_1$ and $|\ \dot{\tau}(t)\ | \le \phi_2$.

*Assumption 8.* Suppose that there exists a non-negative constant $\xi$ such that

$$\begin{aligned} \parallel f_i(u(t)) - f_i(h(t)) \parallel \le \xi \parallel u(t) - h(t) \parallel, \\ 1 \le i \le N,\ i \in Z_+, \end{aligned} \tag{4}$$

for $t \ge t_0$ and $\forall u, h \in R^n$.

*Assumption 9.* Assume that there exist two positive constants $\eta_1, \eta_2$ such that $\eta_1 \le |\ g(t)\ | \le \eta_2$.

*Remark 10. Assumption* 7 is proposed to prevent the occurrence of high-frequency chattering phenomena, which may become an obstacle to identification of network topology, in the evolution of the function $\tau(t)$.

## 3. MAIN RESULTS

In this section, an important proposition is derived. Based on this proposition, the realizability of topology identification for fractional complex networks is proved and two algorithms are proposed to estimate unknown parameters.

### 3.1 A necessary proposition

A proposition is given and proved as follows, which will be used to derive the main results in this paper.

*Proposition 11.* Let ${}_{t_0}^{C}D_t^{\alpha} x(t) = f(t)$, where $x(t), f(t) : R \to R$ and $0 < \alpha < 1$. If there exist two positive constants $\theta_1$ and $\theta_2$ such that $|\ x(t)\ | \le \theta_1$ and $|\ f(t)\ | \le \theta_2$, then there exists a time sequence $\{t_k\}_{k=1}^{\infty}$ such that $|\ f(t_k)\ | \le \frac{1}{t_k^{\beta}}$, where $\beta$ is an arbitrary constant and $0 < \beta < \alpha$. And $t_0 \le t_1 < t_2 < \cdots$, $\lim\limits_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k \le T_k = \inf\limits_{s>0}\{s \mid \frac{s^{\alpha}}{(t_k+s)^{\beta}} \ge 2\theta_1\Gamma(\alpha+1) + \theta_2(t_k - t_0)^{\alpha}\}$.

**Proof.** Assume, for the purpose of contradiction, that there exists a time instant $t^* < \infty$ such that $|\ f(t)\ | > \frac{1}{t^{\beta}}$ for $t \ge t^*$. Further suppose that $f(t) > \frac{1}{t^{\beta}}$ for $t \ge t^*$.

From *Lemmas* 4, 6, the following inequality holds for $t \ge t^*$,

$$\begin{aligned} x(t) &= x(t_0) + {}_{t_0}^{R}I_t^{\alpha} f(t) \\ &= x(t_0) + \frac{1}{\Gamma(\alpha)}\Big[\int_{t_0}^{t^*} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &+ \int_{t^*}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau + \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau \\ &- \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau\Big] \\ &\ge x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &+ {}_{0}^{R}I_t^{\alpha} \frac{1}{t^{\beta}} - \frac{1}{\Gamma(\alpha)} \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau \\ &= x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &+ \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)} t^{\alpha-\beta} - \frac{1}{\Gamma(\alpha)} \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau. \end{aligned} \tag{5}$$

Since $\frac{|f(\tau)|}{(t-\tau)^{1-\alpha}} \le \frac{|f(\tau)|}{(t^*-\tau)^{1-\alpha}}$ for $t \ge t^*$ and $t_0 \le \tau \le t^*$,

$$\begin{aligned} \Big|\ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau\ \Big| &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{|\ f(\tau)\ |}{(t-\tau)^{1-\alpha}} d\tau \\ &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{\theta_2}{(t^*-\tau)^{1-\alpha}} d\tau \\ &= \frac{\theta_2}{\Gamma(\alpha+1)} (t^* - t_0)^{\alpha}, \end{aligned} \tag{6}$$

which implies that $\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau$ is bounded when $\forall t \ge t^*$. Similarly, when $t \ge t^*$,

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau &\le \frac{1}{\Gamma(\alpha)} \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t^*-\tau)^{1-\alpha}} d\tau \\ &= \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)} (t^*)^{\alpha-\beta}, \end{aligned} \tag{7}$$

which implies that $\frac{1}{\Gamma(\alpha)} \int_{0}^{t^*} \frac{1}{\tau^{\beta}(t-\tau)^{1-\alpha}} d\tau$ is bounded.

Then, it follows from inequality (5) that

$$\lim_{t\to\infty} x(t) = \infty. \tag{8}$$

This contradicts the previous hypothesis. Similarly, in the case that $f(t) < -\frac{1}{t^{\beta}}$ for $t \ge t^*$, conflicting results can still be derived. Thus, there exists a time sequence $\{t_k\}_{k=1}^{\infty}$ such that $|\ f(t_k)\ | \le \frac{1}{t_k^{\beta}}$. Meanwhile, $\lim\limits_{k\to\infty} t_k = \infty$.

Next, the conclusion that $t_{k+1} - t_k \le T_k = \inf\limits_{s>0}\{s \mid \frac{s^{\alpha}}{(t_k+s)^{\beta}} \ge 2\theta_1\Gamma(\alpha+1) + \theta_2(t_k - t_0)^{\alpha}\}$ can be proved. Assume, for the purpose of contradiction, that $t_{k+1} > t_k + T_k$, which implies that $|\ f(t)\ | > \frac{1}{(t_k+T_k)^{\beta}}$ for $t \in [t_k, t_k +$

$T_k]$. Further assume that $f(t) > \frac{1}{(t_k+T_k)^\beta}$ for $t \in [t_k, t_k + T_k]$. Based on *Lemmas* 4, 6, the following inequality can be obtained for $t \in [t_k, t_k + T_k]$,

$$\begin{aligned} x(t) &= x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &= x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_k} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &> -\theta_1 - \frac{\theta_2}{\Gamma(\alpha+1)} (t_k - t_0)^\alpha \\ &+ \frac{1}{\Gamma(\alpha)(t_k+T_k)^\beta} \int_{t_k}^{t} \frac{1}{(t-\tau)^{1-\alpha}} d\tau \\ &= -\theta_1 - \frac{\theta_2}{\Gamma(\alpha+1)} (t_k - t_0)^\alpha \\ &+ \frac{(t-t_k)^\alpha}{(t_k+T_k)^\beta \Gamma(\alpha+1)}. \end{aligned} \tag{9}$$

Obviously, $x(t_k + T_k) > -\theta_1 - \frac{\theta_2}{\Gamma(\alpha+1)}(t_k - t_0)^\alpha + \frac{(T_k)^\alpha}{(t_k+T_k)^\beta \Gamma(\alpha+1)} \geq \theta_1$. Similarly, if $f(t) < -\frac{1}{(t_k+T_k)^\beta}$ for $t \in [t_k, t_k + T_k]$, a conclusion can be derived that $x(t_k + T_k) < -\theta_1$. This contradicts the previous hypothesis that $\mid x(t) \mid \leq \theta_1$. Therefore, the conclusion that $t_{k+1} - t_k \leq T_k = \inf_{s>0}\{s \mid \frac{s^\alpha}{(t_k+s)^\beta} \geq 2\theta_1\Gamma(\alpha+1) + \theta_2(t_k - t_0)^\alpha\}$ has been proved.

This completes the proof.

*Remark 12.* It is worth noting that the existence of $\lim_{t\to\infty} x(t)$ does not necessarily imply that $\lim_{t\to\infty} {}_{t_0}^{C}D_t^\alpha x(t) = 0$. For example, the following function is considered,

$$x(t) = \frac{1}{t} cos(t^2),\ t \in (1, \infty).$$

Obviously, $\lim_{t\to\infty} x(t) = 0$ but $\lim_{t\to\infty} \dot{x}(t)$ does not exist. However, according to *Proposition* 11, the weaker conclusion that there exists a time sequence $\{t_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} {}_{t_0}^{C}D_{t_k}^\alpha x(t) = 0$ can be obtained when some additional conditions, like that $x(t), {}_{t_0}^{C}D_t^\alpha x(t)$ are bounded, are satisfied. Since *Proposition* 11 reveals some special relations between the primitive function and its fractional derivative, it may be helpful to some theoretical derivations and practical engineering applications.

### *3.2 Analysis on the realizability of topology identification*

Since $G \neq 0$, there exist two constant vectors $\mu, \zeta \in R^n$ and a positive integer $m \in \{1, 2, \cdots, n\}$ such that $G\mu = \zeta$, $\zeta_m \neq 0$ and $\zeta_i = 0$ $(1 \leq i \leq n,\ i \neq m,\ i \in Z_+)$, where $Rank(G) = Rank(G \vdots \zeta)$ and $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)^T$.

Denote $e_i(t) = (e_{i1}(t), e_{i2}(t), \cdots, e_{in}(t))^T = x_i(t) - y_i(t)$, $1 \leq i \leq N$. Next, the regulation protocol in system (3) is designed as

$$\begin{aligned} u_i(t) &= {}_{t_0}^{C}D_t^\alpha y_i(t) - f_i(y_i(t)) \\ &- \sum_{j=1}^{N} \hat{a}_{ij}(t) G x_j(t - \tau(t)) - d_i e_i(t), \end{aligned} \tag{10}$$

$${}_{t_0}^{C}D_t^\alpha \hat{a}_{ij}(t) = e_i^T(t) G x_j(t - \tau(t)), \tag{11}$$

where $1 \leq i, j \leq N$, $i, j \in Z_+$, $\hat{a}_{ij}(t)$ is an adaptive parameter and $d_i$ is a positive constant. Then, a theorem is given to prove that topology identification of fractional complex network (3) is realizable.

*Theorem 13.* Let $v_s = \mu$ and $v_i = 0$ $(i \in Z_+,\ 1 \leq i \leq N,\ i \neq s)$, where $s$ is a positive integer and $s \in \{1, 2, \cdots, N\}$. Suppose that *Assumptions* 7 − 9 hold and

$$d_i > \xi, \tag{12}$$

where $1 \leq i \leq N$, $i \in Z_+$. Then, under regulation protocols (10)(11), there exists a time sequence $\{t_k^{is}\}_{k=1}^\infty$ $(1 \leq i \leq N,\ i \in Z_+)$ such that

$$\lim_{k\to\infty} \mid \hat{a}_{is}(t_k^{is}) - a_{is} \mid = 0, \tag{13}$$

where $t_0 \leq t_1^{is} < t_2^{is} < \cdots < \infty$, $\lim_{k\to\infty} t_k^{is} = \infty$, $t_{k+1}^{is} - t_k^{is} \leq T_k^{is} = \inf_{\sigma>0}\{\sigma \mid \frac{\sigma^\alpha}{(t_k^{is}+\sigma)^\beta} \geq 2\theta_1\Gamma(\alpha+1) + \theta_2(t_k^{is} - t_0)^\alpha\}$, $\theta_1 = \sqrt{2V(t_0)}$, $\theta_2 = (\xi + d_i + \eta_2 \mid \zeta_m \mid)\sqrt{2V(t_0)} + \sqrt{2V(t_0)} \parallel G \parallel \varphi$, $\varphi = \max\{\max_{s\in[t_0-\phi_1, t_0]} (\sum_{j=1}^{N} \parallel e_j(s) \parallel), \sqrt{2NV(t_0)}\}$, $V(t_0) = \frac{1}{2}\sum_{i=1}^{N} \parallel e_i(t_0) \parallel^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(a_{ij} - \hat{a}_{ij}(t_0))^2$, $\beta$ is an arbitrary constant and $0 < \beta < \alpha$.

Namely, the $s$th column of matrix $A$ can be estimated effectively at time sequence $\{t_k^{is}\}_{k=1}^\infty$ $(1 \leq i \leq N,\ i \in Z_+)$.

**Proof.** Let $\tilde{a}_{ij}(t) = a_{ij} - \hat{a}_{ij}(t)$. Under regulation protocols (10)(11), the error system between (2) and (3) is given by

$$\begin{aligned} {}_{t_0}^{C}D_t^\alpha e_i(t) &= f_i(x_i(t)) - f_i(y_i(t)) \\ &+ \sum_{j=1}^{N} \tilde{a}_{ij}(t) G x_j(t - \tau(t)) - d_i e_i(t), \end{aligned} \tag{14}$$

where $1 \leq i \leq N,\ i \in Z_+$.

The following Lyapunov function for error system (14) is considered,

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{a}_{ij}^2(t). \tag{15}$$

By *Assumption* 8 together with *Lemmas* 3 and 5, we have

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) &\leq \sum_{i=1}^{N} e_i^T(t){}_{t_0}^{C}D_t^{\alpha}e_i(t) \\
&-\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{a}_{ij}(t){}_{t_0}^{C}D_t^{\alpha}\hat{a}_{ij}(t) \\
&=\sum_{i=1}^{N} e_i^T(t)[f_i(x_i(t))-f_i(y_i(t))] \\
&+\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{a}_{ij}(t)e_i^T(t)Gx_j(t-\tau(t)) \\
&-\sum_{i=1}^{N} d_i e_i^T(t)e_i(t) \\
&-\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{a}_{ij}(t)e_i^T(t)Gx_j(t-\tau(t)) \\
&\leq \sum_{i=1}^{N}(\xi-d_i)\parallel e_i(t)\parallel^2 \leq 0.
\end{aligned} \tag{16}$$

According to Gu et al. (2017), it follows from inequality (16) that $\lim_{t\to\infty} \parallel e(t) \parallel = 0$ where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$. And, from *Lemma* 4, $V(t) = V(t_0) + {}_{t_0}^{R}I_t^{\alpha}{}_{t_0}^{C}D_t^{\alpha}V(t) \leq V(t_0)$, which implies that $e_i(t), \tilde{a}_{ij}(t), \hat{a}_{ij}(t)$ are bounded.

Since $v_s = \mu,\ v_i = 0\ (i \in Z_+,\ 1 \leq i \leq N,\ i \neq s)$, error system (14) can be rewritten as follows,

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}e_i(t) &= f_i(x_i(t)) - f_i(y_i(t)) \\
&+\sum_{j=1}^{N}\tilde{a}_{ij}(t)Ge_j(t-\tau(t)) \\
&+\tilde{a}_{is}(t)Gy_s(t-\tau(t)) - d_i e_i(t),
\end{aligned} \tag{17}$$

where $1 \leq i \leq N,\ i \in Z_+$.

From *Assumption* 9, the following inequality holds,

$$\begin{aligned}
\parallel {}_{t_0}^{C}D_t^{\alpha}e_i(t) \parallel &\leq (\xi + d_i) \parallel e_i(t) \parallel + \eta_2 \mid \zeta_m \tilde{a}_{is}(t) \mid \\
&+\sum_{j=1}^{N} \mid \tilde{a}_{ij}(t) \mid\parallel Ge_j(t-\tau(t)) \parallel \\
&\leq (\xi + d_i + \eta_2 \mid \zeta_m \mid)\sqrt{2V(t_0)} \\
&+\sqrt{2V(t_0)} \parallel G \parallel \varphi,
\end{aligned} \tag{18}$$

where the property that

$$\sum_{j=1}^{N} \parallel e_j(t-\tau(t)) \parallel \leq \sqrt{N\sum_{j=1}^{N} \parallel e_j(t-\tau(t)) \parallel^2}$$

is used above. Then ${}_{t_0}^{C}D_t^{\alpha}e_{im}(t)$ is bounded and

$$\begin{aligned}
\mid {}_{t_0}^{C}D_t^{\alpha}e_{im}(t) \mid &\leq \parallel {}_{t_0}^{C}D_t^{\alpha}e_i(t) \parallel \\
&\leq (\xi + d_i + \eta_2 \mid \zeta_m \mid)\sqrt{2V(t_0)} \\
&+\sqrt{2V(t_0)} \parallel G \parallel \varphi,
\end{aligned}$$

where $e_{im}(t)$ is the $m$th component of $e_i(t)$.

According to *Proposition* 11, there exists a time sequence $\{t_k^{is}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} {}_{t_0}^{C}D_{t_k^{is}}^{\alpha}e_{im}(t) = 0$, where $t_0 \leq t_1^{is} < t_2^{is} < \cdots < \infty$, $\lim_{k\to\infty} t_k^{is} = \infty$, $t_{k+1}^{is} - t_k^{is} \leq T_k^{is} = \inf_{\sigma>0}\{\sigma \mid \frac{\sigma^{\alpha}}{(t_k^{is}+\sigma)^{\beta}} \geq 2\theta_1\Gamma(\alpha+1) + \theta_2(t_k^{is}-t_0)^{\alpha}\}$, $\theta_1 = \sqrt{2V(t_0)}$, $\theta_2 = (\xi + d_i + \eta_2 \mid \zeta_m \mid)\sqrt{2V(t_0)} + \sqrt{2V(t_0)} \parallel G \parallel \varphi$, $\beta$ is an arbitrary constant and $0 < \beta < \alpha$.

From equation (17) and *Assumption* 9, the following inequality can be obtained,

$$\begin{aligned}
\mid {}_{t_0}^{C}D_t^{\alpha}e_{im}(t) \mid &\geq \eta_1 \mid \zeta_m \tilde{a}_{is}(t) \mid - (\xi + d_i) \parallel e_i(t) \parallel \\
&-\sqrt{2NV(t_0)} \parallel G \parallel\parallel e(t-\tau(t)) \parallel .
\end{aligned} \tag{19}$$

It follows from inequality (19) that

$$\begin{aligned}
\mid \tilde{a}_{is}(t) \mid \leq \frac{1}{\eta_1 \mid \zeta_m \mid}\{&\mid {}_{t_0}^{C}D_t^{\alpha}e_{im}(t) \mid \\
&+(\xi + d_i) \parallel e_i(t) \parallel \\
&+\sqrt{2NV(t_0)} \parallel G \parallel\parallel e(t-\tau(t)) \parallel\}.
\end{aligned} \tag{20}$$

Obviously, $\lim_{k\to\infty} \mid \tilde{a}_{is}(t_k^{is}) \mid = 0$, which means that the unknown parameter $a_{is}$ can be identified at these special moments $\{t_k^{is}\}_{k=1}^{\infty}$. Therefore, under regulation protocols (10)(11), the $s$th column of the configuration matrix $A$ can be identified.

This completes the proof.

*Remark 14.* It is easy to find that the larger the value of the parameter $d_i$ is, the easier condition (12) is to satisfy and the more favorable it is to the topological identification process. And note that, for a function $z(t): R \to R$, $\lim_{t\to\infty} {}_{t_0}^{C}D_t^{\alpha}z(t) = 0$ does not necessarily imply the existence of $\lim_{t\to\infty} z(t)$. For example, the following equality is considered,

$${}_{t_0}^{C}D_t^{\alpha}z(t) = \frac{1}{t^{\beta}},\ t_0 > 0,$$

where $\beta$ is a constant and $0 < \beta < \alpha < 1$. It is obvious that $\lim_{t\to\infty} {}_{t_0}^{C}D_t^{\alpha}z(t) = 0$ but $\lim_{t\to\infty} z(t) = \lim_{t\to\infty}[z(t_0) + {}_{t_0}^{R}I_t^{\alpha}\frac{1}{t^{\beta}}] = \infty$ (*the derivation of this equality is similar to the derivation of inequalities* (5)(7)(8)). Thus the conclusion that, from (11), $\lim_{t\to\infty} \hat{a}_{ij}(t)$ exists when $\lim_{t\to\infty} e_i(t) = 0$ can not be derived, which means the unknown parameter $a_{ij}$ can not be directly estimated by adaptive protocol (11). But, based on *Theorem* 13, the conclusion has been proved that the unknown topology matrix $A$ can be identified at the time sequences $\{t_k^{is}\}_{k=1}^{\infty}$ $(1 \leq i, s \leq N,\ i, s \in Z_+)$. And the characteristic $t_{k+1}^{is} - t_k^{is} \leq T_k^{is}$ of $\{t_k^{is}\}_{k=1}^{\infty}$ ensures a property that $t_{k+1}^{is} < \infty$ when $t_k^{is} < \infty$, which guarantees the realizability of topology identification for fractional systems in practical engineering applications.

### *3.3 Two algorithms to estimate the unknown topology*

Let $\bar{a}_{ij}(t)$ $(1 \leq i, j \leq N,\ i, j \in Z_+)$ represents the estimator of the unknown parameter $a_{ij}$. Then an algorithm is proposed to identify the unknown parameters in network (3).

*Theorem 15.* Let $v_s = \mu$ and $v_i = 0$ $(i \in Z_+,\ 1 \leq i \leq N,\ i \neq s)$, where $s$ is a positive integer and $s \in \{1, 2, \cdots, N\}$. And let $\bar{a}_{is}(t)$ be defined by

$$\bar{a}_{is}(t) = \hat{a}_{is}(\varepsilon_{is}), \tag{21}$$

where $t \geq t_0$, $1 \leq i \leq N,\ i \in Z_+$ and $\varepsilon_{is} = max\{\arg\min_{t_0\leq\varsigma\leq t}(\mid {}_{t_0}^{C}D_{\varsigma}^{\alpha}e_{im}(t) \mid + \parallel e_i(\varsigma) \parallel + \parallel e(\varsigma - \tau(\varsigma)) \parallel)\}$. Suppose that

*Assumptions* 7−9 hold and the conditions in *Theorem* 13 are satisfied. Then, based on regulation protocols (10)(11), the $s$th column of matrix $A$ can be identified effectively by estimators $\bar{a}_{is}(t)$ $(1 \leq i \leq N,\ i \in Z_+)$. Precisely, the following equality holds,

$$a_{is} = \lim_{t\to\infty} \bar{a}_{is}(t), \tag{22}$$

where $1 \leq i \leq N,\ i \in Z_+$.

**Proof.** According to *Theorem* 13 and *Proposition* 11, $\lim_{t\to\infty} \| e(t) \| = 0$ and there exists a time sequence $\{t_k^{is}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \mid {}_{t_0}^{C}D_{t_k^{is}}^{\alpha} e_{im}(t) \mid = 0$. Next, the following equality can be obtained,

$$\lim_{t\to\infty} \min_{t\geq t_0}\{\mid {}_{t_0}^{C}D_{\varsigma}^{\alpha} e_{im}(t) \mid + \| e_i(\varsigma) \| + \| e(\varsigma - \tau(\varsigma)) \|\} = 0. \tag{23}$$

It is obvious that there exist two positive constants $\varpi_1, \varpi_2$ such that the following inequalities hold,

$$\begin{aligned} &\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid + \| e_i(t) \| \\ &\qquad + \| e(t-\tau(t)) \| \leq \varpi_1 \chi_{is}(t), \\ &\chi_{is}(t) \leq \varpi_2\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid + \| e_i(t) \| \\ &\qquad + \| e(t-\tau(t)) \|\}, \end{aligned} \tag{24}$$

where $\chi_{is}(t) = \frac{1}{\eta_1 |\zeta_m|}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid +(\xi + d_i) \| e_i(t) \| + \sqrt{2NV(t_0)} \| G \| \| e(t-\tau(t)) \|\}$. Therefore,

$$\begin{aligned} &\lim_{t\to\infty} \min_{t\geq t_0}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid + \| e_i(t) \| \\ &\qquad + \| e(t-\tau(t)) \|\} = 0, \\ &if\ and\ only\ if,\ \lim_{t\to\infty} \min_{t\geq t_0}(\chi_{is}(t)) = 0. \end{aligned} \tag{25}$$

It follows from inequality (20) that

$$\begin{aligned} \mid a_{is} - \hat{a}_{is}(t) \mid \leq\ & \frac{1}{\eta_1 \mid \zeta_m \mid}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid \\ & +(\xi + d_i) \| e_i(t) \| \\ & + \sqrt{2NV(t_0)} \| G \| \| e(t-\tau(t)) \|\}. \end{aligned} \tag{26}$$

Then, from inequalities (23)(25)(26), the following inequality holds,

$$\begin{aligned} \lim_{t\to\infty} \mid \bar{a}_{is}(t) - a_{is} \mid &= \lim_{t\to\infty} \mid \hat{a}_{is}(\varepsilon_{is}) - a_{is} \mid \\ &\leq \lim_{t\to\infty} \frac{1}{\eta_1 \mid \zeta_m \mid}\{\mid {}_{t_0}^{C}D_{\varepsilon_{is}}^{\alpha} e_{im}(t) \mid \\ &\quad +(\xi + d_i) \| e_i(\varepsilon_{is}) \| \\ &\quad + \sqrt{2NV(t_0)} \| G \| \| e(\varepsilon_{is} \\ &\quad -\tau(\varepsilon_{is})) \|\} = 0. \end{aligned} \tag{27}$$

This completes the proof.

Note that the condition, $t \to \infty$, is difficult to satisfy in reality because most topology identification processes always take place in a finite period of time. Therefore, in order to have more practical significance, another algorithm will be designed, in which the accuracy function is considered to denote the accuracy of estimated results at any given time.

Next, an assumption is needed.

*Assumption 16.* Suppose that there exists a constant $M > 0$ such that $\mid a_{ij} \mid \leq M$.

*Remark 17.* Note that *Assumption* 16 is very mild. For example, in the Internet, the communication bandwidth between any two routers is always limited and its upper bound can always be known.

An auxiliary function is defined as follows,

$$\begin{aligned} p_{ij}(t) = &\frac{1}{\eta_1 \mid \zeta_m \mid}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid +(\xi + d_i) \| e_i(t) \| \\ &+\psi \| G \| \| e(t-\tau(t)) \|\}, \end{aligned} \tag{28}$$

where $1 \leq i, j \leq N,\ i, j \in Z_+$ and

$$\psi = \sqrt{N \| e(t_0) \|^2 + 2N(M^2N^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \hat{a}_{ij}^2(t_0))}.$$

Let $\omega_{ij}(t)$ $(1 \leq i, j \leq N,\ i, j \in Z_+)$ denote the accuracy function of the estimate of function $\bar{a}_{ij}(t)$ against $a_{ij}$ at time $t$, which is an upper bound of the estimation error $\mid \bar{a}_{ij}(t) - a_{ij} \mid$ between $\bar{a}_{ij}(t)$ and $a_{ij}$. Next an algorithm is presented to estimate the unknown topology of system (3).

*Theorem 18.* Let $v_s = \mu$ and $v_i = 0$ $(i \in Z_+,\ 1 \leq i \leq N,\ i \neq s)$, where $s$ is a positive integer and $s \in \{1, 2, \cdots, N\}$. And let functions $\bar{a}_{is}(t)$ and $\omega_{is}(t)$ be defined by

$$\bar{a}_{is}(t) = \hat{a}_{is}(\varepsilon_{is}), \tag{29}$$

$$\omega_{is}(t) = p_{is}(\varepsilon_{is}), \tag{30}$$

where $t \geq t_0$, $1 \leq i \leq N$, $i \in Z_+$ and $\varepsilon_{is} = max\{\arg\min_{t_0\leq\varsigma\leq t}(p_{is}(\varsigma))\}$. Suppose that *Assumptions* 7−9, 16 hold and the conditions in *Theorem* 13 are satisfied. Then, based on regulation protocols (10)(11), the $s$th column of matrix $A$ can be estimated effectively by estimators $\bar{a}_{is}(t)$ $(1 \leq i \leq N,\ i \in Z_+)$. Precisely, the following equalities hold

$$a_{is} = \lim_{t\to\infty} \bar{a}_{is}(t), \tag{31}$$

$$\lim_{t\to\infty} \omega_{is}(t) = 0, \tag{32}$$

where $1 \leq i \leq N$, $i \in Z_+$ and $\mid a_{is} - \bar{a}_{is}(t) \mid \leq \omega_{is}(t)$ for $t \geq t_0$.

**Proof.** It follows from *Assumption* 16 and inequality (20) that

$$\begin{aligned} \mid a_{is} - \hat{a}_{is}(t) \mid \leq\ & \frac{1}{\eta_1 \mid \zeta_m \mid}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid \\ & +(\xi + d_i) \| e_i(t) \| \\ & + \sqrt{2NV(t_0)} \| G \| \| e(t-\tau(t)) \|\} \\ \leq\ & \frac{1}{\eta_1 \mid \zeta_m \mid}\{\mid {}_{t_0}^{C}D_t^{\alpha} e_{im}(t) \mid \\ & +(\xi + d_i) \| e_i(t) \| \\ & +\psi \| G \| \| e(t-\tau(t)) \|\} \\ =\ & p_{is}(t). \end{aligned} \tag{33}$$

It is obvious that

$$\mid a_{is} - \bar{a}_{is}(t) \mid = \mid a_{is} - \hat{a}_{is}(\varepsilon_{is}) \mid \leq p_{is}(\varepsilon_{is}) = \omega_{is}(t). \tag{34}$$

According to *Theorem* 13 and *Proposition* 11, $\lim_{t\to\infty} \| e(t) \| = 0$ and there exists a time sequence $\{t_k^{is}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \mid {}^C_{t_0}D^{\alpha}_{t_k^{is}} e_{im}(t) \mid= 0$. Next, the following equality can be obtained,

$$\lim_{t\to\infty} \min_{t\geq t_0}\{p_{is}(t)\} = 0. \tag{35}$$

Obviously, from (30), $\lim_{t\to\infty} \omega_{is}(t) = 0$. And it follows from inequality (34) that $\lim_{t\to\infty} \mid \bar{a}_{is}(t) - a_{is} \mid\leq \lim_{t\to\infty} \omega_{is}(t) = 0$.

This completes the proof.

*Remark 19.* In *Theorem* 18, the conclusion is obtained that, via equality (29) and regulation protocols (10)(11), the unknown elements in the $s$th column of matrix $A$ can be estimated effectively. Then, because of the arbitrariness of the value of $s$ in the set $\{1, 2, \cdots, N\}$, all the unknown parameters in $A$ can be identified in the same way.

*Remark 20.* Since the estimation error $\mid \bar{a}_{ij}(t) - a_{ij} \mid$ between $\bar{a}_{ij}(t)$ and $a_{ij}$ is always unknown, the accuracy function $\omega_{ij}(t)$, which is an upper bound of $\mid \bar{a}_{ij}(t) - a_{ij} \mid$ but can always be known, is considered to denote the accuracy of the estimate of function $\bar{a}_{ij}(t)$ against $a_{ij}$ at time $t$ and the reasonableness of using this function to evaluate the validity of estimation algorithm (31) is shown in equalities (32)(34).

*Remark 21.* It is easy to find that regulation protocols (10)(11) are centralized and the individual information of each node in these two algorithms need to be known, such as the inner coupling matrix $G$ and the nonlinear function $f_i$. These may become limiting factors when the algorithms designed in this paper are applied to solve some real-world engineering problems. Therefore, how to design a distributed regulation protocol, which is more easily used to solve the problem about topological identification, will be the focus of our future work.

## 4. NUMERICAL SIMULATIONS

In this section, an example is shown to illustrate the correctness of the obtained theoretical results.

*Example 22.* Consider three-dimensional complex network (3), which consists of six nodes. A nonlinear function is given as follows,

$$f(s(t)) = \begin{pmatrix} -0.8s_1(t) + 2tanh(s_1(t)) - 1.2tanh(s_2(t)); \\ 1.8tanh(s_1(t)) + 1.71tanh(s_2(t)) \\ \qquad + 1.15tanh(s_3(t)); \\ -1.6s_3(t) - 4.75tanh(s_1(t)) \\ \qquad + 1.1tanh(s_3(t)); \end{pmatrix}$$

where $s(t) = (s_1(t), s_2(t), s_3(t))^T$. Let $f_i(s(t)) = f(s(t))$, $1 \leq i \leq 6,\ i \in Z_+$. The weight configuration matrix $A = (a_{ij})_{6\times6}$ is described as follows,

$$A = \begin{bmatrix} -0.3, & 0.1, & 0, & 0, & 0.1, & 0.1; \\ 0.2, & -0.7, & 0.3, & 0, & 0.1, & 0.1; \\ 0, & 0, & -0.2, & 0.05, & 0.15, & 0.05; \\ 0, & 0.1, & 0.6, & -0.8, & 0.1, & 0; \\ 0.2, & 0.2, & 0.5, & 0.05, & -1, & 0.05; \\ 0.1, & 0.2, & 0.3, & 0.4, & 0.5, & -1.5 \end{bmatrix}, \tag{36}$$

where all the elements in this matrix are assumed to be unknown. But suppose that a fact is known that $\mid a_{ij} \mid\leq 10$ for $1 \leq i, j \leq 6,\ i \in Z_+$. The inner coupling matrix $G$ is

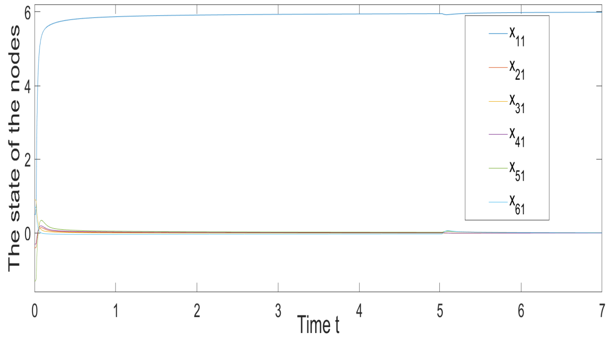


Fig. 1. The state of nodes in network (3) under protocols (10)(11).

$$G = \begin{bmatrix} 2, 0, -1; \\ 0, 1, -1; \\ -1, 0, \ \ 2 \end{bmatrix}. \tag{37}$$

Let $\mu = (1, 1, 1)^T$, then $G\mu = \zeta = (1, 0, 0)^T$. The other parameters in network (3) are assumed to be that $\alpha = 0.65$, $t_0 = 0$ and $\tau(t) = 5$. The initial conditions of system (3) are chosen randomly: $x_1(s) = (0.5, 0.4, 0.7)^T$, $x_2(s) = (-0.4, -1.6, 0.5)^T$, $x_3(s) = (0.9, -0.6, 0.2)^T$, $x_4(s) = (-0.3, -0.8, 1.2)^T$, $x_5(s) = (-1.3, -2.1, 1.8)^T$, $x_6(s) = (0.7, -0.7, 0.7)^T$ and $s \in [-5, 0]$. The initial values of adaptive parameters $\hat{a}_{ij}(t)$ in equations (10)(11) are supposed to be that $\hat{a}_{ij}(t_0) = 0$ for $1 \leq i, j \leq 6,\ i, j \in Z_+$. Let $d_i = 15$ $(1 \leq i \leq 6,\ i \in Z_+)$, which can make condition (12) be satisfied. Let $g(t) = 6$ for $t \geq t_0$. Then auxiliary function (28) can be rewritten as follows,

$$p_{ij}(t) = \frac{1}{6}\{\mid {}^C_{t_0}D^{\alpha}_t e_{i1}(t) \mid +25 \parallel e_i(t) \parallel \\ +\psi \parallel G \parallel\parallel e(t - \tau(t)) \parallel\}, \tag{38}$$

where $\psi = \sqrt{6 \parallel e(t_0) \parallel^2 + 43200}$ and $1 \leq i, j \leq 6,\ i, j \in Z_+$.

Let $v_1 = \mu$ and $v_i = 0$, $2 \leq i \leq 6$, $i \in Z_+$. Based on equality (31), the elements in the first column of matrix $A$ can be identified. Figs. 1,2,3 show the simulation results. Fig. 1 shows the state of nodes in network (3), which indicates that network (3) and auxiliary network (2) can achieve synchronization based on regulation protocols (10)(11). Parameters identification are shown in Fig. 2. The estimators $\bar{a}_{i1}(t)$ $(1 \leq i \leq 6,\ i \in Z_+)$ converge to constants in accordance with the values of unknown parameters $a_{i1}$ $(1 \leq i \leq 6,\ i \in Z_+)$. In Fig. 3, the accuracy functions $\omega_{i1}(t)$ $(1 \leq i \leq 6,\ i \in Z_+)$ converge rapidly to zero, which demonstrates that the method for topology identification proposed in *Theorem* 18 is very effective. And, it can be obtained from Fig. 3 that $\mid a_{i1} - \bar{a}_{i1}(t) \mid\leq 0.43$ $(1 \leq i \leq 6,\ i \in Z_+)$ when $t = 250$.

## 5. CONCLUSION

In this manuscript, topology identification of fractional complex networks is studied. In order to facilitate the derivation of the main conclusions, an important proposition is proposed and proved, which can reveal some special relations between the primitive function and its fractional derivative. Based on this proposition, two algorithms are

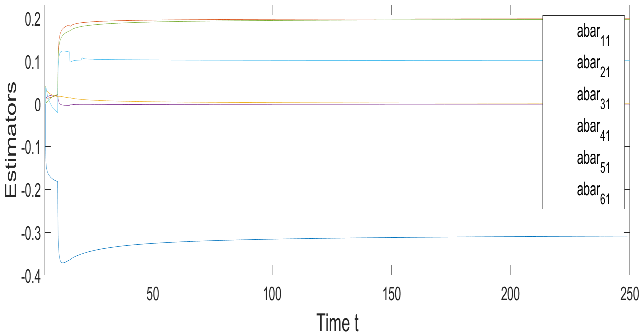


Fig. 2. Time evolution of estimators $\bar{a}_{i1}(t)$ $(1 \leq i \leq 6,\ i \in Z_+)$.

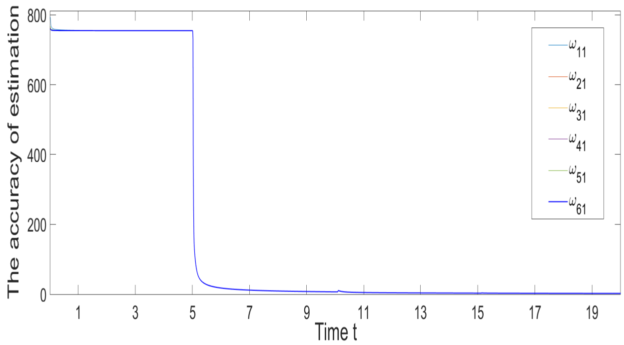


Fig. 3. Time evolution of accuracy functions $\omega_{i1}(t)$ $(1 \leq i \leq 6,\ i \in Z_+)$.

designed, by which the unknown parameters in the original network can be estimated. Moreover, the accuracy function is considered to denote the accuracy of the estimated results and this function can also be used to evaluate the validity of our estimation algorithms.

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