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Dynamics of Generic Linear Agents Over Signed Networks Without Structural Constraints ⋆

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Abstract: Signed networks have been widely used to describe cooperative and competitive interactions in multiagent systems (MASs) so far. Most of the existing dynamics results of MASs on signed networks have a certain constraint on the network topology, that is, the network topology is required to have sufficient connectivity for realizing consensus or bipartite consensus. The highlight of this article is to extend the existing dynamics of MASs to a more general signed network, in which there are no any structural constraints on the topology. This general setting of the network topology unifies most existing models, such as consensus, bipartite consensus, and bipartite containment, which are usually analyzed separately in the same framework using different methods. Relying on a method of constructing cooperative auxiliary digraphs, it is theoretically proved that the agents in closed strong connected components with balanced structure and unbalanced structure gradually reach separately bipartite consensus and consensus, and the agents outside the closed strong connected components gradually enter the convex hull formed by the agents in the closed strong connected components, that is, achieving bipartite containment. Finally, a computer simulation is presented to verify the theoretical discovery.

Keywords: Multiagent systems; Signed networks; Consensus; Bipartite consensus; Bipartite containment

1. INTRODUCTION

In recent decades, coordinated control of multi-agent systems (MASs) has been a hot topic that is of great interest to researchers in various fields. This is because coordinated control system has many advantages over the traditional single-chip system, such as reducing system cost and extending the life of the system. The fundamental problem in coordination control of MASs is to design a distributed protocol based on local relative information to ensure that all agents achieve the final desired states (e.g., see Kozma and Barrat (2008); Huang (2017); Shao *et al.* (2018); Shi *et al.* (2019)).

In most of the existing literatures on coordinated control of MASs, the agents are usually assumed to be cooperative, i.e., the weight of each edge in the interaction topology is positive. However, cooperation and competition among agents generally coexist in most practical scenarios, such as friends and enemies in social networks Wasserman and Faust (1994), trust and distrust in the context of opinion dynamics Altafini (2017). Cooperation and competition among agents can be described by signed networks. Generally, signed networks include structurally balanced networks and structurally unbalanced networks. In structurally balanced networks, the agents are usually divided into two subsets with intra-subset cooperation and inter-subset competition. It has been shown in Qin *et al.* (2016); Zhang and Chen (2016); Zhu *et al.* (2018); Shi (2019) that the whole system will gradually appear a bipartite consensus phenomenon, namely, the these two subsets gradually reach two different consensus states with the same modulus but opposite signs. One study by Valcher and Misra (2014); Meng *et al.* (2016) found that all agents reach a consistent final state with the value being zeros when they interact by structurally unbalanced networks. Moreover, the reference Meng (2017); Zuo *et al.* (2018); Meng and Gao (2019) found that bipartite containment phenomenon will occurs when there are stubborn agents in the system.

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Most noticeably, the above results on the dynamics of MASs on signed networks have a certain limitation on the topology structure, that is, the network topology needs to have sufficient connectivity to achieve consensus, bipartite consensus, or bipartite containment. However, a more general setting, in the sense that the network topology does not have any structural constraints, should be explored due to the complex interaction between agents. This general setting of the network topology has proven to unify most existing models, such as consensus, bipartite consensus, bipartite containment. In this paper, we extend the dynamic model to the case where the network topology is an arbitrary signed network without any structural constraints. Based on a method of constructing cooperative auxiliary digraphs, it is theoretically proved that the agents in closed strong connected components with balanced structure and unbalanced structure gradually reach separately bipartite consensus and consensus, and the agents outside the closed strong connected components gradually enter the convex hull of the agents in the closed strong connected components, that is, achieving bipartite containment.

2. PRELIMINARIES

2.1 Notations

Let $\|S\|_\infty = \max\{\sum_{j=1}^{n}|s_{ij}| \mid i = 1, 2, \ldots, n\}$ denote the infinite norm of a real matrix $S = [s_{ij}] \in \mathbb{R}^{n\times n}$. Matrix S is nonnegative if $s_{ij} \geq 0, i, j = 1, 2, \ldots, n$. A nonnegative matrix S is row-stochastic if $\sum_{j=1}^{n} s_{ij} = 1$, $i = 1, 2, \ldots, n$, and it is sub-stochastic if $\sum_{j=1}^{n} s_{ij} \leq 1$, $i = 1, 2, \ldots, n$. $\mathbb{N}$ denotes the natural number set. $\otimes$ is the Kronecker product. I_n is an n-order identity matrix and $\mathbf{0}$ denotes a matrix whose all elements are equal to 0. Define the signum function $\mathrm{sgn}(x)$ as

$$\mathrm{sgn}(x) = \begin{cases} 1, \ x > 0 \\ 0, \ x = 0 \\ -1, \ x < 0 \end{cases}.$$

Lemma 1. For any real matrix $A \in \mathbb{R}^{p\times p}$, there exists $\beta > 0$ such that $\|A^k\|_\infty \leq \beta k^{p-1}\rho^k$, where ρ denotes the spectral radius of matrix A.

2.2 Signed Network

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be an n-order directed signed network, where $\mathscr{V} = \{1, 2, \ldots, n\}$ and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ denote the node set and edge set, respectively. The directed exchange link from node i to node j is denoted by an edge (i, j). Let $\mathscr{N}_i = \{j \in \mathscr{V} \mid (j, i) \in \mathscr{E}\}$ be the set of node i's neighbors. The adjacency matrix is denoted by $\mathscr{A} = [a_{ij}]$, which satisfies: $a_{ji} \neq 0$ if and only if $(i, j) \in \mathscr{E}$, otherwise $a_{ji} = 0$. Denote the signed Laplacian matrix of $\mathscr{G}$ by $\mathscr{L} = [l_{ij}]$, where

$$l_{ij} = -a_{ij}, \ \forall\, j \neq i,$$
$$l_{ii} = \sum_{j\in\mathscr{N}_i} |a_{ij}|, \ i = 1, 2, \ldots, n.$$

Let $i_0 \to i_1 \to \cdots \to i_r$ denote a path from i_0 to i_r, where $i_0, i_1, \ldots, i_r \in \mathscr{V}$ are different nodes. Denote the distance from node i to node j by $d_{i\to j}$, which is the number of edges in the shortest path from node i to node j. For any different nodes i, j, if there are pathes from i to j and from j to i, then $\mathscr{G}$ is said to be strongly connected. $\mathscr{G}$ is weakly connected if there is at least one path between any two nodes. Digraph $\mathscr{G}_1$ is called a strongly connected component of $\mathscr{G}$ if it is a largest strongly connected subgraph of $\mathscr{G}$, and further, $\mathscr{G}_1$ is a closed strongly connected component of $\mathscr{G}$ if there are no edges reaching the nodes in $\mathscr{G}_1$. In a weakly connected digraph, for any a node i outside the closed strong connected components, there exists at least path from the closed strongly connected components to it.

Generally speaking, signed networks include structurally balanced networks and structurally unbalanced networks. A signed network $\mathscr{G}$ is structurally balanced if all nodes are divided into two non-empty subsets $\mathscr{V}_1$, $\mathscr{V}_2$ such that: 1) $\mathscr{V}_1 \cup \mathscr{V}_2 = \mathscr{V}$ and $\mathscr{V}_1 \cap \mathscr{V}_2 = \emptyset$; 2) for any $(i, j) \in \mathscr{E}$, $a_{ji} > 0$ if nodes i and j are in the same subset, and $a_{ji} < 0$ if nodes i and j belong to different subsets. A signed digraph $\mathscr{G}$ is structurally unbalanced if it is not structurally balanced.

In an n-order signed network, there is at least one closed strong connected component and at most n closed strong connected components. Without loss of generality, it is assumed that the digraph $\mathscr{G}$ has q $(1 \leq q \leq n)$ closed strong connected components $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_q$ with, respectively, the node sets

$$\begin{aligned} \mathscr{V}_1 &= \{r_0 + 1, r_0 + 2, \cdots, r_1\}, \\ \mathscr{V}_2 &= \{r_1 + 1, r_1 + 2, \cdots, r_2\}, \\ &\vdots \\ \mathscr{V}_q &= \{r_{q-1} + 1, r_{q-1} + 2, \cdots, r_q\}, \end{aligned}$$

where $r_0 = 0$ and $r_q \leq n$. The set of nodes outside the closed strong connected components is represented as

$$\mathscr{V}_{q+1} = \{r_q + 1, r_q + 2, \cdots, n\}.$$

Thus, the signed Laplacian matrix of $\mathscr{G}$ takes the partitioned form as

$$\mathscr{L} = \begin{bmatrix} \mathscr{L}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathscr{L}_q & \mathbf{0} \\ \mathscr{R}_1 & \cdots & \mathscr{R}_q & \mathscr{R}_{q+1} \end{bmatrix},$$

where $\mathscr{L}_s$ is the signed Laplacian matrix of the closed strong connected component $\mathscr{G}_s$, $s \in \{1, 2, \cdots, q\}$.

3. MODEL FORMULATION

Consider a system composed of n agents (denoted through 1 to n) evolving on the discrete instant set $\mathcal{T} = \{0, \tau, 2\tau, \cdots, k\tau, \cdots\}$, where τ is the time step-size. The interactions among the agents are depicted by a signed network $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. Each adjacency element in $\mathscr{A}$ satisfy: $a_{ji} > 0$ $(a_{ji} < 0)$ if agent j can receive the cooperative (competitive) information of agent i, otherwise $a_{ji} = 0$. The general linear dynamics of the ith agent can be expressed as

$$x_i[k + 1] = Ax_i[k] + Bu_i[k], \tag{1}$$

where $x_i[k] = [x_i^{(1)}[k], x_i^{(2)}[k], \cdots, x_i^{(p)}[k]]^T \in \mathbb{R}^p$ is the state vector of the ith agent at instant $k\tau$; $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times z}$ are the system matrix and input matrix, respectively. A distributed control input $u_i[k]$ for system (1) is given by

$$u_i[k] = K \sum_{j \in \mathscr{N}_i} |a_{ij}| \big(\text{sgn}(a_{ij}) x_j[k] - x_i[k]\big), \qquad (2)$$

where K is a given feedback gain matrix.

4. A METHOD OF CONSTRUCTING COOPERATIVE AUXILIARY DIGRAPH

In order to analyze the dynamic behavior of the agents over signed networks, we proposes a method of constructing cooperative auxiliary digraph. The details are as follows.

For each agent i with the state $x_i[t]$, we construct an opposing virtual agent i' with the state $-x_i[t]$. If agent i can receive the competitive information from its neighbor j, we can think that the following two scenarios can occur equivalently: 1) agent i can receive the cooperative information of virtual agent j'; 2) virtual agent i' can receive the cooperative information of agent j. Similarly, the situation that agent i can receive the cooperative information of agent j can be regarded as that virtual agent i' can receive the cooperative information of virtual agent j'. Let digraph $\tilde{\mathscr{G}} = \{\tilde{\mathscr{V}}, \tilde{\mathscr{E}}, \tilde{\mathscr{A}}\}$ describe the information interactions among the real and virtual agents, where $\tilde{\mathscr{V}} = \{1, 1', \dots, n, n'\}$ and the details of the edge set $\tilde{\mathscr{E}}$ are shown as follows:

A1) if $(j, i) \in \mathscr{E}$ and $a_{ij} > 0$, then $(j, i) \in \tilde{\mathscr{E}}$ and $(j', i') \in \tilde{\mathscr{E}}$, that is, $[\tilde{\mathscr{A}}]_{2i-1,2j-1} = [\tilde{\mathscr{A}}]_{2i,2j} = a_{ij} > 0$,

A2) if $(j, i) \in \mathscr{E}$ and $a_{ij} < 0$, then $(j, i') \in \tilde{\mathscr{E}}$ and $(j', i) \in \tilde{\mathscr{E}}$, that is, $[\tilde{\mathscr{A}}]_{2i,2j-1} = [\tilde{\mathscr{A}}]_{2i-1,2j} = -a_{ij} > 0$,

where the sequential row indexes of the adjacency matrix $\tilde{\mathscr{A}}$ correspond to nodes $1, 1', \dots, n, n'$, respectively. Divide the nodes in $\tilde{\mathscr{V}}$ into the following subsets

$$\begin{aligned}
\tilde{\mathscr{V}}_1 &= \{r_0 + 1, (r_0 + 1)', \cdots, r_1, r_1'\}, \\
\tilde{\mathscr{V}}_2 &= \{r_1 + 1, (r_1 + 1)', \cdots, r_2, r_2'\}, \\
&\vdots \\
\tilde{\mathscr{V}}_q &= \{r_{q-1} + 1, (r_{q-1} + 1)', \cdots, r_q, r_q'\}, \\
\tilde{\mathscr{V}}_{q+1} &= \{r_q + 1, (r_q + 1)', \cdots, n, n'\}.
\end{aligned}$$

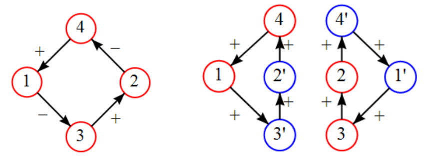
The digraph associated with each set $\tilde{\mathscr{V}}_s$ is denoted by $\tilde{\mathscr{G}}_s$, where $s \in \{1, 2, \dots, q\}$. The Laplacian matrix of digraph $\tilde{\mathscr{G}}$ takes the following form

$$\tilde{\mathscr{L}} = \begin{bmatrix} \tilde{\mathscr{L}}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \tilde{\mathscr{L}}_q & \mathbf{0} \\ \tilde{\mathscr{R}}_1 & \cdots & \tilde{\mathscr{R}}_q & \tilde{\mathscr{R}}_{q+1} \end{bmatrix}.$$

Obviously, for each closed strong connected component $\mathscr{G}_s$, there is a digraph $\tilde{\mathscr{G}}_s$ corresponding to it. The relationship between digraphs $\mathscr{G}_s$ and $\tilde{\mathscr{G}}_s$ is shown below.

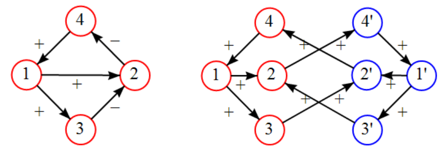
B1) If the closed strong connected component $\mathscr{G}_s$ is structurally balanced, that is, the node set $\mathscr{V}_s$ can be divided into two subsets $\mathscr{V}_{s1}$ and $\mathscr{V}_{s2}$ with intra-subset cooperation and inter-subset competition, then the digraph $\tilde{\mathscr{G}}_s$ contains two strong connected components $\tilde{\mathscr{G}}_{s1}$ and $\tilde{\mathscr{G}}_{s2}$ with no common nodes. Without loss of generality, if we let $\mathscr{V}_{s1} = \{r_{s-1}+1, \dots, r_{s-1}+l_s\}$ and $\mathscr{V}_{s2} = \{r_{s-1}+l_s+1, \dots, r_s\}$, where $l_s \leq r_s - r_{s-1}$, then the digraphs $\tilde{\mathscr{G}}_{s1}$ and $\tilde{\mathscr{G}}_{s2}$ are related to node sets $\tilde{\mathscr{V}}_{s1} = \{r_{s-1}+1, \dots, r_{s-1}+l_s, (r_{s-1}+l_s+1)', \dots, r_s'\}$ and $\tilde{\mathscr{V}}_{s2} = \{(r_{s-1}+1)', \dots, (r_{s-1}+l_s)', r_{s-1}+l_s+1, \dots, r_s\}$, respectively.

B2) If the closed strong connected component $\mathscr{G}_s$ is structurally unbalanced, then the digraph $\tilde{\mathscr{G}}_s$ is strongly connected.



(a) $\mathscr{G}_s$ (b) $\tilde{\mathscr{G}}_s$

Fig. 1. A closed strong connected component $\mathscr{G}_s$ with balanced structure and the corresponding digraph $\tilde{\mathscr{G}}_s$, where "+" and "–" represent cooperation and competition, respectively.



(a) $\mathscr{G}_s$ (b) $\tilde{\mathscr{G}}_s$

Fig. 2. A closed strong connected component $\mathscr{G}_s$ with unbalanced structure and the corresponding digraph $\tilde{\mathscr{G}}_s$.

In Fig. 1 and Fig. 2, we show two examples for the closed strong connected component $\mathscr{G}_s$ and the corresponding digraph $\tilde{\mathscr{G}}_s$.

Remark 1. Through this method, we only need to analyze the dynamics of MASs on corresponding auxiliary digraphs with only cooperative relationships, which can indirectly obtain the dynamics of MASs on signed networks. This method is suitable for the dynamic behavior of arbitrary signed networks.

5. DYNAMICS ANALYSIS

Based on the method of constructing cooperative auxiliary digraph, the properties of row-stochastic matrix and sub-stochastic matrix are exploited to analyze the dynamic behavior of the agents.

The feedback system (1) is first written in a matrix-vector form:

$$x[k+1] = (I_n \otimes A - \mathscr{L} \otimes BK) x[k], \qquad (3)$$

where $x[k] = \big[x_1^T[k], \cdots, x_q^T[k], x_n^T[k]\big]^T$. Based on the construction of digraph $\mathscr{G}$, we define the following model transformation vectors

$$Y_1[k] = \left[x_1^T[k], -x_1^T[k], \cdots, x_{r_1}^T[k], -x_{r_1}^T[k]\right]^T,$$
$$\vdots$$
$$Y_q[k] = \left[x_{r_{q-1}+1}^T[k], -x_{r_{q-1}+1}^T[k], \cdots, x_{r_q}^T[k], -x_{r_q}^T[k]\right]^T,$$
$$Y_{q+1}[k] = \left[x_{r_q+1}^T[k], -x_{r_q+1}^T[k], \cdots, x_n^T[k], -x_n^T[k]\right]^T,$$
$$Y[k] = \left[Y_1^T[k], \cdots, Y_q^T[k], Y_{q+1}^T[k]\right]^T.$$

Then system (3) can be converted equivalently to

$$Y[k+1] = (I_{2n} \otimes A - \tilde{\mathscr{L}} \otimes BK)Y[k]. \tag{4}$$

In this paper, we assume that the input matrix B is of full row rank. When we choose the feedback matrix $K = \gamma B^T(BB^T)^{-1}A$, system (4) can be expressed as

$$Y[k+1] = \left[\left(I_{2n} - \gamma\tilde{\mathscr{L}}\right) \otimes A\right] Y[k], \tag{5}$$

where

$$I_{2n} - \gamma\tilde{\mathscr{L}} = \begin{bmatrix} I_{2r_1} - \gamma\tilde{\mathscr{L}}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & I_{2(r_q - r_{q-1})} - \gamma\tilde{\mathscr{L}}_q & \mathbf{0} \\ -\gamma\tilde{\mathscr{R}}_1 & \cdots & -\gamma\tilde{\mathscr{R}}_q & I_{2(n-r_q)} - \gamma\tilde{\mathscr{R}}_{q+1} \end{bmatrix}.$$

Below we first analyze the dynamics of each closed strong connected component.

Theorem 1. Suppose that the input matrix B is of full row rank and the feedback matrix satisfies $K = \gamma B^T(BB^T)^{-1}A$ with

$$\gamma \le \frac{1}{\max\{\sum_{j\in\mathscr{N}_i} |a_{ij}| \mid i = 1, \ldots, n\}}. \tag{6}$$

For each closed strong connected component $\mathscr{G}_s$, where $s \in \{1, 2, \ldots, q\}$, the following results hold.

1) If $\mathscr{G}_s$ is structurally balanced, then bipartite consensus can be realized. In particular, $\lim_{k\to\infty} x_i[k] = x_j[k]$ if i, j belong to the same subset, and $\lim_{k\to\infty} x_i[k] = -x_j[k]$ if i, j belong to different subsets.
2) If $\mathscr{G}_s$ is structurally unbalanced, then the consensus with state 0 can be realized, namely, $\lim_{k\to\infty} x_i[k] = \mathbf{0}$ for any $i \in \mathscr{V}_s$.

Proof: According to system (5), the dynamics of the agents associated with each closed strong connected component $\mathscr{G}_s$ can be written as

$$Y_s[k+1] = \left[\left(I_{2(r_s - r_{s-1})} - \gamma\tilde{\mathscr{L}}_s\right) \otimes A\right] Y_s[k], \tag{7}$$

where $I_{r_s - r_{s-1}} - \gamma\tilde{\mathscr{L}}_s$ is a row-stochastic matrix under condition (6). Below we consider two different situations.

1) If the digraph $\mathscr{G}_s$ is structurally balanced, then according to the result in **B1)**, we know that the digraph $\tilde{\mathscr{G}}_{s1}$ associated with the node set $\tilde{\mathscr{V}}_{s1} = \{r_{s-1}+1, \ldots, r_{s-1}+l_s, (r_{s-1}+l_s+1)', \ldots, r_s'\}$ is a fully cooperative network and strong connected. According to a known result in Olfati-Saber *et al.* (2007) that the consensus can be realized if the fully cooperative network is strongly connected, we can obtain that the agents associated with the digraph $\tilde{\mathscr{G}}_{s1}$ achieve the consensus, that is,

$$\lim_{k\to\infty} Y_s^*[k] = \left(\mathbf{1}_{r_s - r_{s-1}}\xi_1^T \otimes A^k\right)Y_s^*[0],$$

where $\xi_1 \in \mathbb{R}^{r_s - r_{s-1}}$ is a constant column vector, and

$$Y_s^*[k] = \left[x_{r_{s-1}+1}^T[k], \cdots, x_{r_{s-1}+l_s}^T[k], -x_{r_{s-1}+l_s+1}^T[k], \ldots, -x_{r_s}^T[k]\right]^T.$$

This, in turn, means that bipartite consensus is realized.

2) If the digraph $\mathscr{G}_s$ is structurally unbalanced, then according to the result in **B2)**, we know that $\tilde{\mathscr{G}}_s$ is a fully cooperative network and strongly connected. Thus, we have

$$\lim_{k\to\infty} Y_s[k] = \lim_{k\to\infty} \left(\mathbf{1}_{2(r_s - r_{s-1})}\xi_2^T \otimes A\right) Y_s[0],$$

where $\xi_2 \in \mathbb{R}^{2(r_s - r_{s-1}}$ is a constant column vector. From the construction of $Y_s[k]$, we can see that $\lim_{k\to\infty} x_i[k] = -x_i[k]$ for any $i \in \mathscr{V}_s$. This implies that $\lim_{k\to\infty} x_i[k] = \mathbf{0}$. Therefore, the consensus with state 0 is realized. □

Now we show the dynamics of the agents outside the closed strong connected components.

Theorem 2. Consider system (1) with protocol (2) on a signed network $\mathscr{G}$. Suppose that the input matrix B is of full row rank, then there exists a feedback matrix $K = \gamma B^T(BB^T)^{-1}A$ with the parameter γ satisfying (6) such that the agents outside the closed strong connected components gradually enter the convex hull formed by the agents in the closed strong connected components, where the system matrix A is allowed to be strictly unstable and its spectral radius ρ satisfies:

$$\rho < \frac{1}{\sqrt[P]{1 - (1-\delta)\alpha^{P-1}}}, \tag{8}$$

in which

$$\delta = \max\left\{\Lambda_i\left[I_{2n} - \gamma\tilde{\mathscr{L}}\right] \mid \Lambda_i\left[I_{2n} - \gamma\tilde{\mathscr{L}}\right] < 1, i = 1, \ldots, 2n\right\},$$
$$P = \max\left\{d_{i\to j} \mid i \in \mathscr{V}_1 \cup \mathscr{V}_2 \cup \cdots \cup \mathscr{V}_q, j \in \mathscr{V}_{q+1}\right\},$$
$$\alpha = \min\left\{\gamma|a_{ij}| \mid a_{ij} \neq 0, i, j = 1, 2, \ldots, n\right\}.$$

Proof: Under condition (6), $I_{2n} - \gamma\tilde{\mathscr{L}}$ is a row-stochastic matrix. Without loss of generality, it is assumed that the first w closed strong connected components $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_w$ are structurally balanced and the remaining closed strong connected components $\mathscr{G}_{w+1}, \mathscr{G}_{w+2}, \ldots, \mathscr{G}_q$ are structurally unbalanced, where $w \le q$. According to the result of Theorem 1, we have

$$\begin{aligned} &\lim_{k\to\infty} Y_s[k] = (Z_s\xi_s^T \otimes A^k)Y_s[0], \ s = 1, 2, \ldots, w, \\ &\lim_{k\to\infty} Y_s[k] = \mathbf{0}, \ s = w+1, \ldots, q, \end{aligned} \tag{9}$$

where $Z_s \in \mathbb{R}^{2(r_s - r_{s-1})}$ is a column in which all elements are equal to 1 or -1, $\xi \in \mathbb{R}^{2(r_s - r_{s-1})}$ is a constant column vector. Consequently, system (5) can be written as

$$\lim_{k\to\infty} Y[k] = \lim_{k\to\infty} \left[\left(I_n - \gamma\tilde{\mathscr{L}}\right)^k \otimes A^k\right] Y[0], \tag{10}$$

where

$$\lim_{k\to\infty}(I_n-\gamma\tilde{\mathscr{L}})^k=\begin{bmatrix} Z_1\xi_1^T & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & Z_w\xi_w^T & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \Delta_1 & \cdots & \Delta_w & \Delta_{w+1} & \cdots & \Delta_q & H^k \end{bmatrix}$$

in which $H=I_{2(n-r_q)}-\gamma\tilde{\mathscr{R}}_{q+1}$ is a sub-stochastic matrix, Δ_s, $s=1,2,\ldots,q$ are some nonnegative matrices that are subsequently given.

In order to get this result, we proceed to prove the following equality firstly:

$$\lim_{k\to\infty} H^k\otimes A^k=\mathbf{0}. \tag{11}$$

Let $\mathscr{V}_l=\mathscr{V}_1\cup\mathscr{V}_2\cup\cdots\mathscr{V}_q$. Then there is a directed path $W_{i_0\to i_z}=i_0\to i_1\to\cdots\to i_z$ in the signed network $\mathscr{G}$ for each node i_z, where $i_0\in\mathscr{V}_l$ and $i_1,\ldots,i_z\in\mathscr{V}_{q+1}$. Since $(i_0,i_1)\in\mathscr{E}$, then we have $a_{i_1i_0}\neq 0$. According to the structure of digraph $\tilde{\mathscr{G}}$, we have

$$\sum_{j=1}[H]_{s,j}=1-\gamma|a_{i_1i_0}|\leq\delta<1,\ \ s=2i_1-1,2i_1. \tag{12}$$

Below we consider the edge (i_1,i_2) in two cases. If the weight of (i_1,i_2) is positive, namely, $a_{i_2i_1}>0$, then we have $[H]_{2i_2-1,2i_1-1}=[H]_{2i_2,2i_1}=\gamma a_{i_2i_1}\geq\alpha$. It follows that

$$\sum_{j=1}[H^2]_{s,j}=\sum_{j_1\neq v}[H]_{v,j_1}\sum_{j}[H]_{j_1,j}+[H]_{s,v}\sum_{j}[H]_{v,j}$$
$$\leq 1-(1-\delta)\alpha<1$$

for $s=2i_2-1,v=2i_1-1$ and $s=2i_2,v=2i_1$. And if the weight of (i_1,i_2) is negative, namely, $a_{i_2i_1}<0$, then we have $[H]_{2i_2-1,2i_1}=[H]_{2i_2,2i_1-1}=-\gamma a_{i_2i_1}\geq\alpha$. It thus follows that

$$\sum_{j=1}[H^2]_{s,j}=\sum_{j_1\neq v}[H]_{s,j_1}\sum_{j}[H]_{j_1,j}+[H]_{s,v}\sum_{j}[H]_{v,j}$$
$$\leq 1-(1-\delta)\alpha<1$$

for $s=2i_2-1,v=2i_1$ and $s=2i_2,v=2i_1-1$. Therefore, whether the weight of edge (i_2,i_1) is positive or negative, we can always get

$$\sum_{j=1}[H^2]_{s,j}\leq 1-(1-\delta)\alpha<1,\ \ s=2i_2-1,2i_2. \tag{13}$$

Using the above analysis method for edge (i_2,i_1) to analyze the remaining edges of $W_{i_0\to i_z}$, we can deduce that

$$\sum_{j=1}[H^z]_{s,j}\leq 1-(1-\delta)\alpha^{z-1}<1,\ \ s=2i_z-1,2i_z, \tag{14}$$

where z is the length of $W_{i_0\to i_z}$. According to the definition of P, we have $z\leq P$. Then, it can be obtained that

$$\sum_{j=1}[H^P]_{s,j}\leq 1-(1-\delta)\alpha^{P-1}<1,\ \ s=2i_z-1,2i_z. \tag{15}$$

This means that

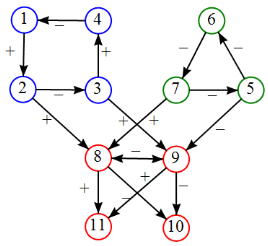


Fig. 3. A signed network $\mathscr{G}$, in which the modulus of all weights are equal to 1, "+" and "–" represent cooperation and competition, respectively.

$$\|H^p\|_\infty\leq 1-(1-\delta)\alpha^{P-1}<1. \tag{16}$$

It follows that

$$\lim_{k\to\infty}\left\|H^k\right\|_\infty=\lim_{c\to\infty}\left\|H^P\right\|_\infty^c\leq\lim_{c\to\infty}\left[1-(1-\delta)\alpha^{P-1}\right]^c. \tag{17}$$

By Lemma 1, one knows that there exists $\beta>0$ such that $\|A^k\|_\infty\leq\beta k^{p-1}\rho^k$. Thus,

$$\lim_{k\to\infty}\left\|A^k\right\|_\infty=\lim_{c\to\infty}\beta(Pc)^{p-1}\rho^{cP}. \tag{18}$$

To prove (11), it suffices to prove that

$$\lim_{c\to\infty}\beta(Pc)^{p-1}\left[\left(1-(1-\delta)\alpha^{P-1}\right)\rho^P\right]^c=0. \tag{19}$$

By (8), we can obtain that $0<\left(1-(1-\delta)\alpha^{P-1}\right)\rho^P<1$. According to the fact that the exponential decay dominates the polynomial increase, the expression (11) holds.

Based on the result (11), the matrix $\lim_{k\to\infty}(I_n-\gamma\tilde{\mathscr{L}})^k$ can be further expressed as

$$\lim_{k\to\infty}(I_n-\gamma\tilde{\mathscr{L}})^k=\begin{bmatrix} Z_1\xi_1^T & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & Z_w\xi_w^T & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \Delta_1 & \cdots & \Delta_w & \Delta_{w+1} & \cdots & \Delta_q & \mathbf{0} \end{bmatrix},$$

where $\Delta_s=-\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_sZ_s\xi_s^T$, $s=1,2,\ldots,w$ and $\Delta_s=-\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_s\mathbf{0}=\mathbf{0}$, $s=w+1,w+2,\ldots,q$. This implies that

$$\lim_{k\to\infty}Y_{q+1}(k)=-\sum_{s=1}^{w}\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_sZ_s\xi_s^TY_s[0]-\sum_{s=w+1}^{q}\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_s\mathbf{0}$$
$$=-\lim_{k\to\infty}\sum_{s=1}^{q}\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_sY_s[k].$$

According to a known fact that the product of row-stochastic matrices is a row-stochastic matrix, we can get that $-\sum_{s=1}^{q}\tilde{\mathscr{R}}_{q+1}^{-1}\tilde{\mathscr{R}}_s$ is a row-stochastic matrix. This also means that the agents outside the closed strong connected components gradually enter the convex hull formed by the agents in the closed strong connected components. □

6. SIMULATION RESULT

Consider a third-order multi-agent system with 11 agents. The interaction details among the agents are

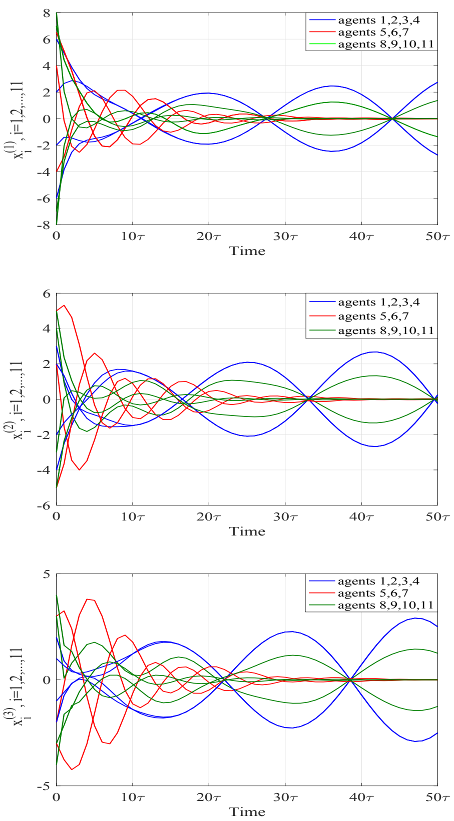


Fig. 4. State trajectories of the agents.

described by a signed network $\mathscr{G}$, which is shown in Fig. 3. Obviously, the weakly connected digraph $\mathscr{G}_1$ has two closed and strongly connected components $\mathscr{G}_1$ with the node set $\mathscr{V}_1 = \{1, 2, 3, 4\}$ and $\mathscr{G}_2$ with the node set $\mathscr{V}_2 = \{5, 6, 7\}$. In addition, it can be seen that $P = 2$. Let $\gamma = \frac{1}{3}$ that satisfies condition (6), then we have $\delta = \frac{2}{3}$ and $\alpha = \frac{1}{3}$. The input matrix B and the system matrix A are expressed respectively by

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0.8816 & 0 & -0.2204 \\ -0.2204 & 0.8816 & 0 \\ 0 & -0.2204 & 0.8816 \end{pmatrix}.$$

We can observe that B is of full row rank, and A is strictly unstable with the $\rho = 1.02$ that satisfies condition (8). Finally, the agents' state trajectories are exhibited in Fig. 4, from which we can observe that the agents in $\mathscr{V}_1$ realize bipartite consensus and the agents in $\mathscr{V}_2$ achieve a consensus state 0, besides, the remaining agents gradually enter the convex hull constructed by the agents in $\mathscr{V}_1 \cup \mathscr{V}_2$.

7. CONCLUSION

The dynamics of general linear MASs over signed networks without structural constraints has been studied in this paper. By adding virtual nodes, a method of constructing cooperative auxiliary digraphs has been developed. In that way, the dynamic behavior analysis of MASs on signed networks is equivalent to that on cooperative auxiliary digraphs. Based on the properties of row-stochastic matrix and sub-stochastic matrix, the dynamics of MASs on cooperative auxiliary digraphs have been analyzed. Finally, a computer simulation has been performed to illustrate the discovered dynamic phenomena.

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