Integral Quadratic Constraints on Linear Infinite-dimensional Systems for Robust Stability Analysis

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Abstract: This paper proposes a framework to assess the stability of an Ordinary Differential Equation (ODE) which is coupled to a 1D-partial differential equation (PDE). The stability theorem is based on a new result on Integral Quadratic Constraints (IQCs) and expressed in terms of two linear matrix inequalities with a moderate computational burden. The IQCs are not generated using dissipation inequalities involving the whole state of an infinite-dimensional system, but by using projection coefficients of the infinite-dimensional state. This permits to generalize our robustness result to many other PDEs. The proposed methodology is applied to a time-delay system and numerical results comparable to those in the literature are obtained.

Keywords: Distributed Parameter Systems; Robustness analysis; IQCs; Coupled ODE/PDE.

1. INTRODUCTION

Many processes are modeled by a finite-dimensional linear system subject to an infinite-dimensional uncertainty (Barreau et al., 2019; Fridman, 2014). Robustness analysis assesses the stability of the resulting coupled system. This paper focuses on an infinite-dimensional type of uncertainties modeled by a linear Partial Differential Equation. Attention has been paid on these special couplings, partly because Time-Delay Systems (TDS) are a subclass thereof (Fridman, 2014; Gu et al., 2003; Safi et al., 2017).

In recent decades, many articles have been dealing with asymptotic stability of TDS, using different robust control tools. A first method is based on the Lyapunov-Krasovskii theorem and requires to find a differentiable functional with suitable properties (Gu et al., 2003). Another approach considers dissipativity (Willems, 1972). More recently, a general framework for dealing with coupled ODE and uncertainties has been proposed by Fu et al. (1998), Jun et al (2001) and Megretski et al (1997). This requires to find IQCs to assess stability of the interconnection.

The methodology presented in Seuret and Gouaisbaut (2015) shows how to construct a Lyapunov functional for TDS based on the projection coefficients of the infinite-dimensional state. The resulting stability test is made up of tractable Linear Matrix Inequalities (LMIs) with a moderate computational burden. This approach has been applied to a transport equation in Safi et al. (2017), to

a heat equation in Baudouin et al. (2019) and to a wave equation in Barreau et al. (2018).

The aim of this paper is to extend the theory developed in the previous references by using IQCs as exposed in Scherer and Veenman (2018). In this context, a filter providing the projection coefficients of the infinite-dimensional state is derived. Thanks to this filter, a new class of IQCs can be generated, not related to a specific class of PDEs. In that sense, the proposed result is adaptable to many linear PDEs and it is promising to extend these ideas to other classes.

In Section 2, the problem is stated and a stability theorem is derived. The filter is obtained in Section 3. Section 4 is dedicated to the application of the previous results to a TDS. Section 5 provides a numerical example with a transport equation. Finally, conclusions are drawn and several perspectives are proposed.

Notation: For real matrices A and B of appropriate dimensions, $\mathrm{col}(A, B) = \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top$, $\mathrm{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\mathrm{He}(A) = A + A^\top$ and $A^\perp$ is a basis of its null-space. $\mathbb{S}^n$ is the set of symmetric matrices. A is (semi-)definite positive if $A \succ 0$ ($A \succeq 0$ resp.). The Euclidean norm of v is defined as $\|v\|^2 = v^\top v$. The following notation is used for matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, P, M$ of appropriate dimensions:

$$\mathcal{L}\left(P, M, \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]\right) = \begin{bmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^\top \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

We also use $C(sI - A)^{-1}B + D = \left\|\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\right\|$.

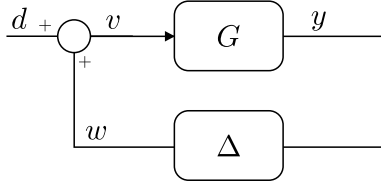$\partial_x^{(i)}$ means the ith distributional derivative with respect

Fig. 1. Block diagram describing the interconnected system (1)-(3).

to the variable x, a short-hand notation is $\partial_x f = f_x$. The space of square integrable functions on $[0,1]$ ($[0,\infty)$, resp.) is denoted by $L_2(0,1)$ (L_2, resp.) and L_{2e} is the space of locally square integrable signals on $[0,\infty)$. We work with Sobolev spaces denoted by

$$H^m(0,1) = \left\{ f \in L_2(0,1) \mid \forall k \le m, \partial_x^{(k)} f \in L_2(0,1) \right\}.$$

The canonical inner product on $L_2(0,1)$ is $\langle f, g \rangle = \int_0^1 f^\top(s) g(s) ds$ for $f, g \in L_2(0,1)$.

2. PROBLEM STATEMENT

The studied system is presented in Fig. 1 and described by the following equations as inspired by the classical IQC framework (Megretski and Rantzer, 1997):

$$\begin{cases} v = w + d, \quad (y,w) \in \Delta, \\ y = Gv. \end{cases} \tag{1}$$

Here, Δ is a relation defined afterwards and G is the linear time-invariant system whose representation is

$$\begin{cases} \dot{X} = AX + Bv, \qquad X(0) = 0_{n,1}, \\ y = CX + Dv, \end{cases} \tag{2}$$

with state, input and output dimensions n, r and p; A, B, C and D are real matrices of appropriate dimensions.

Remark 1. Note that we study the interconnection between an LTI system and a relation, similarly to what is done in the behavioral approach to system theory (Willems and Polderman, 2013). We use a relation since, in our situation, there is no clear way to express Δ as a causal input-output map. This is different from the feedback configuration as studied by Megretski and al. (1997). ■

Before introducing Δ, let us define the following infinite-dimensional system with state $z(x,t) \in \mathbb{R}^l$ for $t \ge 0$:

$$z_t(x,t) = \mathscr{A} z(x,t) = \sum_{k=0}^{m} F_k \partial_x^{(k)} z(x,t), \text{ for } x \in [0,1]. \tag{3}$$

The initial state is $z(\cdot, 0) = 0_{l,1}$. Here $m \in \mathbb{N}$, $m > 0$, $F_m \neq 0$ and the domain of $\mathscr{A}$ is parameterized by $y \in \mathbb{R}^p$:

$$\mathscr{D}_y(\mathscr{A}) = \{ z \in H^m(0,1) \mid K_{\mathscr{B}} \mathscr{B}_m(z) = Ky \} \tag{4}$$

where $K_{\mathscr{B}} \in \mathbb{R}^{ml \times 2ml}$ has full-rank, $K \in \mathbb{R}^{ml \times p}$, and

$$\mathscr{B}_m(z) = \text{col}\left(z(0),\ z(1), \ldots, \partial_x^{(m-1)} z(0),\ \partial_x^{(m-1)} z(1) \right).$$

Note that $\mathscr{B}_m$ is related to the trace operator and maps the input function to its values and those of its derivatives at the boundary points. This operator has already been introduced by Ahmadi et al. (2016) for instance. The equality $K_{\mathscr{B}} \mathscr{B}_m(z) = Ky$ reflects the boundary conditions.

Δ is the relation that is implicitly defined through the boundary conditions of (3). We say that all pairs $(y,w) \in L_{2e}^2$ belong to Δ if there exists a solution $z \in C^1([0,\infty], H^m(0,1))$ of (3) with

$$z(.,t) \in \mathscr{D}_{y(t)}(\mathscr{A}), \quad w(t) = L\mathscr{B}_m(z(.,t)) \tag{5}$$

for all $t \ge 0$, where $L \in \mathbb{R}^{r \times 2ml}$.

Remark 2. Wellposedness is assumed in this paper, as it is usually the case in IQC theory (Megretski et al., 1997) and examples show that Δ is often not empty. Theorem 3.1.7 in combination with Theorem 3.3.4 from Curtain and Zwart (1995) with $KD = 0$ and $d \in L_{2e}$ can be used to obtain general sufficient conditions for Δ to be non-empty. ■

To see that the proposed framework is versatile, the following examples describe how a time-delay system and a coupled ODE/heat equation can be transformed into (1).

Example 1. We consider the time-delay system

$$\begin{cases} \dot{X}(t) = AX(t) + B(X(t-h) + d(t)), \quad t > 0, \\ y(t) = CX(t) + DX(t-h), \qquad t \ge 0, \end{cases}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$, $C = I_n$, $D = 0_{n,n}$ and $h > 0$. This system has been widely studied by Fridman (2014); Sipahi et al. (2011) for instance.

This system can be converted into an ODE that is coupled to a transport equation with speed $\rho = h^{-1}$. The PDE is written as

$$z_t(x,t) = -\rho z_x(x,t), \quad x \in (0,1), \quad t > 0 \tag{6}$$

with $w(t) = z(1,t) = L\mathscr{B}_1(z(t))$ and

$$\mathscr{D}(\mathscr{A}) = \{ z \in H^1(0,1) \mid z(0,t) = y(t) \}.$$

In our framework we have $m = 1, F_0 = 0, F_1 = -\rho I_n$ with $\rho = h^{-1} > 0$ and $l = n$. The boundary conditions are described with $K = I_n$ and $K_{\mathscr{B}} = \begin{bmatrix} I_n & 0_n \end{bmatrix}$, while the signal w is expressed with $L = \begin{bmatrix} 0_n & I_n \end{bmatrix}$.

Example 2. Consider now the interconnection studied in Baudouin et al. (2019) between (2) and a heat equation

$$z_t(x,t) = \gamma z_{xx}(x,t)$$

with $z(0,t) = CX(t)$, $u_x(1,t) = 0$ for $x \in (0,1)$, $t \ge 0$. It can be written in the proposed framework by considering $m = 2, F_0 = 0, F_1 = 0, F_2 = \gamma$. The boundary conditions are such that $K_{\mathscr{B}} = \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{smallmatrix} \right]$ and $K = \left[I \; 0 \right]^\top$. The signal w is generated considering $L = [0 \; I]$.

Remark 3. Note that the system (3) is a generic PDE. Other physical models fitting the framework are, for instance, the linear KdV equation (Miles, 1981) with $m = 3$ or the beam equation with $m = 4$ (Gupta, 1988). ■

We will study the stability of (1) according to the following definition, which is classical in IQC theory.

Definition 1. System (1) is **stable** if there exists γ such that the following inequality holds for all its trajectories:

$$\forall T \ge 0, \quad \int_0^T \|y(s)\|_{\mathbb{R}^n}^2 ds \le \gamma^2 \int_0^T \|d(s)\|_{\mathbb{R}^p}^2 ds. \tag{7}$$

Our goal is to obtain computational tests ensuring that system (1) is stable.

3. ROBUST IQC THEOREM ON SIGNALS

As defined in Megretski and Rantzer (1997), an Integral Quadratic Constraint on $y, \mathscr{B}_m(z) \in L_2$ is expressed as:

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\mathscr{B}_m(z)}(j\omega) \end{bmatrix}^{\top} \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\mathscr{B}_m(z)}(j\omega) \end{bmatrix} d\omega \geq 0. \quad (8)$$

Here, the Fourier transforms are denoted by a $\hat{\cdot}$. Π is a frequency-dependent Hermitian matrix of appropriate dimension and called the multiplier in the sequel. We assume that Π can be written in the form

$$\Pi = \Psi^* M \Psi \quad (9)$$

where Ψ is a transfer matrix with poles in the open left-half plane, $\Psi^*(s) = \Psi(-s)^{\top}$ and $M = M^{\top}$. This multiplier representation is classical and many classes of multipliers admit such a description (Megretski and Rantzer, 1997; Veenman et al., 2016). Using Parseval's theorem and (9), a time-domain version of (8) can be deduced:

$$\int_0^{+\infty} \psi(t)^{\top} M \psi(t) dt \geq 0. \quad (10)$$

Here ψ is the output of the system defined by Ψ with input $y, \mathscr{B}_m(z) \in L_2$ associated to a state-space realization:

$$\Psi = \left[\!\!\left[\begin{array}{c|cc} A_{\Psi} & B_y & B_{\mathscr{B}} \\ \hline C_{\Psi} & D_y & D_{\mathscr{B}} \end{array} \right]\!\!\right].$$

Therefore, Ψ has the following time-domain representation

$$\begin{aligned} \dot{\xi} &= A_{\Psi}\xi + B_y y + B_{\mathscr{B}}\mathscr{B}_m(z), \qquad \xi(0) = 0, \\ \psi &= C_{\Psi}\xi + D_y y + D_{\mathscr{B}}\mathscr{B}_m(z), \end{aligned} \quad (11)$$

where $\xi(t) \in \mathbb{R}^{n_{\xi}}$, $\psi(t) \in \mathbb{R}^{p_{\psi}}$ and $A_{\Psi}, B_y, B_{\mathscr{B}}, C_{\Psi}, D_y$ and $D_{\mathscr{B}}$ are of appropriate dimensions. This filter can be seen as an augmented system providing new signals, which are of interest when finding suitable multipliers M in order to get valid IQCs.

Notice that (10) is an infinite horizon IQC. Considering only truncations of the signal ψ to $[0, T]$ for $T > 0$, a revised notion, called finite horizon IQC with terminal cost, has been proposed in Scherer and Veenman (2018). It introduces a quadratic term depending on $\xi(T)$. This notion, as useful for our purpose, is recalled here.

Definition 2. For the trajectories of (3), (4) and (11) with any $y, \mathscr{B}_m(z) \in L_{2e}$, $M = M^{\top}$ is a multiplier for **a finite horizon IQC with terminal cost** defined by the matrix $Z = Z^{\top}$ if the following holds for all $T \geq 0$:

$$\int_0^T \psi^{\top}(t) M \psi(t) dt + \xi^{\top}(T) Z \xi(T) \geq 0. \quad (12)$$

We then say that M defines a finite horizon IQC with terminal cost Z with respect to the filter Ψ.

Note that, as we are focusing on signals in L_{2e} in the previous definition and not in L_2, the filter Ψ can be unstable. In the sequel, and similarly to what was done by Scherer et al (2018), we introduce the following realization:

$$\Psi \begin{pmatrix} GL \\ I_{2(m+1)l} \end{pmatrix} = \left[\!\!\left[\begin{array}{cc|c} A_{\Psi} & B_y C & B_y DL + B_{\mathscr{B}} \\ 0 & A & BL \\ \hline C_{\Psi} & D_y C & D_y DL + D_{\mathscr{B}} \end{array} \right]\!\!\right] = \left[\!\!\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right]\!\!\right]. \quad (13)$$

All these definitions lead to the following theorem which is an adapted version of Theorem 3 in Scherer et al. (2018).

Theorem 1. Let $P \in \mathbb{S}^{n_{\xi}+n}$ and a multiplier $M = M^{\top}$ be such that the finite horizon IQC with terminal cost

$Z \in \mathbb{S}^{n_{\xi}}$ (defined in (12) with respect to the factorization (9)) holds together with

$$K_{\circ}^{\perp \top} \mathscr{L}\left(P, M, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) K_{\circ}^{\perp} \prec 0, \quad (14)$$

$$P - \begin{bmatrix} Z & 0 \\ 0 & 0_n \end{bmatrix} \succ 0, \quad (15)$$

where

$$K_{\circ} = \begin{bmatrix} 0_{ml,n_{\xi}} & KC & -K_{\mathscr{B}} \end{bmatrix}.$$

Then system (1) is stable in the sense of Definition 1.

Proof : Due to space requirement, this is only a sketch of proof[2]. The calculations are highly inspired by the ones in Scherer and Veenman (2018).

Take first a trajectory $\eta = \text{col}(\xi, X, \mathscr{B}_m, d)$ of (2) and (3) together with (5) and then introduce the dynamical system

$$\begin{cases} \dfrac{d}{dt} \begin{bmatrix} \xi \\ X \end{bmatrix} = \mathcal{A} \begin{bmatrix} \xi \\ X \end{bmatrix} + \tilde{\mathcal{B}} \begin{bmatrix} \mathscr{B}_m(z) \\ d \end{bmatrix}, \qquad \begin{bmatrix} \xi(0) \\ X(0) \end{bmatrix} = 0, \\ \begin{bmatrix} \psi \\ y \\ d \end{bmatrix} = \tilde{\mathcal{C}} \begin{bmatrix} \xi \\ X \end{bmatrix} + \tilde{\mathcal{D}} \begin{bmatrix} \mathscr{B}_m(z) \\ d \end{bmatrix}, \end{cases} \quad (16)$$

with $\tilde{\mathcal{B}} = \begin{bmatrix} \mathcal{B} & B_y D \\ & B \end{bmatrix}$, $\tilde{\mathcal{C}} = \begin{bmatrix} \mathcal{C} \\ 0 & C \\ 0 & 0 \end{bmatrix}$, $\tilde{\mathcal{D}} = \begin{bmatrix} \mathcal{D} & D_y \\ DL & \mathcal{D} \\ 0 & I_r \end{bmatrix}$. Note that, for almost all $t \geq 0$, $\tilde{K}_{\circ}\eta(t) = 0$ where $\tilde{K}_{\circ} = \begin{bmatrix} K_{\circ} & 0 \end{bmatrix}$. Let $\tilde{M}_{\gamma} = \text{diag}(M, \frac{1}{\gamma}I_p, -\gamma I_r)$ and define the following matrix:

$$\tilde{\mathscr{L}}_{\gamma} = \mathscr{L}\left(P, \tilde{M}_{\gamma}, \begin{bmatrix} \mathcal{A} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{bmatrix}\right).$$

One can show that for γ large enough, the following holds:

$$\tilde{K}_{\circ}^{\perp \top} \tilde{\mathscr{L}}_{\gamma} \tilde{K}_{\circ}^{\perp} \prec 0. \quad (17)$$

Let us define $\chi(t) = \left(\tilde{K}_{\circ}^{\perp}\right)^+ \eta(t)$ for almost all $t \geq 0$, where $\left(\tilde{K}_{\circ}^{\perp}\right)^+$ is a right-inverse of $\tilde{K}_{\circ}^{\perp}$. Inequality (17) can be written as

$$\chi^{\top}(t)\tilde{K}_{\circ}^{\perp \top} \tilde{\mathscr{L}}_{\gamma} \tilde{K}_{\circ}^{\perp}\chi(t) = \eta^{\top}(t)\tilde{\mathscr{L}}_{\gamma}\eta(t) \leq 0. \quad (18)$$

Expanding (18) and using (16) yields for almost all $t \geq 0$:

$$\frac{d}{dt}\left(\begin{bmatrix} \xi(t) \\ X(t) \end{bmatrix}^{\top} P \begin{bmatrix} \xi(t) \\ X(t) \end{bmatrix}\right) + \psi^{\top}(t)M\psi(t)$$
$$+ \frac{1}{\gamma}\|y(t)\|_{\mathbb{R}^p}^2 - \gamma\|d(t)\|_{\mathbb{R}^p}^2 \leq 0.$$

An integration between 0 and $T \geq 0$ leads to

$$\begin{bmatrix} \xi(T) \\ X(T) \end{bmatrix}^{\top} P \begin{bmatrix} \xi(T) \\ X(T) \end{bmatrix} + \int_0^T \psi^{\top}(t)M\psi(t)dt$$
$$+ \int_0^T \frac{1}{\gamma}\|y(t)\|_{\mathbb{R}^p}^2 - \gamma\|d(t)\|_{\mathbb{R}^p}^2 dt \leq 0.$$

Using IQC (12), we get for all $T \geq 0$:

$$\begin{bmatrix} \xi(T) \\ X(T) \end{bmatrix}^{\top} P \begin{bmatrix} \xi(T) \\ X(T) \end{bmatrix} - \xi^{\top}(T)Z\xi(T)$$
$$+ \int_0^T \frac{1}{\gamma}\|y(t)\|_{\mathbb{R}^p}^2 - \gamma\|d(t)\|_{\mathbb{R}^p}^2 dt \leq 0.$$

Since (15) holds, we conclude that system (1) is stable. $\square$

Remark 4. Note that there are some differences between the previous theorem and Theorem 1 derived by Megretski

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[2] For the complete proof, the reader can refer to the online version: https://hal.laas.fr/hal-02504830v1.

et al. (1997). For instance, we neither require the signals to be in L_2 nor use the causality of the relation.

The main advantage of the current formulation is that it is more general than in Megretski and Rantzer (1997) and it is possible to recover existing results of the IQC theory, as shown in Scherer and Veenman (2018). ∎

4. THE PROJECTION METHODOLOGY

In this paper, we want to design a filter Ψ which is well-suited for the analysis of infinite dimensional uncertainties such as (3). We will use the projection methodology, an idea firstly introduced for time-delay systems in Seuret and Gouaisbaut (2015). It can be extended to coupled ODE/PDE problems as explained below.

Let $N \geq 0$ and denote by $\{e_k\}_{k \leq N}$ an orthogonal family of $L_2(0,1)$. The projection coefficients of $z(\cdot, t)$ are then defined for $k \in \mathbb{N}$ as $\Omega_k(t) = \int_0^1 z(x,t)e_k(x)dx$.

The filter should then generate the projections coefficients of $z(\cdot, t)$ from its boundary values. Notice that the signals Ω_k can be generated by a marginally stable linear system driven by $\mathscr{B}_m(z)$. Indeed, for $m = 2$, the differentiation of Ω_k with respect to time leads to:

$$\dot{\Omega}_k(t) = \int_0^1 z_t(x,t)e_k(x)dx = \sum_{i=0}^2 F_i \int_0^1 \partial_x^{(i)} z(x,t)e_k(x)dx.$$

If we choose $\{e_k\}_{k \in \mathbb{N}}$ as a polynomial orthogonal family of $L_2(0,1)$, $\frac{d}{dx}e_k$ is a linear combination of strictly lower order polynomials in the same family. Since we are using integration by parts, the Legendre polynomials $\{\mathcal{L}_k\}_{k \in \mathbb{N}}$ are said to be the best polynomial basis of $L_2(0,1)$ according to Boyd and Petschek (2014). This family has the following properties (Courant and Hilbert, 1989):

$$\mathcal{L}_k(1) = 1, \quad \mathcal{L}_k(0) = (-1)^k, \quad \|\mathcal{L}_k\|_{L_2}^2 = (2k+1)^{-1},$$

$$\frac{d}{dx}\mathcal{L}_k(x) = \sum_{j=0}^k \ell_{kj}\mathcal{L}_j(x), \quad \frac{d^2}{dx^2}\mathcal{L}_k(x) = \sum_{j=0}^k \sum_{i=0}^j \ell_{kj}\ell_{ji}\mathcal{L}_i(x),$$

where $\ell_{kj} = (2j+1)(1-(-1)^{k+j})$ if $j \leq k$ and $\ell_{kj} = 0$ otherwise. Note that $\dot{\Omega}_k$ is a linear combination of strictly lower order projection coefficients, which provides a hierarchical structure, expressed explicitly as follows:

$$\dot{\Omega}_k(t) = F_2 \left(z_x(1,t) - (-1)^k z_x(0,t) - \sum_{j=0}^k \ell_{kj}z(1,t) \right.$$
$$\left. + \sum_{j=0}^k (-1)^j \ell_{kj}z(0,t) + \sum_{j=0}^k \sum_{i=0}^j \ell_{kj}\ell_{ji}\Omega_i(t) \right)$$
$$+ F_1 \left(z(1,t) - (-1)^k z(0,t) - \sum_{j=0}^k \ell_{kj}\Omega_j \right) + F_0\Omega_k.$$

The behaviors of the projection coefficients are highly related to the behavior of $z(\cdot, t)$ and can be generated by a linear sub-system whose inputs are the boundary conditions. It is natural to define the state of the filter by stacking the projection coefficients as in

$$\xi_N(t) = \text{col}\left(\Omega_0(t),\ \Omega_1(t),\ \cdots,\ \Omega_N(t)\right).$$

Finally, the filter dynamics can be expressed by (11) in the case $m \in \{1,2\}$, $n_\xi = (N+1)l$ and $p_\psi = 2ml + n_\xi$ with

$$A_\Psi^N = \sum_{i=0}^m (-1)^i \tilde{F}_i^N L_N^i, \quad B_y^N = 0_{n_\xi,p},$$

$$B_\mathscr{B}^N = \begin{bmatrix} B_1^N & \cdots & B_m^N \end{bmatrix}, \quad B_i^N = \sum_{j=i}^m \tilde{F}_j(-L_N)^{j-i}\begin{bmatrix} -\mathbb{1}_N^* & \mathbb{1}_N \end{bmatrix},$$

$$C_\Psi^N = \begin{bmatrix} 0_{2(m+1)l,n_\xi} \\ I_{n_\xi} \end{bmatrix}, \quad D_\mathscr{B}^N = \begin{bmatrix} I_{2(m+1)l} \\ 0_{n_\xi,2(m+1)l} \end{bmatrix}, \quad D_y^N = 0_{p_\psi,p},$$

where

$$L_N = [\ell_{ij}I_l]_{i,j=0..N}, \quad \tilde{F}_i^N = \text{diag}(F_i,\cdots,F_i) \in \mathbb{R}^{n_\xi \times n_\xi},$$

$$\mathbb{1}_N = \begin{bmatrix} I_l & I_l & \cdots & I_l \end{bmatrix}^\top, \quad \mathbb{1}_N^* = \begin{bmatrix} I_l & -I_l & \cdots & (-I_l)^N \end{bmatrix}^\top \in \mathbb{R}^{n_\xi \times l}.$$

Notice that the previous equation can be extended to higher values of m with a proof based on induction.

To summarize, the filter Ψ_N has the form

$$\Psi_N : \begin{array}{ccc} L_{2e}^{p+2ml} & \to & L_{2e}^{p_\psi} \\ \begin{bmatrix} y \\ \mathscr{B}_m(z) \end{bmatrix} & \mapsto & \psi_N = \begin{bmatrix} \mathscr{B}_m(z) \\ \xi_N \end{bmatrix}, \end{array} \tag{20}$$

and from the boundary values of z and its derivatives it computes the Nth projection coefficients of z. Using this filter in Theorem 1 leads to the following definitions:

$$\mathcal{A} = \begin{bmatrix} A_\Psi^N & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_\mathscr{B}^N \\ BL \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C_\Psi^N & 0 \end{bmatrix}, \quad \mathcal{D} = D_\mathscr{B}^N.$$

Note that the filter Ψ_N is depending on the dynamics of the uncertainty. Finding the multiplier M_N is then a consequence of the order N used for the filter and the PDE under consideration. To apply Theorem 1, we can follow the subsequent steps:

(1) First rewrite the system under consideration to fit equations (2) and (3) (see Examples 1 and 2);
(2) Find a class of multiplier M_N and terminal cost matrices Z_N verifying (12) (See Section 5);
(3) Test the feasibility of LMIs (14) and (15) using the multipliers found in the previous step. If they are feasible, then the system is stable (see Section 6).

5. GENERATING IQCS USING PROJECTIONS

Proposing IQCs for infinite-dimensional systems requires different tools than in the finite-dimensional case. We introduce in the first subsection a justification of how the filter designed in the previous subsection can be helpful. Then, the second subsection is dedicated to an application to a transport equation as defined in Example 1.

5.1 General idea

Since $z(\cdot, t) \in H^m([0,1])$, it is difficult to find dissipation inequalities. One solution is to consider an approximation of z on a finite-dimensional space. Let $N \geq 0$ and denote by $(e_k)_{k \leq N}$ an orthogonal family of $L_2(0,1)$. Then, the following holds for $t \geq 0$:

$$\min_{y \in \text{Span}(e_k)_{k \leq N}} \|z(\cdot,t) - y\|_{L_2}^2 = \|z(\cdot,t)\|_{L_2}^2 - \|z_N(\cdot,t)\|_{L_2}^2 \geq 0.$$

Here we have $z_N(\cdot,t) = \sum_{k=0}^N \Omega_k(t)\frac{e_k(\cdot)}{\|e_k\|_{L_2}^2}$. The previous equality shows that z_N is the projection of z on the subspace spanned by the family $(e_k)_{k \leq N}$ and is consequently

the optimal approximation (with respect to the norm $\|\cdot\|_{L_2}$) of z in the former family. Note that for $R \in \mathbb{S}_+^l$, a consequence of the previous statement is the following:

$$\int_0^1 z^\top(x,t)Rz(x,t)dx \geq \sum_{k=0}^N \frac{1}{\|e_k\|_{L_2}^2}\Omega_k^\top(t)R\Omega_k(t). \quad (21)$$

Inequality (21) is referred to as Bessel Inequality and, as a consequence of Parseval's theorem, equality holds when N goes to infinity. The previous inequality is the starting point for building multipliers in a similar way as with Jensen's inequality for time-delay systems (Fridman, 2014). These considerations indicates that Ω_k are signals of interest to characterize the uncertainty Δ with the help of the IQC defined in (12), since the Bessel inequality relates the energies of the PDE and of a finite-dimensional system.

5.2 Application to the transport equation

Example 1 shows how to model a time-delay system using our framework. This section is dedicated to finding multipliers for the uncertainty (6) such that inequality (12) holds. Two multipliers M_N^1 and M_N^2 are obtained using different techniques and an heavy use of (21).

Finite-horizon IQC with terminal cost. Since the transport equation is an hyperbolic equation, it is a lossless system from an energy perspective. This first IQC expresses this physical fact.

Lemma 1. For the transport equation in (6), if $S \succ 0$, then the finite-horizon IQC (12) with terminal cost Z_N^1 and multiplier M_N^1 holds with:

$$M_N^1 = \rho \begin{bmatrix} S & 0 & 0 \\ 0 & -S & 0 \\ \hline 0 & 0 & 0_{n_\xi} \end{bmatrix}, Z_N^1 = -\operatorname{diag}(S, 3S, \ldots, (2N+1)S).$$

Proof : Let us introduce the following notation for $t \geq 0$:

$$\mathcal{I}_1(t) = \int_0^1 z^\top(x,t)Sz(x,t)dx = \|z(\cdot,t)\|_S^2.$$

$\mathcal{I}_1(t)$ is the energy of the PDE at a given time t. Since $\mathcal{I}_1(0) = 0$, the following holds for $T \geq 0$:

$$\mathcal{I}_1(T) = \int_0^1\int_0^T \partial_t\left(z^\top(x,t)Sz(x,t)\right)dtdx$$
$$= \rho\int_0^T z^\top(0,t)Sz(0,t) - z^\top(1,t)Sz(1,t)dt.$$

From (21), we get $\mathcal{I}_1(T) \geq \sum_{k=0}^N(2k+1)\Omega_k^\top(T)S\Omega_k(T)$. Combining with the two previous results yields

$$\rho\int_0^T z^\top(0,t)Sz(0,t) - z^\top(1,t)Sz(1,t)dt$$
$$- \sum_{k=0}^N(2k+1)\Omega_k^\top(T)S\Omega_k(T) \geq 0.$$

Then, the following IQC with terminal cost Z is obtained:

$$\int_0^T \psi_N(t)^\top M_N^1\psi_N(t)dt + \xi(T)^\top Z_N^1\xi(T) \geq 0$$

for all $T \geq 0$. $\qquad\square$

Remark 5. Losslessness does not translate into an equality since we used Bessel inequality. Equality is recovered for $N \to \infty$. $\qquad\blacksquare$

Finite horizon IQC with $Z = 0$. The previous inequality is about the energy balance between the input and the output. In this part, we are interested in the transmission of energy between the input and the PDE itself.

Lemma 2. For $R \succ 0$, The finite horizon IQC (12) with terminal cost $Z = 0$ and with multiplier

$$M_N^2 = \operatorname{diag}(R, 0_l, -R, -3R, \ldots, -(2N+1)R) \quad (22)$$

holds for system (6) together with the filter (19).

Proof : We start this time with an integration by parts:

$$\int_0^1 \operatorname{He}\left((1-x)z^\top(x,t)Rz_x(x,t)\right)dx =$$
$$2\left[(1-x)z^\top(x,t)Rz(x,t)\right]_0^1 + 2\|z(\cdot,t)\|_R^2$$
$$- \int_0^1 \operatorname{He}\left((1-x)z_x^\top(x,t)Rz(x,t)\right)dx.$$

In other words, the following holds:

$$z^\top(0)Rz(0) = \|z(\cdot)\|_R^2 - \int_0^1 \operatorname{He}\left((1-x)z^\top(x)Rz_x(x)\right)dx. \quad (23)$$

In a similar way as previously, define the following:

$$\mathcal{I}_2(t) = \int_0^1(1-x)z^\top(x,t)Rz(x,t)dx \leq \|z(\cdot,t)\|_R^2.$$

Then, we get for $T \geq 0$:

$$\mathcal{I}_2(T) = -2\rho\int_0^T\int_0^1(1-x)z^\top(x,t)Rz_x(x,t)dxdt.$$

Using equation (23) and inequality (21), we get:

$$\mathcal{I}_2(T) \leq \rho\int_0^T z^\top(0)Rz(0) - \sum_{k=0}^N(2k+1)\Omega_k^\top R\Omega_k dt.$$

Since $\mathcal{I}_2(t) \geq 0$ for $t \geq 0$, the previous results lead to:

$$\int_0^T \psi_N(t)M_N^2\psi_N(t)dt \geq 0. \qquad\square$$

Finally, taking $M_N = M_N^1 + M_N^2$ and $Z_N = Z_N^1$, we get that M_N defines an IQC with terminal cost Z_N for the uncertainty described in Example 1. The second step of the methodology has then been successfully applied. The following section is dedicated to an application of Theorem 1 for a time-delay system.

6. NUMERICAL EXAMPLE AND DISCUSSION

We consider the time-delay system in Example 1 with:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-k & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for $k > 0$. This example describes a regenerative chatter taken from Gu et al. (2003); Seuret and Gouaisbaut (2015). This system is a good toy example as finding the values of (k, ρ) making the system stable is not straightforward. Nevertheless, using the Control Toolbox of Matlab® (as in (Seuret and Gouaisbaut, 2015) for instance), the stability areas can be exactly obtained.

A first analysis shows that, in the delay independent case, the maximum allowable k is $k_{max} = 0.3$. Using the Small-Gain theorem leads to the stability for $k \leq 0.104$, for [Megretski, 1997], the maximum k is 0.265, while using the method described in this paper [3], we get 0.299. Even at low order, the proposed methodology is very efficient

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[3] The code is available at https://github.com/mBarreau/IQC.

Stable Intervals	$[0, 0.859]$	$[1.117, 1.264]$	$[2.75, 3.5]$
Theo. 2, N = 0	$[0, 0.062]$	—	—
Theo. 2, N = 2	$[0, 0.854]$	—	—
Theo. 2, N = 5	$[0, 0.859]$	$[1.123, 1.264]$	—
Theo. 2, N = 7	$[0, 0.859]$	$[1.117, 1.264]$	$[2.83, 3.36]$
[Megretski, 1997]	$[0, 0.062]$	—	—
[Veenman, 2016]	$[0, 0.060]$	—	—

Table 1. Table of stable values for ρ^{-1}. The first row depicts the exact stable intervals and rows 2 to 5 are coming from Theorem 1.

for dealing with hyperbolic equations.

We are interested now in the numerical analysis in the case of a fixed $k = 2$. The results are displayed in Table 1. To obtain this table, we used the multiplier obtained in the previous section and solved LMIs (14)-(15) for some h.

First of all, we can note the hierarchy property of Theorem 1 with this example. Indeed, the stable intervals for $N = 0$ are included in the others obtained for higher values of N. And this is also the case for the order 2 and 5. We can note that for $N = 5$ or 7, there are at least two stability pockets detected, meaning that Theorem 1 can recover stable intervals of the form $\rho^{-1} \in [h^-, h^+]$ with a possible non zero h^-. And for $N = 7$, the detected stable intervals are close to the exact ones, meaning that the theorem provides a precise estimate of the stability area. The comparison with other IQC theorems shows that Theorem 1 is different in nature. Indeed, the IQC theorems in Megretski and Rantzer (1997) and Veenman et al. (2016) provide stability for a constant delay between 0 and a maximum value. Consequently, they cannot detect stability pockets but they ensure the robustness for the delay varying between the two obtained bounds.

7. CONCLUSION & PERSPECTIVES

This paper presented a preliminary work on the robust analysis of infinite-dimensional systems coupled to an ODE. It introduces a different way to assess its stability using IQCs. This approach gives insights about how to build a filter deriving the projections coefficients which helps characterizing the uncertainty.

We use a new kind of IQCs, which is valid for many more PDEs and, in that sense, we obtain a robustness result. Moreover, it has been shown with an example that, for the transport equation, it leads to the same result as using one of the most recent Lyapunov functionals in the literature.

Furthermore, this paper offers many perspectives, that were not raised with the Lyapunov approach. For instance, further work concerns a methodology for deriving the IQCs. New tools were introduced and it would be interesting to pursue this direction in order to give a better interpretation of the filter. The interpretation of each IQC in terms of energy would help building new dissipation inequalities that are less conservative.

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