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Distributed State Estimation for a Class of Jointly Observable Nonlinear Systems ⋆

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Abstract: State estimation for a class of nonlinear systems using a network of distributed observers is dealt with in this paper. We propose a network of observers, each of which has its own local measurements which may not be sufficient for observability of the system, while the joint measurements of the sensors in the network guarantees the observability. We derive sufficient conditions on the proposed observers such that the estimated states of each observer asymptotically converge to the states of the system using local measurements and the estimates of a subset of observers in its neighborhood. Accordingly, it is assumed that each observer has access to the estimates of its neighbors via a communication network. The main problem of existing distributed state estimation strategies in the literature is their limitation to linear systems or to nonlinear systems with observable nonlinearities (by considering nonlinearities as states to be estimated). However, based on the proposed strategy, distributed state estimation of a more general class of nonlinear systems can be devised. A numerical example is provided to evaluate the accuracy of the obtained results.

Keywords: Distributed State Estimation, Joint Observability, Lipschitz Nonlinear Systems, Lyapunov Analysis.

1. INTRODUCTION

In recent years, there has been a remarkable growth of large-scale complex systems such as smart power grids, water networks, industrial plants, etc (Lin et al., 2017; Sanz et al., 2012; Stadelmann et al., 2019). The states of such systems are usually monitored by sets of local sensors distributed over the whole system where the sets of the sensors jointly guarantee the observability of the whole system (none of the local sensors may guarantee the observability). If the estimation procedure is conducted in a centralized way, the problem of high computational and communication costs and the delay of information exchange would arise. Therefore, the observation of a large-scale system is desired to be solved in a distributed manner. In this condition, the objective is to estimate the entire states of the system by each local observer whose available information is the local measurements and the state estimates of its neighboring observers received via a communication network.

The main idea of distributed observers existing in the literature is the extension of the classical linear observers to distributed networks by considering relative state estimation of the observers in the design. For instance, in (Olfati-Saber, 2007, 2009; Kamgarpour and Tomlin, 2008), by extending the classical Kalman filter design to distributed sensor networks, distributed estimation schemes were proposed. In (Farina et al., 2010), a distributed estimation algorithm based on a moving horizon estimation paradigm was presented. A suboptimal consensus-based distributed filtering scheme was proposed in (Matei and Baras, 2012). In (Shen et al., 2010) and (Ugrinovskii, 2013), robust distributed $\mathcal{H}_\infty$-consensus filters designs were developed. In (Kim et al., 2016) and (Han et al., 2019), Luenberger-type distributed state observers were presented. In (Mitra and Sundaram, 2018), a distributed observer design based on the idea of observable canonical decomposition was devised. In (Park and Martins, 2017), by introducing augmented states for distributed estimation, necessary and sufficient conditions for distributed estimation were derived, and in (Wang and Morse, 2018) and (Wang et al., 2019), a family of distributed observers to estimate a system states in an arbitrary rate were addressed.

In addition to the above-mentioned studies devoted to linear systems, few studies have been devoted to distributed estimation in nonlinear systems (Ding et al., 2012; Hu et al., 2015; Liang et al., 2016, 2011; Wang et al., 2015, 2016). However, the main assumption of those studies was that the available measurements for each observer guaranteed the observability of all the states of the system. Indeed, the main objective of the introduced approaches in (Ding et al., 2012; Hu et al., 2015; Liang et al., 2016, 2011; Wang et al., 2015, 2016) was distributed filtering such that a desired stochastic performance via a sensor network was achieved. Moreover, in (He et al., 2018) and (He et al., 2020), distributed state estimation in jointly observable nonlinear networks was considered. The main assumption

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of those results was the knowledge of the stochastic properties of the rates of the changes of nonlinearities such that the nonlinearities could be estimated as states. However, that assumption might not have been practical in several practical scenarios, because nonlinearities are functions of the states and the states may not be accessible.

In this paper, a distributed state estimation strategy for a class of nonlinear systems is proposed. We design a network of distributed nonlinear observers, each of which has access to local measurements which are not sufficient for observability of the system, while the ensemble of all the measurements in the network guarantees the observability. Based on the Lyapunov stability criterion and linear matrix inequalities (LMIs) arguments, sufficient conditions on the observers to guarantee local estimation of the states of the system by each observer are obtained. In summary, compared with the existing results in the literature devoted to distributed estimation in nonlinear systems, the main contributions of the paper are as follows:

- In contrast to the results of (Ding et al., 2012; Hu et al., 2015; Liang et al., 2016, 2011; Wang et al., 2015, 2016), we assumed that the local measurements of each observer may not guarantee the observability of the system, and the ensemble of the measurements in the network guarantees the observability (some observers may even have no local measurements).
- Despite the results of (He et al., 2018) and (He et al., 2020), the rates of the changes of the nonlinearities are considered unknown (as the nonlinearities may be functions of unmeasurable states).

It should be noted that in (Battilotti and Mekhail, 2019), distributed state estimation of a class of nonlinear systems with joint observably is considered. However, in that study, the state matrix and the output matrix of the whole system are considered in block diagonal forms, which limited the application of the proposed observation strategy.

The organization of this paper is as follows. In Section 2, the problem is formulated. The proposed design of the distributed observer is presented in Section 3. Simulation results are given in Section 4. Finally, conclusions and future work are discussed in Section 5.

2. PRELIMINARIES

Notation, concepts and some definitions on graph theory are presented in this section.

2.1 Notation

Throughout the paper, the following notations are considered. $\mathbb{R}$ is the set of real numbers. $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of positive and non-negative real numbers, respectively. I_n denotes an $n\times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n\times m$ all-zeros matrix, and $\mathbf{1}_n$ is an $n \times 1$ all-ones vector. $\|\cdot\|$ is the standard Euclidean norm. $\otimes$ denotes the Kronecker product. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^{-r} \in \mathbb{R}^{m\times n}$ represents the right inverse of A such that $AA^{-r} = I_n$. $\simeq$ represents the isomorphic relation between two vector spaces. For a subspace $\mathscr{V} \subseteq \mathscr{X}$, $\mathscr{V}^{\perp}$ denotes the orthogonal complement of $\mathscr{V}$. If $\mathscr{R}, \mathscr{S} \in \mathscr{X}$, we define the subspaces $\mathscr{R} + \mathscr{S} \subseteq \mathscr{X}$ and $\mathscr{R} \cap \mathscr{S} \subseteq \mathscr{X}$ according to

$$\begin{aligned}\mathscr{R} + \mathscr{S} :=& \{r + s : r \in \mathscr{R} \text{ and } s \in \mathscr{S}\},\\ \mathscr{R} \cap \mathscr{S} :=& \{x : x \in \mathscr{R} \text{ and } x \in \mathscr{S}\}.\end{aligned}$$

The symbol $\oplus$ indicates that the subspaces being added are independent. $\dim(\mathscr{V})$ denotes the dimension of the space $\mathscr{V}$. $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a real symmetric matrix and $\lambda_2(\cdot)$ denotes the second smallest eigenvalue of a real symmetric matrix. For a square matrix M, let $M \succ 0$ or $M \succeq 0$ if it is symmetric positive definite or symmetric positive semi-definite, and $\mathrm{diag}(M_1, M_2, \ldots, M_n)$ is a block diagonal matrix composed of the matrices $M_1, M_2, \ldots, M_n$.

2.2 Graph Theory

Communication among the observers is described by an undirected weighted graph denoted by $\mathcal{G} = (\mathbf{N}, \mathcal{E}, \mathcal{A})$ where $\mathbf{N} = \{1, 2, \ldots, N\}$ is a finite nonempty set of nodes of the graph (describing the N sets of observers with local sensors), $\mathcal{E} \subseteq \mathbf{N} \times \mathbf{N}$ represents the edges of the graph (describing communication among the nodes), and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix where a_{ij} is positive if there exists an edge between Node i and Node j, and it is zero otherwise. Moreover, we define an undirected graph as connected if there is path of edges between each two nodes of the graph.

Define the Laplacian matrix associated with the graph $\mathcal{G}$ as $\mathcal{L} := \mathcal{D} - \mathcal{A}$ where the i-th entry of the diagonal matrix $\mathcal{D}$ is given by $d_i = \sum_{j=1}^{N} a_{ij}$. Since $a_{ij} = a_{ji}$, $\mathcal{L}$ is a symmetric matrix. Moreover, if the graph $\mathcal{G}$ is connected, the corresponding Laplacian matrix $\mathcal{L}$ has a simple zero eigenvalue with the corresponding eigenvector $\mathbf{1}_N$, and all the other eigenvalues of $\mathcal{L}$ are positive (Ren et al., 2007).

3. PROBLEM STATEMENT

Consider a class of nonlinear systems in the form

$$\begin{aligned}\dot{x} &= Ax + f(x) + Bu,\\ y_i &= C_i x, \quad i \in \mathbf{N},\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ denotes the input matrix, $f(x) \in \mathbb{R}^n$ is a nonlinear function of the states, and $y_i \in \mathbb{R}^{p_i}$ is the measurement output in the i-th node. Accordingly, $C_i \in \mathbb{R}^{p_i\times n}$ is the output matrix associated with the i-th node. In this condition, the collection of all the outputs can be represented as

$$y = \begin{bmatrix} y_1^{\top} & y_2^{\top} & \ldots & y_N^{\top} \end{bmatrix}^{\top}$$

with $\sum_{i=1}^{N} p_i = p$.

Assumption 1. The communication graph associated with the observers network is connected.

Assumption 2. The matrix A is constructed such that the system is jointly observable, i.e., the pair (C, A) is observable where $C = \begin{bmatrix} C_1^{\top} & \cdots & C_N^{\top} \end{bmatrix}^{\top}$, but (C_i, A) is not necessarily observable for all $i \in \mathbf{N}$.

***Assumption** 3.* We assume that x belongs to a domain $\mathscr{D}$, such that $f(x)$ is Lipschitz on $\mathscr{D}$ as follows (Khalil and Praly, 2014; Astolfi and Marconi, 2015; Shakarami et al., 2018):

$$\|f(x_1) - f(x_2)\| \le \gamma \|x_1 - x_2\|, \forall x_1, x_2 \in \mathscr{D},$$

where $\gamma \in \mathbb{R}_{>0}$.

Considering nonlinear system (1), the objective is to exploit the jointly observability property of the system to design a network of distributed observers such that the estimated states of each observer converge to the states of the system as

$$\lim_{t\to\infty} (\hat{x}_i(t) - x(t)) = \mathbf{0}_{n\times 1}, \quad \forall i \in \mathbf{N}.$$

The main results are stated in the next section.

4. DISTRIBUTED OBSERVERS DESIGN

In this section, the proposed distributed estimation strategy for the system described in (1) is presented. Based on (1), the local observer at Node i is given by:

$$\begin{aligned} \dot{\hat{x}}_i =& A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + f(\hat{x}_i) + Bu \\ &+ \chi P_i^{-1} \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i), \quad i \in \mathbf{N}, \end{aligned} \tag{2}$$

in which $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ denote the observer gains designed as

$$\begin{aligned} L_i =& T_i \begin{bmatrix} P_{io}^{-1} W_{io}^\top \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix} \\ P_i =& T_i \begin{bmatrix} P_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & I_{v_i} \end{bmatrix} T_i^\top \end{aligned} \tag{3}$$

where v_i is the dimension of the unobservable subspace of the pair (C_i, A), and $P_{io} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$ is a positive definite matrix, which is designed later together with $W_{io} \in \mathbb{R}^{(n-v_i)\times p_i}$. Moreover, $T_i \in \mathbb{R}^{n\times n}$ is a similarity transformation matrix which can be represented as $T_i = [T_{io}\ T_{iu}]$ where $T_{iu} \in \mathbb{R}^{n\times v_i}$ is an orthonormal basis of the unobservable subspace of (C_i, A), and $T_{io} \in \mathbb{R}^{n\times(n-v_i)}$ is an orthonormal basis such that Im T_{io} is orthogonal to Im T_{iu} for all $i \in \mathbf{N}$. According to the definition of the matrix T_i, one gets (Kim et al., 2016)

$$\begin{aligned} T_i^\top A T_i =& \begin{bmatrix} A_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ A_{ir} & A_{iu} \end{bmatrix}, \\ C_i T_i =& [C_{io}\ \ \mathbf{0}_{p_i\times v_i}] \end{aligned} \tag{4}$$

where the pair (C_{io}, A_{io}) is observable for all $i \in \mathbf{N}$. The idea of performing a local observer decomposition is inspired by the work of (Kim et al., 2016). This is crucial to achieve the convergence of the estimation error, which is shown later in the analysis.

***Lemma** 1.* Consider the jointly observable system given in (1) and the transformation matrix T_i. Define the space of the system states as $\mathscr{X} \simeq \mathbb{R}^n$, and let $T_o := [T_{1o}\ T_{2o}\ \ldots\ T_{No}]$. Then,

$$\text{Im } T_o = \mathscr{X}.$$

***Proof**.* According to the construction of the transformation matrix T_i, we have

$$(\text{Im } T_{io})^\perp = \bigcap_{k=1}^{n} \text{Ker } C_i A^{k-1}. \tag{5}$$

Since $T_o := [T_{1o}\ T_{2o}\ \ldots\ T_{No}]$, it follows that (Wonham, 1985)

$$\text{Im } T_o = \left(\left(\sum_{i=1}^{N} \text{Im } T_{io} \right)^\perp \right)^\perp = \left(\bigcap_{i=1}^{N} (\text{Im } T_{io})^\perp \right)^\perp. \tag{6}$$

Now, from (5) and (6), one gets

$$\text{Im } T_o = \left(\bigcap_{i=1}^{N} \left(\bigcap_{k=1}^{n} \text{Ker } C_i A^{k-1} \right) \right)^\perp. \tag{7}$$

Since

$$\bigcap_{i=1}^{N} \left(\bigcap_{k=1}^{n} \text{Ker } C_i A^{k-1} \right) = \bigcap_{k=1}^{n} \left(\bigcap_{i=1}^{N} \text{Ker } C_i A^{k-1} \right),$$

from (7), it follows that

$$\text{Im } T_o = \left(\bigcap_{k=1}^{n} \text{Ker } (C A^{k-1}) \right)^\perp. \tag{8}$$

According to Assumption 2, as the pair (C, A) is observable, from (8), we have Im $T_o = 0^\perp = \mathscr{X}$ (Wonham, 1985), which completes the proof. ■

Before presenting the main results, we define the following matrices:

$$\begin{aligned} \tilde{P} &= [P_1\ P_2\ \ldots\ P_N], \\ K_{io} &= -(A_{io}^\top P_{io} + P_{io} A_{io} + C_{io}^\top W_{io} + W_{io}^\top C_{io}), \\ K &= \text{diag}(K_{1o}, K_{2o}, \ldots, K_{No}), \\ M &= T_o^{-r} \sum_{i=1}^{N} \Big(T_{iu} A_{ir} T_{io}^\top + T_{io} A_{ir}^\top T_{iu}^\top \\ &\quad + T_{iu}(A_{iu} + A_{iu}^\top) T_{iu}^\top \Big) \left(T_o^{-r}\right)^\top, \\ \Lambda_i &= (A + L_i C_i)^\top P_i + P_i (A + L_i C_i), \\ \Lambda &= \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_N), \\ \Lambda_P &= \left[\Lambda_1 + \gamma(P_1^2 + I_n)\ \ldots\ \Lambda_N + \gamma(P_N^2 + I_n)\right], \\ Q &= -\gamma N I_n - \sum_{i=1}^{N} \left(\Lambda_i + \gamma P_i^2\right), \\ \bar{P} &= \text{diag}\left(P_1^2 + I_n, P_2^2 + I_n, \ldots, P_N^2 + I_n\right). \end{aligned} \tag{9}$$

***Theorem** 1.* Consider the nonlinear system described in (1) under Assumptions 1-3 and the distributed observer given in (2) and (3). Then, the estimation errors $e_i = \hat{x}_i - x, \forall i \in \mathbf{N}$, converge to zero if the matrices P_{io} and W_{io}, $i \in \mathbf{N}$, are obtained from the solution of the following LMI:

$$\begin{aligned} \begin{bmatrix} I_{Nn} & \sqrt{\gamma}\tilde{P}^\top \\ \sqrt{\gamma}\tilde{P} & T_o (K - M) T_o^\top - \gamma N I_n \end{bmatrix} \succ 0, \\ P_{io} \succ 0, \end{aligned} \tag{10}$$

and the gain χ is chosen as follows:

$$\chi > \frac{\left\| \Lambda + \gamma \bar{P} + \Lambda_P^\top Q^{-1} \Lambda_P \right\|}{2\lambda_2(\mathcal{L})}. \tag{11}$$

***Proof**.* Considering the nonlinear system with the form of (1) and based on the devised observer described in (2), the differential equation describing $e_i = \hat{x}_i - x,\ i \in \mathbf{N}$, is given by

$$\dot{e}_i = (A + L_iC_i)e_i + f(\hat{x}_i) - f(x) + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(e_j - e_i), \quad i \in \mathbf{N}. \tag{12}$$

To analyze the evolution of the estimation errors of all the nodes, we consider the following Lyapunov candidate:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i.$$

Since $P_{io} \succ 0$, $\forall i \in \mathbf{N}$, V is positive definite. The time derivation of V along (12) yields

$$\begin{aligned}\dot{V} = &\sum_{i=1}^{N} e_i^\top \left((A + L_iC_i)^\top P_i + P_i(A + C_iL_i)\right) e_i \\ &+ 2\chi \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(e_j - e_i\right)^\top e_i \\ &+ 2\sum_{i=1}^{N} e_i^\top P_i\left(f(\hat{x}_i) - f(x)\right).\end{aligned} \tag{13}$$

According to Assumption 3 and since $e_i = \hat{x}_i - x$, one gets

$$2\sum_{i=1}^{N} e_i^\top P_i\left(f(\hat{x}_i) - f(x)\right) \le 2\gamma \sum_{i=1}^{N} \|e_i\|\|P_ie_i\|. \tag{14}$$

Since $2\|e_i\|\|P_ie_i\| \le e_i^\top e_i + e_i^\top P_i^2 e_i$, from (14), it can be written that

$$2\sum_{i=1}^{N} e_i^\top P_i\left(f(\hat{x}_i) - f(x)\right) \le \gamma e^\top \bar{P} e \tag{15}$$

where $\bar{P} \in \mathbb{R}^{Nn\times Nn}$ is a block diagonal matrix given in (9) and $e := \begin{bmatrix} e_1^\top & e_2^\top & \dots & e_N^\top \end{bmatrix}^\top$. From (13) and (15), it follows that

$$\dot{V} \le e^\top \Lambda e - 2\chi e^\top\left(\mathcal{L} \otimes I_n\right)e + \gamma e^\top \bar{P} e \tag{16}$$

where $\Lambda \in \mathbb{R}^{Nn\times Nn}$ is a block diagonal matrix in which $\Lambda_i \in \mathbb{R}^{n\times n}$ is a symmetric matrix defined in (9). Define the error space as $\mathscr{E} \simeq \mathbb{R}^{Nn}$. According to Assumption 1 and the properties of the Laplacian matrix of a connected undirected graph, the kernel of $\mathcal{L} \otimes I_n$ has the form of $\mathbf{1}_N \otimes \omega$ where $\omega \in \mathbb{R}^n$ is an arbitrary vector. Now, by letting $\mathscr{E}_c \subseteq \mathscr{E}$ $(\dim(\mathscr{E}_c) = n)$ as the null space of $\mathcal{L} \otimes I_n$, we consider another subspace $\mathscr{E}_r \subseteq \mathscr{E}$ $(\dim(\mathscr{E}_r) = Nn - n)$, which is the orthogonal complement of $\mathscr{E}_c$ of the space $\mathscr{E}$, i.e., $\mathscr{E}_c \oplus \mathscr{E}_r = \mathscr{E}$. Suppose the vectors e_c and e_r are elements of the subspaces $\mathscr{E}_c$ and $\mathscr{E}_r$ respectively, i.e., $e_c \in \mathscr{E}_c$ and $e_r \in \mathscr{E}_r$. Therefore, from (16), one gets

$$\begin{aligned}\dot{V} \le &e_c^\top(\Lambda + \gamma\bar{P})e_c + 2e_r^\top(\Lambda + \gamma\bar{P})e_c \\ &+ e_r^\top(\Lambda + \gamma\bar{P})e_r - 2\chi e_r^\top(\mathcal{L} \otimes I_n)e_r.\end{aligned} \tag{17}$$

A necessary condition to guarantee the negative definiteness of $\dot{V}$ is to ensure $e_c^\top(\Lambda + \gamma\bar{P})e_c$ is negative definite $\forall e_c \in \mathscr{E}_c$. Since e_c has the form of $\mathbf{1}_N \otimes \omega$, according to the definition of Λ and $\bar{P}$ in (9), we can derive

$$e_c^\top(\Lambda + \gamma\bar{P})e_c = \omega^\top\left(\gamma N I_n + \sum_{i=1}^{N}(\Lambda_i + \gamma P_i^2)\right)\omega. \tag{18}$$

From (3) and (4), we have

$$\Lambda_i = T_i\begin{bmatrix} -K_{io} & A_{ir}^\top \\ A_{ir} & A_{iu} + A_{iu}^\top \end{bmatrix} T_i^\top \tag{19}$$

where $K_{io} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$ is given in (9). Now, since $T_i = [T_{io}\ T_{iu}]$, (19) can be rewritten as follows:

$$\begin{aligned}\Lambda_i = &-T_{io}K_{io}T_{io}^\top + T_{iu}A_{ir}T_{io}^\top + T_{io}A_{ir}^\top T_{iu}^\top \\ &+ T_{iu}(A_{iu} + A_{iu}^\top)T_{iu}^\top.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{N}\Lambda_i = &-T_oKT_o^\top + \sum_{i=1}^{N}\left(T_{iu}A_{ir}T_{io}^\top + T_{io}A_{ir}^\top T_{iu}^\top\right) \\ &+ \sum_{i=1}^{N} T_{iu}(A_{iu} + A_{iu}^\top)T_{iu}^\top\end{aligned} \tag{20}$$

where $K \in \mathbb{R}^{\left(Nn - \sum_{i=1}^N v_i\right)\times\left(Nn - \sum_{i=1}^N v_i\right)}$ is a block diagonal matrix given in (9) and T_o is defined in Lemma 1. According to Lemma 1, rank $T_o = n$, and hence T_o is a row independent matrix. Therefore, there exists T_o^{-r} such that $T_oT_o^{-r} = I_n$. Therefore, from (20), it can be said that

$$\sum_{i=1}^{N}\Lambda_i = -T_o\left(K - M\right)T_o^\top$$

in which M is defined in (9). Therefore,

$$\begin{aligned}&-\gamma N I_n - \sum_{i=1}^{N}\left(\Lambda_i + \gamma P_i^2\right) \\ &= T_o\left(K - M\right)T_o^\top - \gamma N I_n - \gamma\tilde{P}\tilde{P}^\top\end{aligned} \tag{21}$$

where $\tilde{P}$ is as defined in (9). According to (10) and the Schur Complement (see (Boyd et al., 1994) for the Schur Complement), one can conclude that

$$T_o\left(K - M\right)T_o^\top - \gamma N I_n - \gamma\tilde{P}\tilde{P}^\top \succ 0. \tag{22}$$

From (21) and (22), it follows that (18) is negative definite. Moreover, since $\mathscr{E}_r$ is the orthogonal complement of $\mathscr{E}_c$, it can be said that $\forall e_r \in \mathscr{E}_r$ and $\forall \omega \in \mathbb{R}^n$,

$$e_r^\top(\mathbf{1}_N \otimes \omega) = 0.$$

Thus, as $\mathbf{1}_N$ is the eignevector associated with the zero eigenvalue of $\mathcal{L}$, one can get that $\forall e_r \in \mathscr{E}_r$ (Olfati-Saber and Murray, 2004),

$$e_r^\top(\mathcal{L} \otimes I_n)e_r \ge \lambda_2(\mathcal{L})e_r^\top e_r.$$

Based on the above-mentioned issues, (17) can be satisfied if

$$\dot{V} \le -\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix} Q & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\lambda_2(\mathcal{L})I_{Nn} - \left(\Lambda + \gamma\bar{P}\right)\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix}$$

where $Q \in \mathbb{R}^{n\times n} \succ 0$ is defined in (9). Now, according to the Schur Complement, to guarantee the negative definiteness of $\dot{V}$, the following condition should be satisfied:

$$2\chi\lambda_2(\mathcal{L})I_{Nn} - \Lambda - \gamma\bar{P} - \Lambda_P^\top Q^{-1}\Lambda_P \succ 0$$

where the inequality condition (11) guarantees that. Accordingly, V asymptotically converges to zero, and the proof is completed. ■

Remark 1. It is worth mentioning that, of course, the results of Theorem 1 apply to linear systems when $f(x) = \mathbf{0}_{n\times 1}$. In this condition, since $\gamma = 0$, the LMI condition (10) becomes

$$\begin{bmatrix} I_{Nn} & 0 \\ 0 & T_o\left(K - M\right)T_o^\top \end{bmatrix} \succ 0.$$

Since (C_{io}, A_{io}) is observable $\forall i \in \mathbf{N}$, K_{io} given in (9) can be any symmetric positive definite matrix with proper

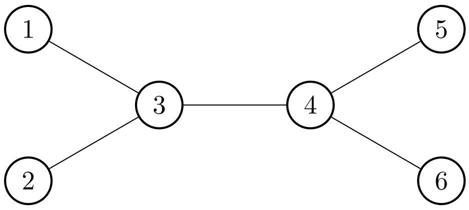

Fig. 1. Network communication graph.

selection of P_{io} and W_{io}. Hence, (10) is always solvable since we can always let

$$\lambda_{\min}(K) = \min_{i\in\mathbf{N}} \lambda_{\min}(K_{io}) > \|M\|. \tag{23}$$

Thus, by designing K based on (23), the distributed estimation problem is always feasible for a linear system satisfying Assumption 1 and Assumption 2.

***Remark** 2.* It is worth mentioning that the sufficient condition on the distributed observer design is the joint observability of the whole system by all the available measurements in the network. As a result, the proposed strategy in this paper still guarantees the convergence of the all estimation errors in the network even when for some nodes $i \in \mathbf{N}$, $C_i = 0$. In such cases, A_{io}, A_{ir}, and T_{io} do not exist, and $A_{iu} = I_n$ with $T_{iu} = I_n$. Accordingly, $L_i = \mathbf{0}_{n\times p_i}$ and $P_i = I_n$.

5. SIMULATION RESULTS

In this section, the effectiveness of the proposed distributed observer design is shown in a numerical example. We consider a jointly observable nonlinear system, which is in the form of (1) when

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C_1 = [0\ 0\ 1], C_2 = [0\ 1\ 1], C_3 = [0\ 0\ 0],$$
$$C_4 = [1\ 0\ 0], C_5 = [0\ 1\ 0], C_6 = [0\ 0\ 0].$$

By defining $x = [x(1)\ x(2)\ x(3)]^\top$, the nonlinear term is given by

$$f(x) = \begin{bmatrix} 2\sin x(1) \\ 0.1x(2)\cos x(2) \\ 1.5\sin x(3)\cos x(3) \end{bmatrix}.$$

By considering Assumption 3, the domain $\mathscr{D}$ is assumed such that $\gamma = 2$. Moreover, the nodes are assumed to be connected by the unweighted undirected communication graph depicted in Fig. 1 whose Laplacian matrix is given by

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

implying that $\lambda_2(\mathcal{L}) = 0.4384$.

The distributed observers are designed as (2), and based on Theorem 1, the corresponding gains are obtained as follows:

$$L_1 = [0\ 7.86\ \ -15.72]^\top,\ P_1 = \mathrm{diag}(1, 2.52, 2.52),$$
$$L_2 = [0\ \ -15.72\ 0]^\top,\ P_2 = \mathrm{diag}(1, 2.52, 2.52),$$
$$L_3 = \mathbf{0}_{3\times 1},\quad P_3 = I_3,$$
$$L_4 = [-16.67\ 0\ \ -2.97]^\top,\ P_4 = \mathrm{diag}(2.49, 2.52, 2.52),$$
$$L_5 = [0\ 0\ 7.86]^\top,\ P_5 = \mathrm{diag}(1, 2.52, 2.52),$$
$$L_6 = \mathbf{0}_{3\times 1},\quad P_6 = I_3,$$

and $\chi = 784$. Under these conditions, the observers estimates along with the real states of the system are illustrated in Fig. 2. According to Fig. 2, even if $(C_i, A), i \in \{1, 2, 3, 5, 6\}$, are not observable, all the local observers are capable of providing asymptotic state estimation following the proposed nonlinear observer design.

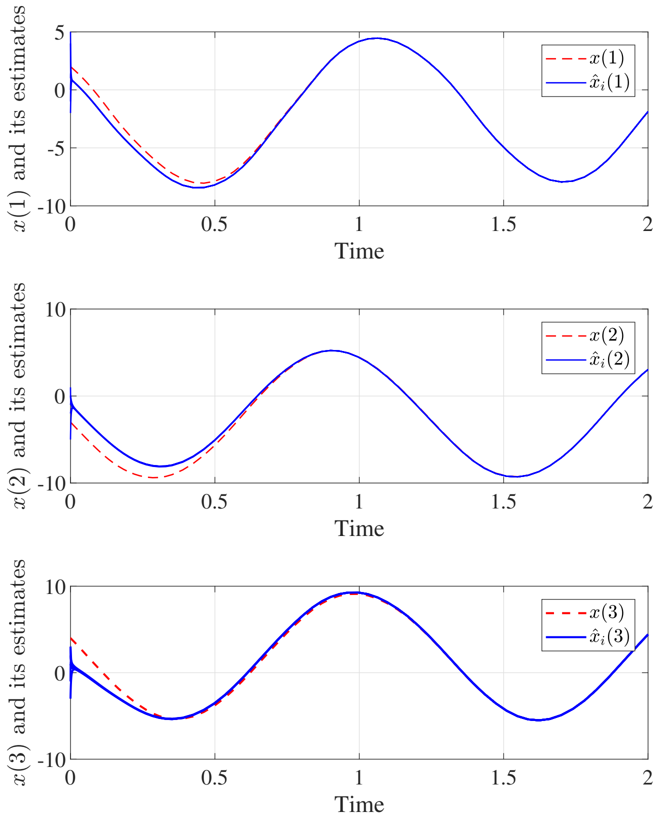

Fig. 2. Observers estimates and the states of the system.

6. CONCLUSIONS AND FUTURE WORK

A distributed observer design for a class of nonlinear systems with distributed sensors was proposed in this paper. Each observer just had access to local measurements of some sensors which might not have been sufficient for observabilily of the system. It was shown that if the joint of the measurements of all the sensors guaranteed the observability and the proposed distributed observers could exchange their estimates through a connected communication graph, they were able to estimate all the states of the system by only using local measurements.

The extension of the idea to more general classes of nonlinear systems and to networks of observers with faulty sensors are other open problems in this area to be considered as future work. Moreover, distributed state estimation when the communication graph being time-varying is also an interesting topic, which is worth investigating in the future.

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