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Constructive backstepping for a class of delay systems based on functionals of complete type

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Abstract: In this paper we consider an alternative approach of the backstepping control strategy with the introduction of artificial delays combined with Lyapunov-Krasovskii functionals of complete type, thus allowing a constructive approach for the design of asymptotically stabilizing controls of linear systems with delay in the input and state that are too long for being neglected.

Keywords: Linear systems, Delay systems, Lyapunov methods, Stability of delay systems

1. INTRODUCTION

Backstepping is a well known technique of design of globally asymptotically stabilizing controls. It applies to broad classes of systems, which include for instance time-varying systems, systems with nonlinear terms, systems with delay and uncertainties. For more than thirty years, this tool has been developed in many contributions such as for instance Coron and Praly (1991), Kokotovic (1992), Kokotovic et al. (1992), Karafyllis (2002), Krstic et al. (1995), Tsinias (1989). This popular technique has been successfully applied to solve many engineering problems. Despite being such an effective technique, it can still be improved and extended in particular because the presence of delays may be an obstacle to the construction of stabilizing control laws using the classical version of backstepping. Thus, in recent years a new version of backstepping has been proposed. It relies on the introduction of artificial delays in the control or dynamic extensions used in it. This new developments are presented in particular in the contributions Mazenc et al. (2019), Mazenc and Malisoff (2016), Mazenc et al. (2018), Mazenc et al. (2016).

Several works design controls under input delays following the predictor feedback design for systems with long input delays; see Kharitonov and Niculescu (2003), Bekiaris-Liberis and Krstic (2013), Zhou (2014), Kharitonov (2015). In the present work we propose an extension of backstepping for coupled linear time invariant systems with delays in state and input our contribution combine backstepping with the introduction of artificial delays and the design of a Lyapunov-Krasovskii functional of complete type; see Krasovskii (1956), Kharitonov (2013) from which robustness issues can be derived.

Notation and classical definitions. For $h > 0$, we let $C([-h, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$ valued functions defined on $[-h, 0]$. We denote this set as C_{in} and we call it the set of all initial functions. $PC([-h, 0], \mathbb{R}^n)$ is the space of $\mathbb{R}^n$ valued piecewise continuous functions on $[-h, 0]$. For a continuous function $\varphi : [-h, \infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define φ_t by $\varphi_t(m) = \varphi(t + m)$ for all $m \in [-h, 0]$, we introduce the norm $|\varphi|_h = \sup |\varphi(\theta)|_{\theta \in [-h,0]}$. We let I_n denote the identity matrix of dimension n. The Euclidean norm $| \cdot |$ is used for vectors and the corresponding induced norm for matrices.

For linear time-delay systems of the form

$$\dot{\xi}(t) = A_0 \xi(t) + A_1 \xi(t - h). \tag{1}$$

The matrix $U(\tau)$ is a Lyapunov matrix of system (1) associated with a symmetric positive definite matrix W if it satisfies the following properties:

(1) *Dynamic property*

$$\frac{dU(\tau)}{d\tau} = U(\tau)A_0 + U(\tau - h)A_1, \qquad \tau \geq 0. \tag{2}$$

(2) *Symmetry property*

$$U(-\tau) = U(\tau)^T. \tag{3}$$

(3) *Algebraic property*

$$-W = U(0)A_0 + A_0^T U(0) + U(-h)A_1 + A_1^T U(h). \tag{4}$$

Organization of the paper. In Section 2, we introduce a family of systems in feedback form with delays and state the problem to be solved. In Section 3, we present our main results. In Section 4, an illustrative example shows the stabilizing effect of the proposed control law. Section 5 concludes the paper with some brief remarks.

2. SYSTEM AND ASSUMPTIONS

In this section, we present the control problem for the systems with delay that we study. The control design we propose is based on a backstepping strategy using an artificial delay combined with the use of complete type Lyapunov functional.

We consider a system in feedback form with two pointwise delays of the form:

$$\begin{cases} \dot{\xi}(t) = A_0\xi(t) + A_1\xi(t-h) + B\eta(t-\tau), \\ \dot{\eta}(t) = M\eta(t) + f(t) + u(t), \end{cases} \quad (5)$$

with $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, $B, M, A_i \in \mathbb{R}^n \times \mathbb{R}^n$ for $i = 0, 1$, $u \in \mathbb{R}^n$ is the input, $h \geq 0$ and f is a continuous and known function.

We introduce the following assumption:

Assumption 1 There exist two matrices K_0 and K_1 such that the origin of the system

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) \quad (6)$$

with $H_0 = A_0 + BK_0$ and $H_1 = A_1 + BK_1$, is globally exponentially stable.

Problem 1 Under Assumption 1, design a stabilizing control law for system (5).

For later use, let us introduce the notation

$$\eta_s(t) = K_0\xi(t+\tau) + K_1\xi(t+\tau-h) \quad (7)$$

for all $t \geq \tau$.

3. MAIN RESULT

In this section, we propose a stabilizing control law for the system (5) and we carry out the corresponding closed-loop stability analysis.

Theorem 1. Let system (5) satisfy Assumption 1. There is a real scalar $r > 0$ such that this system in closed-loop with the feedback law

$$\begin{aligned} u(t) =& N\eta(t) - \frac{M+N+kI_n}{r}\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\,dl \\ &+\frac{1}{r}[\eta_s(t) - e^{-kr}\eta_s(t-r)] - f(t), \end{aligned} \quad (8)$$

where N is a matrix such that $M + N$ is Hurwitz and k a positive real number, is globally exponentially stable.

Proof. The proof starts with the definition of the operator:

$$\bar{\eta}(t) = \eta(t) - \frac{1}{r}\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\,dl. \quad (9)$$

Which is well defined for all $t \geq \tau + r$. Simple calculations give

$$\begin{cases} \dot{\xi}(t) = A_0\xi(t) + A_1\xi(t-h) + B\bar{\eta}(t-\tau) \\ \qquad +\frac{B}{r}\int_{t-\tau-r}^{t-\tau} e^{k(l-t+\tau)}\eta_s(l)dl, \\ \dot{\bar{\eta}}(t) = M(\bar{\eta}(t) + \frac{1}{r}\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\,dl) + u(t) \\ \qquad +\frac{1}{r}\left[e^{-kr}\eta_s(t-r) - \eta_s(t)\right] \\ \qquad +\frac{k}{r}\left[\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\,dl\right] + f(t). \end{cases} \quad (10)$$

Observe that the control law (8) can be rewritten as:

$$\begin{aligned} u(t) =& N\bar{\eta}(t) - \frac{M+kI_n}{r}\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\,dl \\ &+\frac{1}{r}[\eta_s(t) - e^{-kr}\eta_s(t-r)] - f(t). \end{aligned} \quad (11)$$

The closed-loop system reduces to:

$$\begin{cases} \dot{\xi}(t) = A_0\xi(t) + A_1\xi(t-h) + B\bar{\eta}(t-\tau) \\ \qquad +\frac{B}{r}\int_{t-r-\tau}^{t-\tau} e^{k(l-t+\tau)}\eta_s(l)\,dl, \\ \dot{\bar{\eta}}(t) = (M+N)\bar{\eta}(t). \end{cases} \quad (12)$$

Hence the $\bar{\eta}$-subsystem in (12) is exponentially stable. We now study the ξ-subsystem

$$\begin{aligned} \dot{\xi}(t) =& A_0\xi(t) + A_1\xi(t-h) + \frac{B}{r}\int_{t-r-\tau}^{t-\tau} e^{k(l-t+\tau)}\eta_s(l)\,dl \\ &+\epsilon(t) \end{aligned} \quad (13)$$

with $\epsilon(t) = B\bar{\eta}(t-\tau)$. We observe that

$$\begin{aligned} \dot{\xi}(t) =& A_0\xi(t) + A_1\xi(t-h) + B\eta_s(t-\tau) \\ &+\frac{B}{r}\int_{t-r-\tau}^{t-\tau} [e^{k(l-t+\tau)}\eta_s(l) - \eta_s(t-\tau)]\,dl \\ &+\epsilon(t). \end{aligned} \quad (14)$$

Using (7) and the definitions of H_0 and H_1 (see Assumption 1) we obtain

$$\begin{aligned} \dot{\xi}(t) =& H_0\xi(t) + H_1\xi(t-h) \\ &+\frac{B}{r}\int_{t-r-\tau}^{t-\tau} [e^{k(l-t+\tau)}\eta_s(l) - \eta_s(t-\tau)]\,dl + \epsilon(t) \end{aligned} \quad (15)$$

for all $t \geq \tau$. We deduce that

$$\begin{aligned} \dot{\xi}(t) =& H_0\xi(t) + H_1\xi(t-h) + \epsilon(t) \\ &+\frac{B}{r}\int_{t-r}^{t} [K_0(e^{k(l-t)}\xi(l) - \xi(t)) \\ &+K_1(e^{k(l-t)}\xi(l-h) - \xi(t-h))]\,dl \end{aligned} \quad (16)$$

for all $t \geq \tau$. As a consequence,

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) + \zeta(t,\xi_t) \quad (17)$$

where

$$\begin{aligned} \zeta(t,\xi_t) =& \frac{B}{r}\int_{t-r}^{t} K_0\left[e^{k(l-t)}\xi(l) - \xi(t)\right] \\ &+K_1\left[e^{k(l-t)}\xi(l-h) - \xi(t-h)\right]dl + \epsilon(t). \end{aligned} \quad (18)$$

This functional can be rewritten as

$$\begin{aligned} \zeta(t,\xi_t) =& \frac{B}{r}\int_{t-r}^{t}\left[K_0\int_{t}^{l}\frac{d}{ds}(e^{k(s-t)}\xi(s))ds\right. \\ &\left.+K_1\int_{t}^{l}\frac{d}{ds}(e^{k(s-t)}\xi(s-h))ds\right]dl + \epsilon(t). \end{aligned} \quad (19)$$

Then

$$\begin{aligned} \zeta(t,\xi_t) =& -\frac{BK_0}{r}\int_{t-r}^{t}\int_{t}^{l} ke^{k(s-t)}\xi(s)\,dsdl \\ &+\frac{BK_0}{r}\int_{t-r}^{t}\int_{t}^{l} e^{k(s-t)}\dot{\xi}(s)\,dsdl \\ &-\frac{BK_1}{r}\int_{t-r}^{t}\int_{t}^{l} ke^{k(s-t)}\xi(s-h)\,dsdl \\ &+\frac{BK_1}{r}\int_{t-r}^{t}\int_{t}^{l} e^{k(s-t)}\dot{\xi}(s-h)\,dsdl + \epsilon(t). \end{aligned} \quad (20)$$

In view of Assumption A1, the Lyapunov matrix $U(\theta)$, $\theta \in [-h, 0]$ of system (6) associated with matrix $W = W_0 + W_1 + hW_2$, where W_i are symmetric positive definite matrices for $i = 0, 1, 2$ exists and is unique. Furthermore, the functional defined as

$$\begin{aligned} V(\xi_t) &= \xi^T(t)U(0)\xi(t) \\ &+2\xi^T(t)\int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta \\ &+\int_{-h}^{0} \xi^T(t+\theta_1)H_1^T\int_{-h}^{0} U(\theta_1-\theta_2)H_1\xi(t+\theta_2)\ d\theta_2 d\theta_1 \\ &+\int_{-h}^{0} \xi^T(t+\theta)[W_1+(\theta+h)W_2]\xi(t+\theta)\ d\theta \end{aligned} \tag{21}$$

admits a quadratic lower bound. Its derivative along the trajectories of system (17) is

$$\begin{aligned} \dot V(\xi_t) &= -\xi^T(t)W_0\xi(t) - \xi^T(t-h)W_1\xi(t-h) \\ &-\int_{-h}^{0} \xi^T(t+\theta)W_2\xi(t+\theta)\ d\theta + 2\zeta^T(t,\xi_t)\times \\ &\left[U(0)\xi(t) + \int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta\right]. \end{aligned} \tag{22}$$

Let us define

$$l(\xi_t) = U(0)\xi(t) + \int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta.$$

Then,

$$\begin{aligned} \dot V(\xi_t) &= -\xi^T(t)W_0\xi(t) - \xi^T(t-h)W_1\xi(t-h) \\ &-\int_{-h}^{0} \xi^T(t+\theta)W_2\xi(t+\theta)\ d\theta + 2\zeta^T(t,\xi_t)l(\xi_t). \end{aligned} \tag{23}$$

Using the inequality $2a^Tb \le a^TMa + b^TM^{-1}b$ which holds for an arbitrary symmetric positive definite matrix $M \in \mathbb{R}^{n\times n}$ and $a, b \in \mathbb{R}^n$, we arrive at

$$\begin{aligned} \dot V(\xi_t) &\le -\lambda_{min}(W_0)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 \\ &-\lambda_{min}(W_2)\int_{-h}^{0} |\xi(t+\theta)|^2\ d\theta + \frac{1}{\delta}|\zeta(t,\xi_t)|^2 \\ &+\delta|l(\xi_t)|^2 \end{aligned} \tag{24}$$

for a positive real scalar $\delta > 0$. Let us define

$$\alpha_0 = |U(0)| \quad \text{and} \quad \alpha_1 = \sup_{\theta\in[-h,0]} |U(-h-\theta)H_1|. \tag{25}$$

Then the inequality

$$|l(\xi_t)|^2 \le 2\alpha_0^2|\xi(t)|^2 + 2\alpha_1^2\int_{-h}^{0} |\xi(t+\theta)|^2\ d\theta \tag{26}$$

holds and

$$\begin{aligned} |\zeta(t,\xi_t)|^2 &\le \frac{c_0}{r^2}\left|\int_{t-r}^{t}\int_{t}^{l} ke^{k(s-t)}\xi(s)dsdl\right|^2 \\ &+\frac{c_0}{r^2}\left|\int_{t-r}^{t}\int_{t}^{l} e^{k(s-t)}\dot\xi(s)dsdl\right|^2 \\ &+\frac{c_1}{r^2}\left|\int_{t-r}^{t}\int_{t}^{l} ke^{k(s-t)}\xi(s-h)dsdl\right|^2 \\ &+\frac{c_1}{r^2}\left|\int_{t-r}^{t}\int_{t}^{l} e^{k(s-t)}\dot\xi(s-h)dsdl\right|^2 + 2|\epsilon(t)|^2. \end{aligned} \tag{27}$$

with $c_0 = 8|BK_0|^2$ and $c_1 = 8|BK_1|^2$. Therefore, using (26) and (27) into (23) yields

$$\begin{aligned} \dot V(\xi_t) &\le -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 \\ &-\lambda_{min}(W_1)|\xi(t-h)|^2 \\ &-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\xi(t+\theta)|^2\ d\theta \\ &+\frac{c_0}{\delta}k^2r\int_{t-r}^{t} |\xi(s)|^2ds \\ &+\frac{c_0}{\delta}r\int_{t-r}^{t} |\dot\xi(s)|^2ds \\ &+\frac{c_1}{\delta}k^2r\int_{t-h-r}^{t-h} |\xi(s)|^2ds \\ &+\frac{c_1}{\delta}r\int_{t-h-r}^{t-h} |\dot\xi(s)|^2ds + \frac{2}{\delta}|\epsilon(t)|^2. \end{aligned} \tag{28}$$

Now, consider the functional

$$\begin{aligned} V_1(\xi_t) &= V(\xi_t) + \frac{c_0}{\delta}r\int_{t-r}^{t}\int_{m}^{t} |\dot\xi(s)|^2dsdm \\ &+\frac{c_1}{\delta}r\int_{t-h-r}^{t-h}\int_{m}^{t-h} |\dot\xi(s)|^2dsdm \\ &+\frac{c_0}{\delta}k^2r\int_{t-r}^{t}\int_{m}^{t} |\xi(s)|^2dsdm \\ &+\frac{c_1}{\delta}k^2r\int_{t-h-r}^{t-h}\int_{m}^{t-h} |\xi(s)|^2dsdm \\ &+\frac{c_1}{\delta}r^2\int_{t-h}^{t} |\dot\xi(m)|^2dm. \end{aligned} \tag{29}$$

Using (28), we get

$$\begin{aligned} &\dot V_1(\xi_t) \le \\ &-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2)|\xi(t)|^2 \\ &-(\lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2)|\xi(t-h)|^2 \\ &-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\xi(t+\theta)|^2d\theta \\ &+\frac{8}{\delta}|B|^2(|K_0|+|K_1|)r^2|\dot\xi(t)|^2 + \frac{2}{\delta}|\epsilon(t)|^2. \end{aligned} \tag{30}$$

By (13), we obtain

$$\begin{aligned} &\dot V_1(\xi_t) \le \\ &-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2)|\xi(t)|^2 \\ &-(\lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2)|\xi(t-h)|^2 \\ &-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\xi(t+\theta)|^2d\theta \\ &+\frac{8}{\delta}|B|^2(|K_0|+|K_1|)r^2\left|A_0\xi(t) + A_1\xi(t-h)\right. \\ &\left.+\frac{B}{r}\int_{t-r-\tau}^{t-\tau} e^{k(l-t+\tau)}\eta_s(l)dl + \epsilon(t)\right|^2 + \frac{2}{\delta}|\epsilon(t)|^2 \end{aligned} \tag{31}$$

with $q = 8|B|^2(|K_0|+|K_1|)$, hence

$$\begin{aligned} \dot V_1(\xi_t) &\le -\Big(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2 \\ &-\frac{4qr^2}{\delta}|A_0|^2\Big)|\xi(t)|^2 - \Big(\lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2 \\ &-\frac{4qr^2}{\delta}|A_1|^2\Big)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \\ &\int_{-h}^{0} |\xi(t+\theta)|^2d\theta + \frac{c_0qr}{\delta}\int_{t-r}^{t} |e^{k(l-t)}|^2|\xi(l)|^2dl \\ &+\frac{c_1qr}{\delta}\int_{t-h-r}^{t-h} |e^{k(l+h-t)}|^2|\xi(l)|^2dl \\ &+\frac{2}{\delta}(2qr^2+1)|\epsilon(t)|^2. \end{aligned} \tag{32}$$

Simple calculations give:

$$\begin{aligned}\dot{V}_1(\xi_t) \le& -(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2\\ &-\frac{4qr^2}{\delta}|A_0|^2)|\xi(t)|^2 - (\lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2\\ &-\frac{4qr^2}{\delta}|A_1|^2)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times\\ &\int_{-h}^{0}|\xi(t+\theta)|^2d\theta + \frac{c_0qr}{\delta}\int_{t-r}^{t}|\xi(l)|^2dl\\ &+\frac{c_1qr}{\delta}\int_{t-h-r}^{t-h}|\xi(l)|^2dl + \frac{2}{\delta}(2qr^2+1)|\epsilon(t)|^2.\end{aligned} \tag{33}$$

Let us now define

$$\begin{aligned}V_2(\xi_t) =& V_1(\xi_t) + \frac{c_0qr}{\delta}\int_{t-r}^{t}\int_{m}^{t}|\xi(l)|^2dldm\\ &+\frac{c_1qr}{\delta}\int_{t-h-r}^{t-h}\int_{m}^{t-h}|\xi(l)|^2dldm.\end{aligned} \tag{34}$$

Simple calculations give:

$$\begin{aligned}&\dot{V}_2(\xi_t) \le\\ &-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2 - \frac{4qr^2}{\delta}|A_0|^2\\ &-\frac{c_0qr^2}{\delta})|\xi(t)|^2\\ &-(\lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2 - \frac{4qr^2}{\delta}|A_1|^2\\ &-\frac{c_1qr^2}{\delta})|\xi(t-h)|^2\\ &-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0}|\xi(t+\theta)|^2d\theta\\ &+\frac{c_0qr}{\delta}\int_{t-r}^{t}|\xi(l)|^2dl + \frac{c_1qr}{\delta}\int_{t-h-r}^{t-h}|\xi(l)|^2dl\\ &+\frac{2}{\delta}(2qr^2+1)|\epsilon(t)|^2.\end{aligned} \tag{35}$$

Recall that W_0, W_1, W_2 and $U(\theta)$, $\theta \in [-h, 0]$ depend of W. Furthermore, α_0 and α_1 depend on $U(\theta)$ as well. Since W_0, W_1 and W_2 are symmetric positive definite matrices there always exist δ and r small enough so that the terms multiplied by δ and $\frac{r}{\delta}$ do not destroy the negativity of the factors of $|\xi(t)|^2$, $|\xi(t-h)|^2$ and $\int_{t-h}^{t}|\xi(\theta)|^2\, d\theta$ in the inequality (35). Hence, there exist δ and r such that,

$$\begin{aligned}a_1 =& \lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{c_0}{\delta}k^2r^2\\ &-\frac{4qr^2}{\delta}|A_0|^2 - \frac{c_0qr^2}{\delta} \ge 0,\end{aligned} \tag{36}$$

$$a_2 = \lambda_{min}(W_1) - \frac{c_1}{\delta}k^2r^2 - \frac{4qr^2}{\delta}|A_1|^2 - \frac{c_1qr^2}{\delta} \ge 0, \tag{37}$$

$$a_3 = \lambda_{min}(W_2) - 2\delta\alpha_1^2 \ge 0 \tag{38}$$

and we obtain the inequality

$$\begin{aligned}\dot{V}_2(\xi_t) \le& -a_1|\xi(t)|^2 - a_2|\xi(t-h)|^2\\ &-a_3\int_{-h}^{0}|\xi(t+\theta)|^2d\theta + \frac{(4qr^2+2)}{\delta}|\epsilon(t)|^2.\end{aligned} \tag{39}$$

Since $\epsilon(t) = B\bar{\eta}(t)$ and $\bar{\eta}(t)$ is exponentially stable, $\epsilon(t)$ converges exponentially to the origin, then we deduce that $\xi(t)$ also converges exponentially to the origin. This allows us to conclude.

4. ILLUSTRATIVE EXAMPLE

Consider the system

$$\begin{cases}\dot{\xi}(t) = 2\xi(t) + \xi(t-0.25) + \eta(t-0.5),\\ \dot{\eta}(t) = \eta(t) + u(t),\end{cases} \tag{40}$$

with $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$. When $W_0 = 1$, $W_1 = 0.02$ and $W_2 = 0.03$, the conditions (36)-(38) are satisfied for

$$\delta = 0.087 \quad \text{and} \quad r = 0.00072. \tag{41}$$

The integral term in the control law (8) satisfies the equality

$$\int_{t-r}^{t} e^{k(l-t)}\eta_s(l)\, dl = [w(t) - e^{-kr}w(t-r)], \tag{42}$$

where $w(t)$ is the solution of the differential equation

$$\dot{w}(t) = -kw(t) + \eta_s(t). \tag{43}$$

The control law (8) can be rewritten as

$$\begin{aligned}u(t) =& N\eta(t) - \frac{M+N+kI_n}{r}[w(t) - e^{-kr}w(t-r)]\\ &+\frac{1}{r}[\eta_s(t) - e^{-kr}\eta_s(t-r)] - f(t).\end{aligned} \tag{44}$$

One can observe that the expression obtained for the control law change a distributed delayed term for a pointwise one. Clearly, it is easier to implement.

The state variables $\eta(t)$, $\xi(t)$ of the system (40) in closed loop with the control law (44) are respectively depicted on Figures 1 and 2.

The control law (44) can be implemented for values of r larger than those obtained by the conditions (36)-(38) which are very conservative. For these values, we obtain a better performance, for example for $r = 0.06$ the simulation results of the state variables $\eta(t)$, $\xi(t)$ and the control law (44) are presented on Figures 3 and 4.

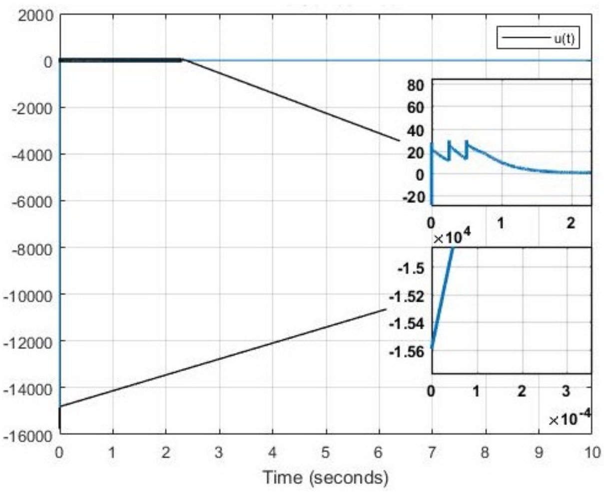

Fig. 1. Control law (44) with $N = -5$, $k = 0.1$ and $r = 0.00072$

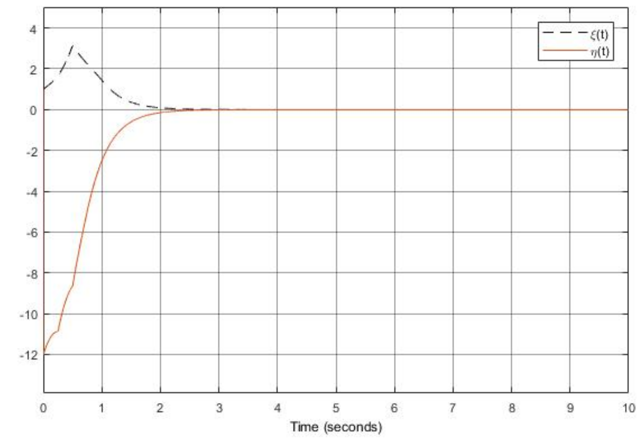

Fig. 2. States variables $\eta(t)$ and $\xi(t)$ using the control law (44) with $r = 0.00072$

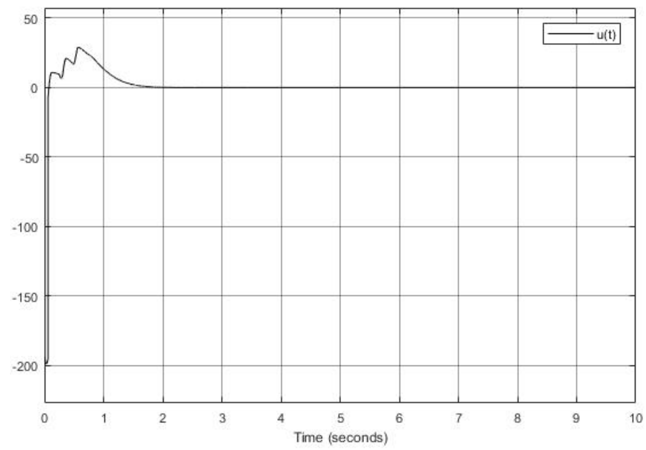

Fig. 3. Control law (44) with $N = -5$, $k = 0.1$ and $r = 0.06$

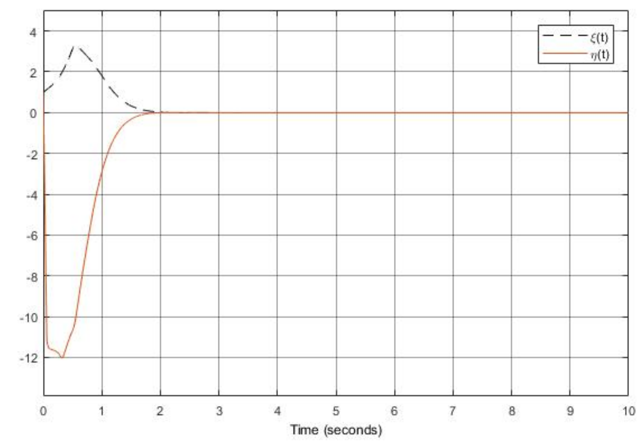

Fig. 4. States variables $\eta(t)$ and $\xi(t)$ using the control law (44) with $r = 0.06$

For comparison purposes, we can try out the computation for the classical backstepping approach for system (40). Following the usual procedure, we take $z(t) = \eta(t) - \eta_s(t)$ and we introduce

$$V_c(\xi_t, \eta_t) = V(\xi_t) + z^T(t-\tau) \cdot z(t-\tau) \tag{45}$$

as a Lyapunov functional candidate. We obtain the control law

$$\begin{aligned} u_c(t) = &-BU(0)\xi(t+\tau) \\ &-B\int_{-h}^{0} U(-h-l)H_1\xi(t+\tau+l)dl \\ &-kz(t) + K_0A_0\xi(t+\tau) + K_0A_1\xi(t+\tau-h) \\ &+K_0B\eta(t) + K_1A_0\xi(t+\tau-h) - M\eta(t) \\ &+K_1A_1\xi(t+\tau-2h) + K_1B\eta(t-h) - f(t) \end{aligned} \tag{46}$$

where $k > 0$.

Simulations results for system (40) in closed-loop with the control law (46) and $k = 10$ are shown below. The state variables $\eta(t)$, $\xi(t)$ and the control law (46) are respectively depicted on Figures 5 and 6.

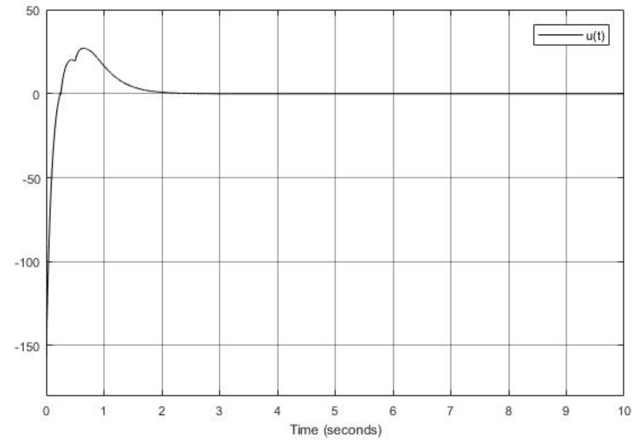

Fig. 5. Control law $u_c(t)$

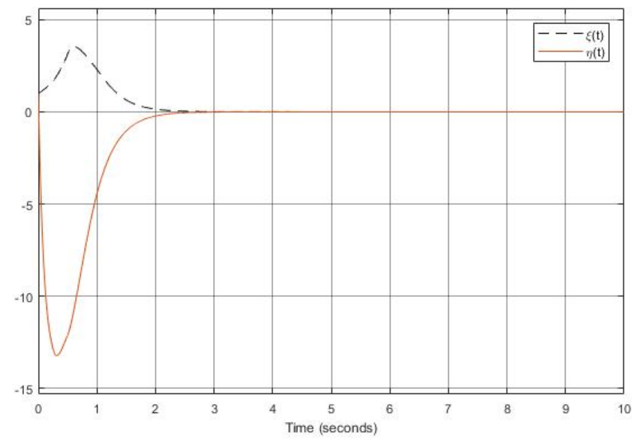

Fig. 6. States variables $\eta(t)$ and $\xi(t)$ using the control law (46)

The comparison of Figures 3,4 and 5,6, shows that the delay based control law that avoids integral terms in the control achieves a closed loop behavior similar to the backstepping classical approach.

5. CONCLUSIONS

In this paper we proposed a new backstepping control design based on the use of artificial delays for time invariant linear systems with two delays. A critical feature of this approach is that the approach is constructive and produces a strict Lyapunov-Krasovskii functional from which robustness properties can be derived. The presented example shows that the control law generates a satisfactory closed-loop response. Much remains to be done. We plan to extend our design to families of time-varying systems and to systems whose measurements are affected by disturbances.

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