Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Positive Real Properties and Physical Realizability Conditions For a Class of Linear Quantum Systems ⋆

Aline I. Maalouf, Ian R. Petersen *

* *The Research School of Engineering, The Australian National University, Canberra, ACT 2601(e-mail: Aline.Maalouf@anu.edu.au,i.r.petersen@gmail.com)*

Abstract: Theoretical developments in the field of quantum optics and quantum superconducting electrical circuits involving continuous measurement based feedback control as well as coherent control are an important prerequisites for advances in the domain of quantum technology. Within these perspectives, this paper considers positive real properties for a class of quantum systems whose quantum stochastic differential equation model involves annihilation operators only and then relates them to corresponding bounded real properties and consequently to physical realizability conditions developed earlier by the authors. Based on the positive real properties of these quantum systems, it is anticipated that it is possible to use the Brune algorithm in order to find an electrical circuit that can physically implement these quantum systems. This theory, in the case of one-port circuits, may be useful for the implementation of microwave circuits related to quantum filters found in the field of quantum computing.

Keywords: Quantum control, Positive Real Properties, Physical Realizability

1. INTRODUCTION

Quantum computing is a new and revolutionary technology that aims to store and process information based on the principles of quantum physics. The main advantage of quantum computing is that it can handle certain types of hard problems that classical computers cannot. However, the hardware implementation of quantum algorithms is still under development. More explicitly, the main idea behind quantum computing is to remove the binary bits of classical computing and use instead their corresponding quantum bits, which are referred to as qubits. A qubit is a quantum system for which it is possible to prepare and control its states with the possibility of implementing them in large numbers and then couple them to one another in a predefined manner.

Many technologies for building quantum computers have been proposed. However, one of the most successful ones involves super-conducting quantum circuit technology, operating at extremly low temperatures. This enables super-conducting qubits function in a way that allows the building of highly scaled complex quantum systems. These devices have indeed many advantages over other technologies. They can be fabricated by applying lithographic methods, and their physical properties can be managed at the design and implementation phases. In addition, their control is relatively simple, based mainly on high speed (GSample/sec) microwave signals that can be produced using electronics at the room temperature. Moreover, they can be cascaded with the entanglement necessary for quantum logic, by using microwave signals distributed via low-loss super-conducting transmission lines or resonant cavities Schoelkopf (2016).

Recent research has improved significantly the critical performance metrics for these qubits, especially the lifetime over which they can faithfully store a bit of quantum information Schoelkopf (2016). This allows for a sufficiently low error rate to enable scaling. Although these developments are considered to be important, other significant technical challenges remain, limiting the development of larger quantum computers Schoelkopf (2016). In fact, because the quantum information in these systems is stored by the presence or absence of a single quantum of electrical energy in a resonant circuit element, it is necessary to avoid many subtle types of dissipation that can cause decoherence Schoelkopf (2016).

For example, in quantum integrated circuits, it is necessary to take extreme caution in order to avoid any small level of radiation, cross-talk in circuits, and the influence of microscopic amounts of lossy materials or defects. It is important to mention here that in contrast to standard integrated circuits, quantum devices will achieve increasing complexity and scaling by developing robust methods for fabrication on a relatively large (millimeter to centimeter) scales rather than by extreme miniaturization Schoelkopf (2016).

Within this perspective, microwave circuits for quantum systems are therefore an area of research that needs to be developed to facilitate advances in super-conducting quantum processors. For example, a surface code has

⋆ This work was supported by the Australian Research Council under grant DP180101805 and the Airforce Office of Scientific Research under agreement number FA2386-16-14065.

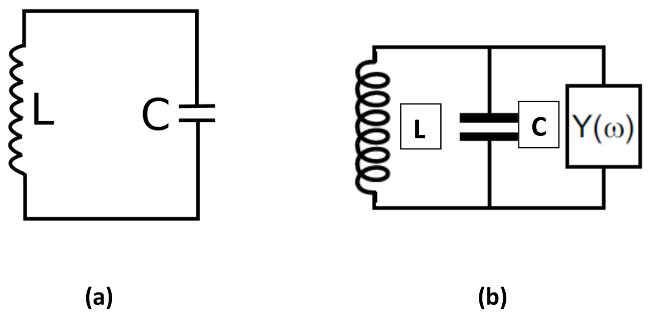

Fig. 1. (a) LC oscillator. (b) LC oscillator connected to an admittance $Y(\omega)$ in parallel with the circuit.

been designed in order to realize fault-tolerant quantum computers Kitaev (2003). This type of surface code is an error correction code with a threshold error value around 1%. This is achievable for locally coupled qubits. However, the configuration of the surface code is not an easy task. This has been reported to be best achieved by using high Q-factor resonators in addition to other components: e.g., see DiVincenzo (2009). One attempt in that direction has been mentioned in Houck et al. (2008) where microwave filters have been used in the quantum processors in order to control the spontaneous emission rates of qubits. Within these perspectives, some other research activities have been carried by Bronn et al. (2015), Sete et al. (2015) which have led to a new design of a certain type of filters namely, 'Bucy filters'. These types of filters are capable of maintaining a measurable coupling of qubit resonators while suppressing their emissions spontaneously. These results were crucial in the development of multi-mode structures for microwave circuits. In fact, these modes were used to engineer the qubit interactions along with filters designed in order to achieve high-fidelity of two-qubits gates McKay et al. (2015).

In this paper, we use a Hamiltonian framework, which provides a precise path to go from the classical to the quantum description of a given system Vool and Devoret (2017). It is shown how it can be applied to electrical circuits by means of the positive real properties of the class of quantum systems under consideration Vool and Devoret (2017). Although it is straightforward to apply the Hamiltonian framework to the LC oscillator of Figure 1, it is much more tedious to do so for more complicated circuits Vool and Devoret (2017).

This framework allows for the establishment of a theoretical relationship between microwave circuits as discussed earlier with the positive real properties of the class of annihilation operator only systems developed by the authors. This will consequently result in a corresponding physical implementation of the class of annihilation operator only systems in terms of classical electrical circuits leading to the construction of microwave circuits.

2. STATE-SPACE DESCRIPTION OF A CLASS OF ANNIHILATION OPERATOR LINEAR QUANTUM SYSTEMS

In the physics community, many models have been proposed in order to mathematically represent linear quantum systems, such as those represented by using linear quantum differential equations Gardiner and Collett (1985), Gardiner and Zoller (2000), Hudson and Parthasarathy (1984). These types of systems are found in the domain of quantum optics. The quantum noise in these types of systems is used to model the effect of boson fields as well as heat baths along with optical and phonon fields. The models are obtained by taking expectations in order to get a master equation and then using specific Lindblad generators along with some positive maps in order to complete the models James (2005). By that means, linear and nonlinear quantum differential equations are derived. In that case, many systems in quantum optics can be defined by means of differential equations; for instance, see Walls and Milburn (2008), Gardiner and Zoller (2000) and Bachor and Ralph (2004).

An important type of quantum system can be described by using creation and annihilation operators in the Heisenberg picture involving harmonic oscillators that are coupled to optical fields; for instance, see Wiseman and Milburn (2010), Walls and Milburn (2008) and Gardiner and Zoller (2000). A specific type of quantum systems are governed by Wiener fields (e.g.; see James et al. (2008)). In that case, the question of whether the quantum system in question can be represented by a quantum harmonic oscillator is related to the physical realizability conditions that were developed in James et al. (2008). Moreover, in Maalouf and Petersen (2009), Maalouf and Petersen (2011b) and Maalouf and Petersen (2011c), the lossless bounded real property of annihilation operator quantum systems has been connected to the physical realizability of these types of systems. The annihilation operator quantum systems considered in this paper can be represented by using quantum probability theory Bouten et al. (2007) as in Maalouf and Petersen (2011b) and Maalouf and Petersen (2011a). In that case, the quantum differential equations (QSDEs) describing the systems under consideration are of the form

$$\begin{aligned} da(t) &= Fa(t)dt + Gdw(t); \qquad a(0) = a_0 \\ dy(t) &= Ha(t)dt + Jdw(t) \end{aligned} \tag{1}$$

where $J \in \mathcal{C}^{n_y \times n_w}$, $H \in \mathcal{C}^{n_y \times n}$, $F \in \mathcal{C}^{n \times n}$, $G \in \mathcal{C}^{n \times n_w}$. Also, n_y, n_w, n are positive integers.

In addition, the vector of annihilation operators $a(t)$ is $a(t) = [a_1(t) \cdots a_n(t)]^T$. In that case, w represents the input fields and has the following partition:

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t). \tag{2}$$

Here, $\beta_w(t)$ and $w(t)$ are a self-adjoint adapted vector and the quantum noise signal respectively (Please refer to Bouten et al. (2007), K.R.Parthasarathy (1992) and Hudson and Parthasarathy (1984)). The Ito table of the quantum noise $\tilde{w}(t)$ is

$$d\tilde{w}(t)d\tilde{w}^{\dagger}(t) = F_{\tilde{w}}dt \tag{3}$$

(see Belavkin (1992) and K.R.Parthasarathy (1992)) where $F_{\tilde{w}}$ is a Hermitian positive definite matrix. Here, the notation † represents the adjoint transpose vector of operators. Also, the noise components satisfy the following commutation relations:

$$[d\tilde{w}(t), d\tilde{w}^*(t)] = d\tilde{w}(t)d\tilde{w}^\dagger(t) - (d\tilde{w}^*(t)d\tilde{w}^T(t))^T = T_w dt \quad (4)$$

Here, T_w is a Hermitian complex matrix. The signals involving noises are operators on a Fock space (e.g; see Belavkin (1992) and K.R.Parthasarathy (1992)). The process $\beta_w(t)$ represents the variables of fields interacting with the system (1). Hence, $\beta_w(0)$ should be an operator on a Hilbert space that is different from that of a_0 and the noises. The assumption is made that $\beta_w(t)$ and $a(t)$ commute with each other for any $t \geq 0$. In addition, being an adapted field, $\beta_w(t)$ and $d\tilde{w}(t)$ commute together for all $t \geq 0$. The following assumption is made on the system (1): $n_w = n_y$. Equation (1) is an annihilation operator quantum differential equation where the integration is considered to be quantum Ito integration with respect to $dw(t)$. Note that $a(t)$ is adapted, and the commutator of $d\tilde{w}(t)$ with $a(t)$ is zero. If $\beta_w(t)$ represents the currents and $y(t)$ represent the output voltages of the quantum network in question then $n_w = n_y$ and the resulting impedance transfer function is:

$$Z_c(s) = J + H(sI - F)^{-1}G. \quad (5)$$

It is important to mention here that complex realizations are considered such that the matrices J, H, G, F are all complex.

3. POSITIVE-REAL PROPERTIES FOR ANNIHILATION OPERATOR QUANTUM SYSTEMS

An important question arises as to whether a given impedance matrix $Z(s)$ is physically realizable. For the case of a one-port network, Brune in Brune (1931) showed that if $Z(s)$ satisfies some Positive Real (PR) conditions, then it is possible to find a physical circuit having an impedance of $Z(s)$ across its terminals .

In an independent approach, the authors developed PR conditions that correspond to the quantum system given by (1). These PR conditions were then related to the Bounded Real conditions developed in Maalouf and Petersen (2011a). Then in a related paper, the authors used the Brune algorithm proposed in Brune (1931) in order to find an electric circuit equivalent to the system given in (1), which is crucial in the development of microwave circuits for such quantum systems. By that means, in this paper, the PR conditions developed for the system in (1) are illustrated along with their relationship to the bounded real properties for the system in question and how they relate to the physical realizability conditions proposed in Maalouf and Petersen (2011a). Then, the authors showed how these PR conditions relate to the Brune algorithm that is used to construct an equivalent electrical circuit made up of resistors, inductors, capacitors and transformers that describes the annihilation operator quantum system as given by (1).

3.1 Positive Real Lemma

Theorem 1. Anderson and Moore (2007) Let $Z_c(.)$ be an $n_y \times n_y$ rational function matrix of the complex variable s, where $Z_c(\infty) < \infty$. Also, assume that $Z_c(s)$ has a minimal realization $\{J, H, G, F\}$. Then the existence of a P_a (Hermitian positive definite matrix) along with matrices L and W_a satisfying

$$P_aF + F^\dagger P_a = -LL^\dagger; \quad P_aG = H - LW_a; \quad W_a^\dagger W_a = J + J^\dagger \quad (6)$$

is equivalent to $Z_c(s)$ being positive real.

3.2 Positive Real Lemma in the Lossless Case

For the case when $Z_c(s)$ is lossless, the matrices L and W_a become zero. The corresponding Positive Real Lemma is referred to as the Lossless Positive Real Lemma as follows:

Theorem 2. Anderson and Moore (2007) Let $Z_c(.)$ be an $n_y \times n_y$ rational function matrix of the complex variable s, with $Z_c(\infty) < \infty$. Also, assume that $Z_c(s)$ has a minimal realization $\{J, H, G, F\}$. Then $Z_c(s)$ being lossless positive real is equivalent to the existence of a positive definite Hermitian matrix P_a such that

$$P_aF + F^\dagger P_a = 0; \quad P_aG = H; \quad J + J^\dagger = 0. \quad (7)$$

4. PHYSICAL REALIZABILITY AND THE LOSSLESS POSITIVE REAL PROPERTY

4.1 The relationship between (S,L,H) properties and QSDEs

The type of annihilation operator quantum systems considered in this paper of the form (1), could be also described in terms of a Hamiltonian operator, a coupling operator and a scattering matrix. Within this perspective, the type of quantum systems considered in this paper can be represented by G, $\mathcal{A}_G$ (variable space) and $\mathcal{H}_G$ (Hilbert space). The Hamiltonian $\mathcal{H} \in \mathcal{A}_G$ describes the energy of G. The quantum stochastic process W is composed of m field channels which drive the system (1)

$$W = \begin{pmatrix} W_1 \\ \vdots \\ W_m \end{pmatrix}. \quad (8)$$

The process W has the following second order Ito products:

$$dW_i(t)dW_j(t)^* = \delta_{ij}dt; \quad dW_i(t)^*dW_j(t) = 0; \quad dW_i(t)dW_j(t) = 0; \quad dW_i(t)^*dW_j(t)^* = 0$$

where $W_i(t)^*$ is the adjoint of $W_j(t)$ defined on a Fock space. This means that the process W is canonical. The scattering matrix couples the different fields together. In most cases, we let $S = I$. On the other hand, coupling between the fields and the system is represented by means of the coupling operators L as follows

$$L = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix} \quad (9)$$

where $L_j \in \mathcal{A}_G$. In the notation $G = (S, L, H)$, S, L are the scattering matrix and the coupling operator respectively whereas the system's Hamiltonian is given by H. The corresponding Schrodinger equation with $U_a(0) = I$ and $S = I$ is as follows:

$$dU_a(t) = \left\{ dW^\dagger L - L^\dagger dW - \frac{1}{2} L^\dagger L dt - iH dt \right\} U_a(t). \quad (10)$$

Equation (10) specifies the motion of the system in question evolving unitarily with respect the principles of quantum mechanics. In that case, the notation † refers to the Hilbert space adjoint. Given an operator $a_i \in \mathcal{A}_G$, its Heisenberg evolution is given by $a_i(t) = j_t(a_i) = U_a(t)^* a_i U_a(t)$ and satisfies

$$\begin{aligned} da_i(t) = &(\mathcal{L}_L(t)(a_i(t)) - i\,[a_i(t), H(t)]) \quad (11) \\ &+ \left[L(t)^\dagger, a_i(t)\right] dW(t). \end{aligned}$$

Hence, $[A, B]$ refers to the commutator of two operators. In equation (11),

$$\mathcal{L}_L(a_i) = \frac{1}{2} L^\dagger [a_i, L] + \frac{1}{2} \left[L^\dagger, a_i\right] L. \quad (12)$$

In this paper, $L(t)$ does not depend on creation operators $a_i(t)^*$ but depends only on the annihilation operators $a_i(t)$. Therefore, $[a_i, L] = 0$ and then

$$\mathcal{L}_L(a_i) = \frac{1}{2} \left[L^\dagger, a_i\right] L. \quad (13)$$

The system's generator G is given by

$$\mathcal{G}_G(a_i) = -i\,[a_i, H] + \mathcal{L}_L(a_i). \quad (14)$$

When a is a vector of n annihilation operators $a_i, \dots, a_n$, we can write

$$a = \begin{pmatrix} a_i \\ \vdots \\ a_n \end{pmatrix}, \quad (15)$$

$$\mathcal{L}_{L_i}(a) = \frac{1}{2} \left[L^\dagger, a_i\right] L \quad (16)$$

and

$$\mathcal{G}_{G_i}(a) = -i\,[a_i, H] + \mathcal{L}_{L_i}(a). \quad (17)$$

Therefore, we can write

$$\begin{aligned} da_i(t) = &(\mathcal{L}_{L_i}(t)(a_i(t)) - i\,[a_i(t), H(t)]) \\ &+ \left[L(t)^\dagger, a_i(t)\right] dW(t) \end{aligned}$$

and

$$\begin{aligned} da(t) = &(\mathcal{L}_L(t)(a(t)) - i\,[a(t), H(t)]) \\ &+ \left[L(t)^\dagger, a(t)\right] dW(t) \end{aligned}$$

where

$$\mathcal{L}_L(a) = \begin{pmatrix} \mathcal{L}_{L_1}(a) \\ \vdots \\ \mathcal{L}_{L_n}(a) \end{pmatrix} \quad (18)$$

and

$$[a, H] = \begin{pmatrix} [a_1, H] \\ \vdots \\ [a_n, H] \end{pmatrix}. \quad (19)$$

Let

$$\begin{aligned} A(a(t), a(t)^\dagger) &= \begin{pmatrix} A_1(a(t), a(t)^\dagger) \\ \vdots \\ A_n(a(t), a(t)^\dagger) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{L_1}(a) \\ \vdots \\ \mathcal{L}_{L_n}(a) \end{pmatrix} - i \begin{pmatrix} [a_1, H] \\ \vdots \\ [a_n, H] \end{pmatrix} \\ &= \mathcal{L}_L(a) - i\,[a, H] \\ &= \mathcal{G}_G(a) \end{aligned} \quad (20)$$

and

$$\begin{aligned} B(a(t), a(t)^\dagger) &= \begin{pmatrix} B_1(a(t), a(t)^\dagger) \\ \vdots \\ B_n(a(t), a(t)^\dagger) \end{pmatrix} \quad (21) \\ &= \begin{pmatrix} [L^\dagger, a_1] \\ \vdots \\ [L^\dagger, a_n] \end{pmatrix} \\ &= \left[L^\dagger, a\right]. \quad (22) \end{aligned}$$

Hence,

$$\begin{aligned} da(t) &= A(a(t), a(t)^\dagger)dt + B(a(t), a(t)^\dagger)dW(t) \\ da(t)^* &= A(a(t), a(t)^\dagger)^* dt + B(a(t), a(t)^\dagger)^* dW(t)^*. \end{aligned}$$

The notation * and † are the complex conjugate and the complex conjugate transpose respectively. Also, $y(t) = j_t(W(t)) = U_a(t)^* W(t) U_a(t)$ with

$$\begin{aligned} dy(t) &= C(a(t))dt + D(t)dW(t) \\ dy(t)^* &= C(a(t))^* dt + D(t)^* dW(t)^* \end{aligned}$$

where $D(t) = I$ and $C(a(t)) = L(t)$. Hence, the following (QSDEs) are a description of the system G

$$\begin{aligned} d\bar{a}(t) &= \bar{A}(a(t), a(t)^\dagger)dt + \bar{B}(a(t), a(t)^\dagger)d\bar{W}(t); \\ d\bar{y}(t) &= \bar{C}(a(t), a(t)^\dagger)dt + \bar{D}(a(t), a(t)^\dagger)d\bar{W}(t) \end{aligned} \quad (23)$$

where $\bar{a}(t) = \begin{bmatrix} a \\ a^* \end{bmatrix}$, $\bar{A}(a, a^\dagger) = \begin{bmatrix} A(a, a^\dagger) \\ A(a, a^\dagger)^* \end{bmatrix}$, $\bar{C}(a, a^\dagger) = \begin{bmatrix} C(a) \\ C(a)^* \end{bmatrix}$, $\bar{B}(a, a^\dagger) = \begin{bmatrix} B(a, a^\dagger) & 0 \\ 0 & B(a, a^\dagger)^* \end{bmatrix}$, $\bar{D}(t) = \begin{bmatrix} D(t) & 0 \\ 0 & D(t)^* \end{bmatrix}$ and $d\bar{W}(t) = \begin{bmatrix} dW(t) \\ dW(t)^* \end{bmatrix}$.

Since in this paper, we are mainly interested in annihilation operators systems only, the system in (23) is described by

$$\begin{aligned} da(t) &= A(a(t), a(t)^\dagger)dt + B(a(t), a(t)^\dagger)dW(t); \\ dy(t) &= C(a(t), a(t))dt + D(a(t), a(t)^\dagger)dW(t) \end{aligned} \quad (24)$$

where

$$\begin{aligned} A(a(t), a(t)^\dagger) &= \mathcal{L}_L(a) - i\,[a, H] = \mathcal{L}_G(a); \\ B(a(t), a(t)^\dagger) &= \left[L^\dagger, a\right] S; \\ C(a) &= L; \\ D(t) &= S; \end{aligned}$$

In that case,

$$\mathcal{L}_L(a) = \frac{1}{2}L^\dagger [a, L] + \frac{1}{2}\left[L^\dagger, a\right] L. \quad (25)$$

For the case when $L = \Lambda a$, $H = a^\dagger M a$, (Λ is $n_w \times n$ matrix and M is a $n \times n$ Hermitian matrix) (see Maalouf and Petersen (2011b), Maalouf and Petersen (2011a) for instance), we can write:

$$\begin{aligned}\mathcal{L}_L(a) &= \frac{1}{2}\left[a^\dagger\Lambda^\dagger, a\right]\Lambda a;\\ &= -\frac{1}{2}\Lambda^\dagger\Theta_a\Lambda a. \quad (26)\end{aligned}$$

Also,

$$\begin{aligned}[a, H] &= \left[a, a^\dagger M a\right];\\ &= \left[a, a^\dagger\right] M a;\\ &= \Theta_a M a \quad (27)\end{aligned}$$

Hence,

$$\begin{aligned}A(a, a^\dagger) &= A(a);\\ &= -\frac{1}{2}\Lambda^\dagger\Theta_a\Lambda a - i\Theta_a M a;\\ &= \left[-\frac{1}{2}\Lambda^\dagger\Theta_a\Lambda a - i\Theta_a M\right] a. \quad (28)\end{aligned}$$

Since Λ, M are constant matrices, then, we can write $A(a) = Fa$ where $F = -\frac{1}{2}\Lambda^\dagger\Theta_a\Lambda - i\Theta_a M$. Also,

$$\begin{aligned}B(a, a^\dagger) = \left[L^\dagger, a\right] &= \left[a^\dagger\Lambda^\dagger, a\right] S\\ &= -\Theta_a\Lambda^\dagger S = G. \quad (29)\end{aligned}$$

In addition,

$$\begin{aligned}C(a) = L &= \Lambda a;\\ &= Ha = G \quad (30)\end{aligned}$$

with $H = \Lambda$. Also,

$$D(t) = S. \quad (31)$$

Therefore, the annihilation operator quantum system given by (1) is given by:

$$\begin{aligned}da(t) &= Fa(t)dt + Gdw(t); \qquad a(0) = a_0\\ dy(t) &= Ha(t)dt + Jdw(t)\end{aligned}$$

where

$$\begin{aligned}F &= -\frac{1}{2}\Lambda^\dagger\Theta_a\Lambda - i\Theta_a M;\\ G &= -\Theta_a\Lambda^\dagger S;\\ H &= \Lambda;\\ J &= D = S = I. \quad (32)\end{aligned}$$

4.2 Transfer Function for a Quantum System Satisfying The Physically Realizability Condition

As mentioned earlier, the transfer function for (1) is given by:

$$Z_c(s) = J + H(sI - F)^{-1}G. \quad (33)$$

When the system (1) is physically realizable then conditions (32) are satisfied with

$$\begin{aligned}L &= \Lambda a \text{ (L is the coupling operator)};\\ H &= a^\dagger M a; \text{(H is the Hamiltonian)};\\ S &= I\text{(scattering matrix)} \quad (34)\end{aligned}$$

Therefore, the quantum system's transfer function is as follows:

$$Z_a(s) = S - \Lambda\left(sI + \frac{1}{2}\Lambda^\dagger\Theta_a\Lambda + i\Theta_a M\right)^{-1}\Theta_a\Lambda^\dagger S. \quad (35)$$

4.3 Transfer Function of a Bounded Real Annihilation Operator Linear Quantum System

The following theorem provides the form of the transfer function of a bounded real annihilation operator linear quantum system.

Proposition 3. A transfer function matrix $Z_a(s)$ corresponding to a lossless bounded real quantum system (S, L, H) of the form (1) can be represented in the following fractional form:

$$Z_a(s) = \left(\Theta_a^{-1} - \Sigma_a(s)\right)\left(\Theta_a^{-1} + \Sigma_a(s)\right)^{-1} S \quad (36)$$

where

$$\Sigma_a(s) = \frac{1}{2}\Lambda(sI + i\Theta_a M)^{-1}\Lambda^\dagger. \quad (37)$$

Remark: In this paper, we will be interested in the question of when the transfer function (37) is physically realizable.

4.4 Lossless Positive Real versus Lossless Bounded Real

The following theorem provides a relationship between a lossless bounded real and a lossless positive real transfer function for a minimal annihilation operator quantum system (1).

Theorem 4. Suppose that the quantum system (1) represented by (S, L, H) is minimal. Then the following hold:

(1) Suppose $\Sigma_a(s) = J_{\Sigma_a} + H_{\Sigma_a}(sI - F_{\Sigma_a})^{-1}G_{\Sigma_a}$ is lossless positive real with $(J_{\Sigma_a} + J_{\Sigma_a}^\dagger)$ nonsingular. Then, the transformed transfer function: $Z_a(s) = \mathcal{M}(\Sigma_a)(s) = (\Theta_a^{-1} - \Sigma_a(s))(\Theta_a^{-1} + \Sigma_a(s))^{-1}S$ is lossless bounded real.
(2) Suppose $Z_a(s)$ is a lossless bounded real transfer function $Z_a(s) = J_{z_a} + H_{z_a}(sI - F_{z_a})^{-1}G_{z_a}$ with $det(I + Z_a(s)) \neq 0$ and $(J_{z_a} + J_{z_a}^\dagger)$ nonsingular. Then the transformed function: $\Sigma_a(s) = \mathcal{M}(Z_a(s))(s) = (\Theta_a^{-1} - Z_a(s))(\Theta_a^{-1} + Z_a(s))^{-1}$ is lossless positive real.

A Relationship Between Physical Realizability And Lossless Positive Real Properties The following theorem illustrates a relationship between the physical realizability condition for quantum systems involving annihilation operators only and the lossless positive real property.

Theorem 5. Suppose that the quantum system (1) presented by (S, L, H) is minimal. Then, the system (1) is physically realizable if and only if the transfer function matrix system having $\Sigma_a(s)$ defined in (37) is lossless positive real and $J = I$.

Theorem 5 follows from Theorem 4, Theorem 3 and Theorem 6.6 in Maalouf and Petersen (2011b).

Remark: Note that by using the bounded real properties of the impedance $Z_a(s)$ as defined in (35), it is possible to implement physically the linear quantum system (1) by means of quantum optical cavities. Whereas, by using the positive real properties of the impedance $\Sigma_a(s)$, it is possible to implement physically the linear quantum system (1) by means of electrical circuits using the Brune algorithm (to be published in a separate paper). Both $Z_a(s)$ and $\Sigma_a(s)$ are related by means of a linear transformation as given by Theorem 4. This justifies the contribution of this paper in the implementation of microwave circuits involving quantum filters that are used in the field of quantum computing.

5. CONCLUSION

In this paper, the positive real properties (lossless positive real) for a class of annihilation operator linear quantum systems have been developed and related to the physical realizability conditions developed earlier by the authors. This relationship is fundamental in order to be able to physically implement this class of quantum systems by means of superconducting microwave circuits.

REFERENCES

Anderson, B.D.O. and Moore, J.B. (2007). *Optimal Control: Linear Quadratic Methods*. Dover Publications, New York, USA.

Bachor, H.A. and Ralph, T.C. (2004). *A Guide to Experiments in Quantum Optics*. Wiley-VCH, Weinheim, Germany.

Belavkin, V.P. (1992). Quantum continual measurements and a posteriori collapse on CCR. *Commun. Math. Phys*, 146, 611–635.

Bouten, L., van Handel, R., and James, M.R. (2007). An introduction to quantum filtering. *SIAM Journal of Control and Optimization*, 46, 2199–2241.

Bronn, N.T., Magesan, E., Masluk, N.A., Chow, J.M., Gambetta, J.M., and Steffen, M. (2015). Reducing spontaneous emission in circuit quantum electrodynamics by a combined readout/filter technique. *IEEE Transactions on Applied Superconductivity*, 25.

Brune, O. (1931). Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency. *Doctoral thesis, MIT*.

DiVincenzo, D.P. (2009). Fault-tolerant architectures for superconducting qubits. *Phys. Scr.*, T137, 014020.

Gardiner, C.W. and Collett, M.J. (1985). Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Phys. Rev. A*, 31, 3761–3774.

Gardiner, C.W. and Zoller, P. (2000). *Quantum Noise*. Springer-Verlag, Berlin, third edition.

Houck, A.A., Schreier, J.A., Johnson, B.R., Chow, J.M., Koch, J., Gambetta, J.M., Schuster, D.I., Frunzio, L., Devoret, M.H., Girvin, S.M., and Schoelkopf, R.J. (2008). Controlling the spontaneous emission of a superconducting transmon qubit. *Phys. Rev. Lett.*, 101, 080502.

Hudson, R.I. and Parthasarathy, K.R. (1984). Quantum Ito's formula and stochastic evolution. *Commun. Math. Phys.*, 93, 301–323.

James, M.R. (2005). Control theory: From classical to quantum optimal, stochastic, and robust control. *The Australian National University, Lecture Notes Collection, FreeScience.info ID2207, http://www.freescience.info/go.php?pagename=books id=2207.*

James, M.R., Nurdin, H.I., and Petersen, I.R. (2008). H^∞ control of linear quantum stochastic systems. *IEEE Trans. Automat. Contr.*, 53, 1787–1803.

Kitaev, A.Y. (2003). Fault-tolerant quantum computation by anyons. *Ann. Phys.*, 303, 2–30.

K.R.Parthasarathy (1992). *A introduction to quantum stochastic calculus*. Birkhauser, Berlin.

Maalouf, A.I. and Petersen, I.R. (2011a). Bounded real properties for a class of annihilation-operator linear quantum systems. *IEEE Transactions on Automatic Control*, 56(4), 786–801.

Maalouf, A.I. and Petersen, I.R. (2011b). Coherent H^∞ control for a class of annihilation operator linear quantum systems. *IEEE Transactions on Automatic Control*, 56(2), 309–319.

Maalouf, A.I. and Petersen, I.R. (2011c). Finite horizon H^∞ control for a class of linear quantum measurement delayed systems: A dynamic game approach. *Proceedings of the American Control Conference (ACC '11)*, 4340–4347.

Maalouf, A.I. and Petersen, I.R. (2009). LQG control for a class of linear quantum systems. *Proceedings of the European Control Conference ECC.*

McKay, D.C., Naik, R., Reinhold, P., Bishop, L.S., and Schuster, D.I. (2015). High contrast qubit interactions using multimode cavity qed. *Phys. Rev. Lett.*, 114, 080501.

Schoelkopf, R. (2016). Quantum computing with superconducting circuits. *in Proceedings of the 2016 IEEE International Interconnect Technology Conference / Advanced Metallization Conference (IITC/AMC)*, 43–44.

Sete, E.A., Martinis, J.M., and Korotkov, A.N. (2015). Quantum theory of a bandpass purcell filter for qubit readout. *Physical Review A*, 92.

Vool, U. and Devoret, M. (2017). Introduction to quantum electromagnetic circuits. *International Journal of Circuit Theory and Applications*, 45(7), 897–934.

Walls, D. and Milburn, G.J. (2008). *Quantum Optics*. Springer-Verlag, Berlin.

Wiseman, H.M. and Milburn, G.J. (2010). *Quantum Measurement and Control*. Cambridge University Press, Cambridge, United Kingdom.