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Finite-horizon Anisotropy-based Estimation with Packet Dropouts ⋆

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Abstract: In this paper, the estimation problem is studied for a class of linear discrete time-varying system with packet dropout in the framework of anisotropy-based theory. The extended vector of fragment of the disturbance sequence is from the set of random vectors with bounded anisotropy. The packet dropout effect is considered to be random and described by a binary switching sequence with Bernoulli distribution. The input-to-error dynamics is obtained for multiplicative noise system with mutually independent noises and input disturbance. By using anisotropy-based approach, the estimation problem is reduced to optimization one with convex constraints. The developed method provides the (sub)optimal estimator ensuring the boundedness of anisotropic norm for input-to-output error system. Numerical example is provided to demonstrate efficiency of proposed approach.

Keywords: dropouts, estimation and filtering, multiplicative noise systems, time-varying systems, anisotropy.

1. INTRODUCTION

Widely used $\mathcal{H}_2$ and $\mathcal{H}_\infty$ approaches to disturbance rejection is based on fulfilment some assumptions for stochastic properties of disturbances. From this, $\mathcal{H}_2$ estimation provides the best performance only for certain class of input disturbances (Gaussian disturbances with zero mean and given covariance matrix), but doesn't guarantee robustness with respect to changes in input distribution. And $\mathcal{H}_\infty$ estimation as well as $\mathcal{H}_\infty$ theory contains excessive conservatism since it operates with the so called the worst case disturbance, so that's a pure idealization. The more real case of disturbance is connected with disturbance having uncertainty in stochastic properties. The set of possible (or valid) disturbances are usually chosen within some class defined by certain properties. In anisotropy-based theory, such property is related to boundedness of anisotropy of a random vector. The definition was introduced by Igor Vladimirov in Vladimirov et al. (1995), and it is based on concepts of information and probability theory (its deepest meaning, however, is far beyond these two theories). The anisotropy of a random vector quantitatively measures the difference between the disturbance of one and of Gaussian distributed vector. The advantage of using the anisotropy appears when one deals with uncertain disturbance, and has, for example, no tools for specify it. As well as $\mathcal{H}_\infty$ theory, anisotropy-based theory implies the induced norm as a cost function – the anisotropic norm. The remarkable property of this consists of achieving the values of scaled $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms as limiting cases when specific parameter goes to zero and infinity, correspondingly. In the first researches within anisotropy-based theory, Igor Vladimirov and co-authors set out many problems: analysis of linear discrete time-invariant and time-varying systems Vladimirov et al. (1996a, 2001), control design including both time invariant and time varying cases Vladimirov et al. (1996b); Tchaikovsky et al. (2017), and others. Some problems were considered and solved for non-centered disturbances Kustov et al. (2017); Yurchekov (2018a). All existing methods and approaches of anisotropy-based theory were applied to linear discrete *nonrandom* systems. In contrary, multiplicative noise systems description differs from standard linear systems description in known way, hence some modification of anisotropic norm calculation has to appear. Anisotropy-based analysis for time-varying multiplicative noise systems was considered in Belov et al. (2019), where boundedness condition of anisotropic norm was derived. This result has important practical application. For sensor networks, the estimation problem in finite time horizon can be solved in terms of convex optimization problem. Although the first attempt of control design was considered in Yurchekov (2018b), the precise computation of anisotropic norm for time-varying multiplicative noise systems was described in Belov et al. (2019). The important contribution of anisotropy-based theory is applying the information criterion of disturbances. For this reason, anisotropy-based estimation provides better performance in comparison with $\mathcal{H}_\infty$, and less conservatism than $\mathcal{H}_2$ estimation.

2. PRELIMINARIES

Let us consider the Hilbert space $\mathbb{L}_2^m$ and appropriate inner product $\langle x, y\rangle = \mathbf{E}[x^\top y]$ for arbitrary vectors x and $y \in \mathbb{L}_2^m$, where $\mathbf{E}[\cdot]$ denotes the expectation. Denote by $\|x\| = \sqrt{\langle x, x\rangle}$ the norm of m-dimensional random vector x. For any real-valued random $p\times m$-matrices X, Y from the Hilbert space $\mathbb{L}_2^{p\times m}$, define the inner product as follows: $\langle X, Y\rangle = \mathrm{tr}\left(\mathbf{E}[X^\top Y]\right)$, where $\mathrm{tr}(\cdot)$ stands for the

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trace of the square matrix. Likewise, the norm in $\mathbb{L}_2^{p\times m}$ be as follows: $\|X\| = \sqrt{\langle X, X\rangle}$.

For any random matrix $F \in \mathbb{L}_2^{p\times m}$ and some input vector W from Hilbert space $\mathbb{L}_2^m$ let us define the root-mean square (RMS) gain as $\mathbf{Q}(F, W) = \|FW\|/\|W\|$, where $FW \in \mathbb{L}_2^p$. When F and W are considered to be mutually independent, one can rewrite the RMS gain in the following manner:

$$\mathbf{Q}(F, W) = \sqrt{\frac{\mathrm{tr}(\Lambda\Sigma)}{\mathrm{tr}(\Sigma)}}, \quad (1)$$

where $\Lambda = \mathbf{E}[F^\top F] \in \mathbb{R}^{m\times m}$, $\Sigma = \mathbf{E}[WW^\top] \in \mathbb{R}^{m\times m}$.

The upper bound of (1) is associated with stochastic version of the maximum singular value of F, i.e.

$$\max_{W\in\mathbb{L}_2^m} \mathbf{Q}(F, W) = \|F\|_\infty,$$

where stochastic norm $\|F\|_\infty$ is equal to square root of maximum eigenvalue of $\mathbf{E}[F^\top F]$. On the other hand, when vector W is such that $|W|^{-1}W$ has uniform distribution on unit $(m-1)$-dimensional sphere $\mathbb{S}_m = \{x \in \mathbb{R}^m : |x| = 1\}$ (for example, if W is Gaussian with zero mean and scalar covariance matrix), the RMS gain becomes equal to $\|F\|_2/\sqrt{m}$, where $\|F\|_2 = \sqrt{\mathrm{tr}\Lambda}$.

Initially, the definition of anisotropy for a random vector $W \in \mathbb{L}_2^m$ was suggested in Vladimirov et al. (1995) but later modified in Vladimirov et al. (2001) as follows:

$$\mathbf{A}(W) = \min_{\lambda>0} \mathbf{D}(f\|p_{m,\lambda}) = \frac{m}{2}\ln\left(\frac{2\pi e}{m}\|W\|^2\right) - \mathbf{h}(W)$$

where $\mathbf{h}(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw$ presents the differential entropy of W. Here, $f(w)$ is the probability density function (pdf) of W, $p_{m,\lambda}(w)$ is the pdf of the Gaussian m-dimensional random vector of the form

$$p_{m,\lambda}(w) = (2\pi\lambda)^{-m/2}\exp\left(-\frac{|w|^2}{2\lambda}\right),\ w \in \mathbb{R}^m.$$

The mean value of such vectors is equal to zero and covariance matrix is of the form λI_m, $\lambda > 0$. The functional $\mathbf{D}(f\|p_{m,\lambda})$ denotes the Kullback-Leibler divergence (or relative entropy) of random vector W with pdf f w.r.t. another one with pdf $p_{m,\lambda}$.

The set of random vectors with bounded anisotropy is denoted as

$$\mathbb{W}_a = \{W \in \mathbb{L}_2^m : \mathbf{A}(W) \leqslant a\},$$

and anisotropic norm of $F \in \mathbb{L}_2^{p\times m}$ is a partial case of stochastic norm and introduced as

$$|\!|\!|F|\!|\!|_a = \sup_{W\in\mathbb{W}_a} \mathbf{Q}(F, W). \quad (2)$$

For nonround systems (i.e. systems associated with non-round operator), the following property holds true:

$$\|F\|_2/\sqrt{m} = \lim_{a\to+0} |\!|\!|F|\!|\!|_a \leqslant |\!|\!|F|\!|\!|_a \leqslant \lim_{a\to+\infty} |\!|\!|F|\!|\!|_a = \|F\|_\infty.$$

Out of this, results of anisotropy-based theory is always "between" ones of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories.

In Kustov (2018), the case of anisotropy-based analysis for linear discrete time-varying (LDTV) systems with random matrices is considered. It is shown that the covariance matrix of input that ensures maximum of RMS gain for system F is of the form $\Sigma(q) = (I_m - q\Lambda)^{-1}$, where q is a certain parameter, and $\Lambda = \mathbf{E}[F^\top F]$. The scalar parameter $q \in [0; \|F\|_\infty^{-2})$ is the unique solution of the special-type equation

$$-\frac{1}{2}\ln\det\frac{m\Sigma(q)}{\mathrm{tr}\Sigma(q)} = a.$$

Given the relation between worst case covariance matrix and parameter q, it is possible to rewire (1) as

$$|\!|\!|F|\!|\!|_a = \sqrt{\frac{\mathrm{tr}(\Lambda\Sigma(q))}{\mathrm{tr}(\Sigma(q))}}. \quad (3)$$

The partial case of anisotropic norm calculation for LDTV multiplicative noise system is considered in Belov et al. (2019). The systems considered are of the following form:

$$\begin{aligned} x(k+1) &= \sum_{i=0}^{M} \xi_{1,i}(k)A_i(k)x(k) + \sum_{i=0}^{M} \xi_{2,i}(k)B_i(k)w(k), \\ z(k) &= \sum_{i=0}^{M} \xi_{3,i}(k)C_i(k)x(k) + \sum_{i=0}^{M} \xi_{4,i}(k)D_i(k)w(k), \end{aligned} \quad (4)$$

where $x(k)$, $w(k)$, $z(k)$ denote the state, input and output, respectively. Matrices $A_i(k) \in \mathbb{R}^{n_x\times n_x}$, $B_i(k) \in \mathbb{R}^{n_x\times m_w}$, $C_i(k) \in \mathbb{R}^{p_z\times n_x}$, $D_i(k) \in \mathbb{R}^{p_z\times m_w}$ are time-depended, and set of random scalar variables $\xi_{j,i}(k)$ are mutually independent in all j, i, k. The additional condition $\xi_{j,0}(k) = 1$, $j = \overline{1,4}$, together with $\mathbf{E}[\xi_{j,i}(k)] = 0$ for $i > 0$ corresponds to the case that mean values of matrices in (4) equal to $A_0(k), B_0(k), C_0(k), D_0(k)$. It is shown in Belov et al. (2019) that if there exist solutions of coupled difference Riccati equations

$$\begin{aligned} R_1(k) &= \sum_{i=0}^{M}\left(A_i^\top(k)R_1(k+1)A_i(k) + qC_i^\top(k)C_i(k)\right), \\ R_2(k) &= A_0^\top(k)R_2(k+1)A_0(k) + L^\top(k)S^{-1}(k)L(k), \end{aligned}$$

where

$$\begin{aligned} S(k) &= \big(I_{m_w} - B_0^\top(k)R_2(k+1)B_0(k) \\ &\quad - \sum_{i=0}^{M}\left(qD_i^\top(k)D_i(k) + B_i^\top(k)R_1(k+1)B_i(k)\right)\big)^{-1}, \\ L(k) &= S(k)\big(B_0^\top(k)R_1(k+1)A_0(k) \\ &\quad + B_0^\top(k)R_2(k+1)A_0(k) + qD_0^\top(k)C_0(k)\big), \end{aligned}$$

with bounded conditions $R_1(N+1) = 0$, $R_2(N+1) = 0$, then anisotropic norm of system (4) is expressed by means of special functions $\mathcal{N}(q)$ and $\mathcal{A}(q)$ as $|\!|\!|F|\!|\!|_a = \mathcal{N}(\mathcal{A}^{-1}(a))$, where anisotropy of extended vector of input sequence does not exceed given $a > 0$. Here,

$$\mathcal{N}(q) = \sqrt{\frac{\Phi(q)-1}{q\Phi(q)}}, \quad \mathcal{A}(q) = \frac{l_w}{2}\left(\ln\Phi(q) - \Psi(q)\right), \quad (5)$$

and $\mathcal{A}(q) = a$ holds true for unique $q \in [0; \|F\|_\infty^{-2})$. Functions $\Phi(q), \Psi(q)$ in (5) are defined in terms of solutions of Riccati equations and Lyapunov-type equation

$$\begin{aligned} \Upsilon(k+1) &= B_0(k)S(k)B_0^\top(k) \\ &\quad + (A_0(k)+B_0(k)L(k))\Upsilon(k)(A_0(k)+B_0(k)L(k))^\top \end{aligned}$$

with initial condition $\Upsilon(0) = 0$, as following:

$$\Phi(q) = \frac{1}{l_w} \sum_{k=0}^{N} \operatorname{tr}\left(S(k) + L(k)\Upsilon(k)L^\top(k)\right),$$

$$\Psi(q) = \frac{1}{l_w} \sum_{k=0}^{N} \ln\det S(k), \quad l_w = m_w(N+1).$$

Based on this result, the boundedness condition of anisotropic norm was obtained, and then suboptimal problem was proposed and solved.

3. PROBLEM STATEMENT

Consider the linear discrete time-varying system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)w(t), \quad x(0) = 0, \\ z(t) &= M(t)x(t) + N(t)w(t), \\ y(t) &= \lambda(t)C(t)x(t) + D(t)w(t) \\ &\quad +(1-\lambda(t))y(t-1), \end{aligned} \tag{6}$$

on finite time interval $t = \overline{0, N}$. Matrices $A(t) \in \mathbb{R}^{n_x \times n_x}$, $B(t) \in \mathbb{R}^{n_x \times m_w}$, $M(t) \in \mathbb{R}^{p_z \times n_x}$, $N(t) \in \mathbb{R}^{p_z \times m_w}$, $C(t) \in \mathbb{R}^{p_y \times n_x}$ and $D(t) \in \mathbb{R}^{p_y \times m_w}$ are nonrandom matrices-function of time. Vectors $x(t)$, $w(t)$, $z(t)$ and $y(t)$ denote the state, the input, the output to be estimated, and the measurement, respectively. Extended vector $W_{0:N} = (w_0^{\mathrm{T}}, \ldots, w_N^{\mathrm{T}})^{\mathrm{T}}$ belongs to the set $\mathbb{W}_a$, i.e. $\mathbf{A}(W_{0:N}) \leqslant a$. Random variable $\lambda(t)$ has the Bernoulli distribution with known probabilities $P(\lambda(t) = 1) = p$ and $P(\lambda(t) = 0) = 1 - p$ corresponding to normal operation and case when measurement failure occurs. The additional term $(1 - \lambda(t))y(t-1)$ in measurement output describes the correction when failure at time instant t occurs.

The problem of this paper is to design estimator for (6) (i.e. to obtain estimation $\widehat{z}(t)$ of output $z(t)$) such that anisotropic norm of closed input-to-error system is bounded by parameter $\gamma > 0$ chosen as small as possible.

4. MAIN RESULTS

The system (6) contains vectors from three different time instants $t-1, t, t+1$. It is possible to avoid this after system modification so it will contain time instances $t+1$ and t only. For that, define the extended state as follows:

$$\overline{x}(t) = \begin{pmatrix} x(t) \\ y(t-1) \end{pmatrix} \in \mathbb{R}^{n_x + p_y}.$$

Then, the system (6) can be rewritten in standard form

$$\begin{aligned} \overline{x}(t+1) &= \overline{A}^\lambda(t)\overline{x}(t) + \overline{B}(t)w(t), \quad \overline{x}(0) = 0, \\ z(t) &= \overline{M}(t)\overline{x}(t) + \overline{N}(t)w(t), \\ y(t) &= \overline{C}^\lambda(t)\overline{x}(t) + \overline{D}(t)w(t), \end{aligned} \tag{7}$$

where

$$\overline{A}^\lambda(t) = \begin{bmatrix} A(t) & 0 \\ \lambda(t)C(t) & (1-\lambda(t))I_{p_y} \end{bmatrix}, \; \overline{B}(t) = \begin{bmatrix} B(t) \\ D(t) \end{bmatrix},$$

$$\overline{M}(t) = [M(t) \quad 0], \; \overline{N}(t) = N(t),$$

$$\overline{C}^\lambda(t) = \left[\, \lambda(t)C(t) \quad (1-\lambda(t))I_{p_y} \,\right], \; \overline{D}(t) = D(t).$$

One can note that $\overline{A}^\lambda(t)$ and $\overline{C}^\lambda(t)$ are matrices linear in the scalar random variable $\lambda(t)$.

To obtain the estimation of output $z(t)$ and state vector $x(t)$, let us choose the estimation model as

$$\begin{aligned} \widehat{\overline{x}}(t+1) &= W(t)\widehat{\overline{x}}(t) + H(t)(y(t) - \widehat{y}(t)), \quad \widehat{\overline{x}}(0) = 0, \\ \widehat{z}(t) &= \overline{M}(t)\widehat{\overline{x}}(t), \end{aligned}$$

where $\widehat{y}(t)$ is chosen as

$$\widehat{y}(t) = \mathbf{E}\left[\overline{C}^\lambda(t)\right]\widehat{\overline{x}}(t) = \overline{C}^p(t)\widehat{\overline{x}}(t) \tag{8}$$

according to expectation of matrix $\overline{C}^\lambda(t)$. Here, we use $\overline{C}^p(t)$ to denote the following matrix:

$$\overline{C}^p(t) = \mathbf{E}\left[\overline{C}^\lambda(t)\right] = \left[\, pC(t) \quad (1-p)I_{p_y} \,\right].$$

Similar to the notation above, let us introduce the matrix

$$\overline{A}^p(t) = \mathbf{E}\left[\overline{A}^\lambda(t)\right] = \begin{bmatrix} A(t) & 0 \\ pC(t) & (1-p)I_{p_y} \end{bmatrix}.$$

Then, the state error $e(t) = \overline{x}(t) - \widehat{\overline{x}}(t)$ has the following dynamics:

$$\begin{aligned} e(t+1) &= \overline{A}^\lambda(t)\overline{x}(t) - W(t)\overline{x}(t) \\ &\quad - H(t)\left(\overline{C}^\lambda(t) - \overline{C}^p(t)\right)\overline{x}(t) \\ &\quad + \left(W(t) - H(t)\overline{C}^p(t)\right)e(t) \\ &\quad + \left(\overline{B}(t) - H(t)\overline{D}(t)\right)w(t). \end{aligned}$$

So that, the input-to-error system can be presented as:

$$\begin{aligned} e(t+1) &= \left(\overline{A}^\lambda(t) - \overline{A}^p(t) + \overline{A}^p(t) - W(t)\right)\overline{x}(t) \\ &\quad - H(t)\left(\overline{C}^\lambda(t) - \overline{C}^p(t)\right)\overline{x}(t) \\ &\quad + \left(W(t) - H(t)\overline{C}^p(t)\right)e(t) \\ &\quad + \left(\overline{B}(t) - H(t)\overline{D}(t)\right)w(t), \\ \widetilde{z}(t) &= \overline{M}(t)e(t) + \overline{N}(t)w(t), \end{aligned}$$

where $\widetilde{z}(t) = z(t) - \widehat{z}(t)$.

Now we also introduce the following matrices $\overline{A}^\circ$ and $\overline{C}^\circ$:

$$\begin{aligned} \overline{A}^\circ(t) &= \overline{A}^\lambda(t) - \overline{A}^p(t) \\ &= (\lambda(t) - p)\begin{bmatrix} 0 & 0 \\ C(t) & -I_{p_y} \end{bmatrix} = (\lambda(t) - p)\widetilde{A}(t), \\ \overline{C}^\circ(t) &= \overline{C}^\lambda(t) - \overline{C}^p(t) \\ &= (\lambda(t) - p)\left[\, C(t) \quad -I_{p_y} \,\right] = (\lambda(t) - p)\widetilde{C}(t), \end{aligned}$$

and provide the following type of error system dynamics:

$$\begin{aligned} e(t+1) &= \left(\overline{A}^p(t) - W(t)\right)\overline{x}(t) \\ &\quad + \left(\overline{A}^\circ(t) - H(t)\overline{C}^\circ(t)\right)\overline{x}(t) \\ &\quad + \left(W(t) - H(t)\overline{C}^p(t)\right)e(t) \\ &\quad + \left(\overline{B}(t) - H(t)\overline{D}(t)\right)w(t). \end{aligned}$$

Then, we complete the dynamics of error with one for the extended state:

$$\overline{x}(t+1) = \left(\overline{A}^p(t) + \overline{A}^\circ(t)\right)\overline{x}(t) + \overline{B}(t)w(t).$$

To assemble two types of dynamics in one, define the extended vector

$$\zeta(t) = \begin{pmatrix} \overline{x}(t) \\ e(t) \end{pmatrix}.$$

The combined dynamics of the input-to-output error system is of the form:

$$\zeta(t+1) = \begin{bmatrix} \overline{A}^p(t) & 0 \\ \overline{A}^p(t) - W(t) & W(t) - H(t)\overline{C}^p(t) \end{bmatrix} \zeta(t) + \begin{bmatrix} \overline{A}^\circ(t) & 0 \\ \overline{A}^\circ(t) - H(t)\overline{C}^\circ(t) & 0 \end{bmatrix} \zeta(t) + \begin{bmatrix} \overline{B}(t) \\ \overline{B}(t) - H(t)\overline{D}(t) \end{bmatrix} w(t),$$
$$\widetilde{z}(t) = \begin{bmatrix} 0 & \overline{M}(t) \end{bmatrix} \zeta(t) + \overline{N}(t) w(t).$$

Before implement the result of Belov et al. (2019), let us change the notations of matrices in the state-space realisation above according to the following:

$$\begin{aligned} \zeta(t+1) &= (\mathcal{A}_0(t) + \xi(t)\mathcal{A}_1(t))\zeta(t) + \mathcal{B}(t)w(t), \\ \widetilde{z}(t) &= \mathcal{M}(t)\zeta(t) + \mathcal{N}(t)w(t), \end{aligned} \tag{9}$$

where

$$\mathcal{A}_0(t) = \begin{bmatrix} \overline{A}^p(t) & 0 \\ \overline{A}^p(t) - W(t) & W(t) - H(t)\overline{C}^p(t) \end{bmatrix},$$
$$\mathcal{A}_1(t) = \begin{bmatrix} \widetilde{A}(t) & 0 \\ \widetilde{A}(t) - H(t)\widetilde{C}(t) & 0 \end{bmatrix},$$
$$\mathcal{B}(t) = \begin{bmatrix} \overline{B}(t) \\ \overline{B}(t) - H(t)\overline{D}(t) \end{bmatrix},$$
$$\mathcal{M}(t) = \begin{bmatrix} 0 & \overline{M}(t) \end{bmatrix}, \quad \mathcal{N}(t) = \overline{N}(t),$$

and $\xi(t) = \lambda(t) - p$ is the scalar random variable with Bernoulli distribution.

Let us introduce the set of backward-time difference Riccati inequalities

$$\begin{aligned} \mathcal{R}(t) \succ\, & \mathcal{A}_0^{\mathrm{T}}(t)\mathcal{R}(t+1)\mathcal{A}_0(t) + \sigma^2\mathcal{A}_1^{\mathrm{T}}(t)\mathcal{R}(t+1)\mathcal{A}_1(t) \\ & + q\mathcal{M}^T(t)\mathcal{M}(t) + \mathcal{L}^{\mathrm{T}}(t)\mathcal{S}^{-1}(t)\mathcal{L}(t) \end{aligned} \tag{10}$$

with boundary condition $\mathcal{R}(N+1) = 0$. If this series of inequalities is solvable with respect to q and $\mathcal{R}(t)$, $t = \overline{0,N}$, together with the inequality

$$\sum_{t=0}^{N} \ln\det \mathcal{S}^{-1}(t) \geqslant 2a + m_w(N+1)\ln(1 - q\gamma^2), \tag{11}$$

then anisotropic norm of system (9) is bounded by γ, see Belov et al. (2019) for details. Here, we use $\sigma^2 = p(1-p)$, and matrices $\mathcal{S}(t)$, $\mathcal{L}(t)$ are the following:

$$\mathcal{S}(t) = (I_{m_w} - \mathcal{B}^{\mathrm{T}}(t)\mathcal{R}(t+1)\mathcal{B}(t) - q\mathcal{N}(t)^{\mathrm{T}}\mathcal{N}(t))^{-1},$$
$$\mathcal{L}(t) = \mathcal{S}(t)(\mathcal{B}^{\mathrm{T}}(t)\mathcal{R}(t+1)\mathcal{A}_0(t) + q\mathcal{N}(t)^{\mathrm{T}}\mathcal{M}(t)),$$

where scalar parameter $q \in \big[0; \|F\|_\infty^{-2}\big)$. All matrices $\mathcal{R}(t), \mathcal{S}(t)$ are positive definite for $t = \overline{0,N}$.

Next step is concerned with the change of variables. Let us denote $\eta^2 = q^{-1}$ and $R(t) = q^{-1}\mathcal{R}(t)$, then (10) and (11) have to change too. In new variables, these inequalities become as follows:

$$\begin{aligned} R(t) \succ\, & \mathcal{A}_0^{\mathrm{T}}(t)R(t+1)\mathcal{A}_0(t) + \sigma^2\mathcal{A}_1^{\mathrm{T}}(t)R(t+1)\mathcal{A}_1(t) \\ & + \mathcal{M}^T(t)\mathcal{M}(t) + L^{\mathrm{T}}(t)S^{-1}(t)L(t), \end{aligned} \tag{12}$$

$$\sum_{t=0}^{N} \ln\det S^{-1}(t) \geqslant 2a + m_w(N+1)\ln(\eta^2 - \gamma^2), \tag{13}$$

where

$$S(t) = \big(\eta^2 I_{m_w} - \mathcal{B}^{\mathrm{T}}(t)R(t+1)\mathcal{B}(t) - \mathcal{N}(t)^{\mathrm{T}}\mathcal{N}(t)\big)^{-1},$$
$$L(t) = S(t)\big(\mathcal{B}^{\mathrm{T}}(t)R(t+1)\mathcal{A}_0(t) + \mathcal{N}(t)^{\mathrm{T}}\mathcal{M}(t)\big).$$

The more convenient method to solve inequalities (12) is to perform transformation to linear matrix inequalities. Using Schur complement formula, one can get the following:

$$\begin{bmatrix} R(t) - \mathcal{M}^{\mathrm{T}}(t)\mathcal{M}(t) & * & * & * \\ \mathcal{N}^{\mathrm{T}}(t)\mathcal{M} & \eta^2 I_{m_w} - \mathcal{N}^{\mathrm{T}}(t)\mathcal{N}(t) & * & * \\ R(t+1)\mathcal{A}_0(t) & -R(t+1)\mathcal{B}(t) & R(t+1) & * \\ \sigma R(t+1)\mathcal{A}_1(t) & 0 & 0 & R(t+1) \end{bmatrix} \succ 0 \tag{14}$$

for all $t = \overline{0, N-1}$. Note that (14) contains $R(t+1)$ multiplied by $\mathcal{A}_0(t), \mathcal{A}_1(t)$ and $\mathcal{B}(t)$, which are depended on unknown matrices $W(t)$ and $H(t)$. To avoid nonlinearity, let us introduce new matrix variables

$$U(t) = \begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times p_y} \\ W(t) & H(t) \end{bmatrix}, \quad X(t) = R(t+1)U(t).$$

Applying variables $X(t)$, $t = \overline{0,N}$, the inequalities (14) can be converted to

$$\begin{bmatrix} R - \mathcal{M}^{\mathrm{T}}\mathcal{M} & * & * & * \\ \mathcal{N}^{\mathrm{T}}\mathcal{M} & \eta^2 I_{m_w} - \mathcal{N}^{\mathrm{T}}\mathcal{N} & * & * \\ R_+\mathcal{A}_{00} + X\mathcal{A}_{01} & -R_+\mathcal{B}_0 - X\mathcal{B}_1 & R_+ & * \\ R_+\mathcal{A}_{10} + X\mathcal{A}_{11} & 0 & 0 & R_+ \end{bmatrix} \succ 0, \tag{15}$$

where for sake of simplicity, $R = R(t)$, $R_+ = R(t+1)$, $\mathcal{N} = \mathcal{N}(t)$, $\mathcal{M} = \mathcal{M}(t)$, $X = X(t)$,

$$\mathcal{A}_{00} = \begin{bmatrix} \overline{A}^p(t) & 0 \\ \overline{A}^p(t) & 0 \end{bmatrix}, \; \mathcal{A}_{01} = \begin{bmatrix} -I_{n_x+p_y} & I_{n_x+p_y} \\ 0 & -\overline{C}^p(t) \end{bmatrix},$$
$$\mathcal{A}_{10} = \begin{bmatrix} \sigma\widetilde{A}(t) & 0 \\ \sigma\widetilde{A}(t) & 0 \end{bmatrix}, \; \mathcal{A}_{11} = \begin{bmatrix} 0 & 0_{(n_x+p_y)\times(n_x+p_y)} \\ -\sigma\widetilde{C}(t) & 0 \end{bmatrix},$$
$$\mathcal{B}_0 = \begin{bmatrix} \overline{B}(t) \\ \overline{B}(t) \end{bmatrix}, \; \mathcal{B}_1 = \begin{bmatrix} 0_{(n_x+p_y)\times m_w} \\ -\overline{D}(t) \end{bmatrix}.$$

Thus, we additionally add the following constraint on matrices $R(t)$:

$$R(t) = \begin{bmatrix} R_1(t) & 0 \\ 0 & R_2(t) \end{bmatrix},$$

and the following constraint on matrices $X(t)$:

$$X(t) = \begin{bmatrix} 0 & 0 \\ X_1(t) & X_2(t) \end{bmatrix}.$$

So, if both series of matrices $R(t)$ and $X(t)$ are successfully found then, according to

$$\begin{bmatrix} 0 & 0 \\ W(t) & H(t) \end{bmatrix} = R^{-1}(t+1)X(t),$$

one has

$$W(t) = R_2^{-1}(t+1)X_1(t), \quad H(t) = R_2^{-1}(t+1)X_2(t). \tag{16}$$

To fulfil the positive definiteness condition on matrix $S(t)$ in (12) and (13), we introduce matrix $\Phi(t)$ such that $0 \prec \Phi(t) \prec \mathcal{S}^{-1}(t)$, and matrix $\Psi(t) = \eta^2\Phi(t)$. Doing this, the inequality for matrix $S(t)$ transforms to the following inequality for matrix $\Psi(t)$:

$$\begin{bmatrix} \eta^2 I_{m_w} - \Psi - \mathcal{N}^{\mathrm{T}}\mathcal{N} & * \\ R_+\mathcal{B}_{00} + X\mathcal{B}_{01} & R_+ \end{bmatrix} \succ 0, \tag{17}$$

where time dependence of $\Psi = \Psi(t)$ is omitted as in (15). The inequality (17) holds true for every time instant $t = \overline{0, N-1}$. Special-type inequality (13) should be modified as follows:

$$\sum_{t=0}^{N} \ln\det \Psi(t) \geqslant 2a + m_w(N+1)\ln(\eta^2 - \gamma^2). \tag{18}$$

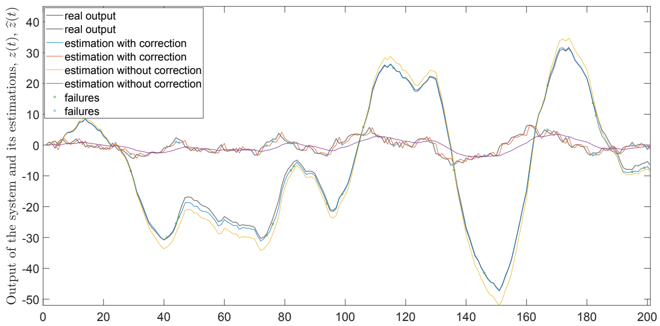

Fig. 1. Output $z(t)$ and its estimation $\widehat{z}(t)$ for two cases: with and without correction when failure occurs. The time instances when failure occurs on some sensor shown as circles on the corresponding curve.

To satisfy the boundary condition $R(N+1) = 0$, we need to fulfil the following inequalities:

$$\begin{bmatrix} R(N) - \mathcal{M}^{\mathrm{T}}(N)\mathcal{M}(N) & * \\ \mathcal{N}^{\mathrm{T}}(N)\mathcal{M}(N) & \eta^2 I_{m_w} - \mathcal{N}^{\mathrm{T}}(N)\mathcal{N}(N) \end{bmatrix} \succ 0, \tag{19}$$

$$\eta^2 I_{m_w} - \Psi(N) - \mathcal{N}^{\mathrm{T}}(N)\mathcal{N}(N) \succ 0. \tag{20}$$

It is important to note that inequalities (15)–(20) are linear in variables R, X, Ψ, η^2, γ^2, so it is possible to use MATLAB software for solving optimization problem

$$\gamma^2 \to \min_{R,X,\Psi,\eta^2,\gamma^2}$$

subject to convex constraints (15), (17)–(20). After solving the problem above, one can get the matrix of the estimator by inverse change of variables (16), provided that convex optimization problem is feasible.

5. ILLUSTRATIVE EXAMPLES

Let us demonstrate the method proposed with some simple example. Consider system (6) with matrices

$$A = \begin{bmatrix} 0.95 & 1 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0.1 \end{bmatrix},$$

and failure probability $P(\lambda(t) = 0) = 1 - p = 0.05$. The system is supposed to operate on time interval $k = 0, \ldots, N$ with $N = 200$, and anisotropy of input disturbance is chosen as $a = 1$. The optimization problem was solved in Matlab using Yalmip (Löfberg (2004)) and SeDuMi (Sturm (1999)). The upper bound γ for anisotropic norm for the case where correction has been considered is found to be equal $\gamma \simeq 1.77$. Additionally, the same has been done for this system but without correction, the computed upper bound of anisotropic norm is equal to $\gamma \simeq 17.18$. The simulation results are presented in the Fig. 1.

6. CONCLUSION

For linear discrete time-varying system with randomly occurring packet dropouts and their correction, the anisotropic estimation problem was solved. The method proposed is based on the results analysis for systems with multiplicative noises. Solving final optimization problem with convex constraints gives the matrices of the desired estimator. To demonstrate the method efficiency, the illustrative example is presented.

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