Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

NILT and Prony technique for new definitions of fractional calculus for modeling very slow decay phenomena

Rui-Yang Cai
College of Information Science and Technology
Donghua University
Shanghai 201620, PR China
Email:2151359@mail.dhu.edu.cn

Yu-Quan Chen
Department of Automation
University of Science and Technology of China
Hefei 230026, PR China
Email:cyq@mail.ustc.edu.cn

Yang-Quan Chen
School of Engineering (MESA-Lab)
University of California, Merced
Merced, California, 95343, USA
Email:ychen53@ucmerced.edu

Chun-Hai Kou*
Department of Applied Mathematics
Donghua University
Shanghai 201620, PR China
Email:kouchunhai@dhu.edu.cn

***Abstract*—Based on the data from the Tesla users about the degradation of Tesla Model S battery capacity, this paper introduces two new definitions of fractional integrals and derivatives to describe these very slow decay phenomena, which fill the blanks in this field. Several existing definitions of fractional calculus are reviewed at first, together with their drawbacks in describing very slow decay. Then, two new definitions governed by new kernel functions are introduced. With the aids of Numerical Inverse Laplace Transform (NILT), Prony technique and MATLAB command stmcb(), mathematical properties of these two fractional operators are investigated.**

Keywords: ***Very slow decay; New fractional operators; Numerical Inverse Laplace Transform; Prony technique***

I. INTRODUCTION

The last decades have witnessed a remarkable development in fractional calculus as shown by a number of mathematical monographs dedicated to it [1]–[8]. Based on the relation between CTRW framework and anomalous phenomena, authors in [9]–[11] draw the conclusion that a new wave has been set off on fractional systems, whose heavy-tailed distribution and memory property mostly match the characteristics of anomalous phenomena. The applications of fractional calculus have also widely spread into physics [12]–[14], biology [15], [16], fractional thermoelasticity [17], aerodynamics [18], bioengineering [19], etc. It's known that the nature of fractional integral/derivative operators is a kind of convolution form. Hence, one may define different definitions with proper kernel functions. A survey paper [20] listed almost all the existing definitions of fractional order integrals and derivatives that appeared in mathematics, physics and engineering. However, some complaints have been made for the cumbersome mathematical expressions and the loss of generality on some of existing definitions.

*Corresponding author. Email: kouchunhai@dhu.edu.cn

*This research has received funding from the Natural Science Foundation of Shanghai (No. 19ZR1400500).

In fact, fractional calculus is of particular convenience at the very start. However, if such effects aren't present, adopting a new definition is necessary. Therefore, in this article, two new kinds of definitions for fractional calculus are suggested with two kernel functions involving the logarithmic function to characterize the very slow decay. The first one is suitable to use the Laplace transform and its inverse, while the second can degenerate to classical integral and derivative, which overcome the loss of generality problem well. The motivation of this new approach is to model the capacity decay of Tesla Model S battery.

In Section II, some common definitions are reviewed with their drawbacks in describing a class of phenomena stated. Section III focuses on two new kinds of definitions, together with analyzing their mathematical properties by employing NILT, Prony technique and stmcb() command.

II. OVERVIEW OF EXISTING DEFINITIONS FOR FRACTIONAL CALCULUS

At first, we recall three definitions of fractional calculus for continuous functions. Then, from a practical example, i.e., the data from the Tesla users about the degradation of Tesla Model S battery capacity, we discuss the drawbacks of the three definitions in theoretical analysis and in real world applications. We declare that among the rest of this paper, we only consider about order $\alpha \in (0, 1)$.

Definition II.1. *[1] The α-th order Riemann-Liouville fractional integral and Caputo fractional derivative of a continuous function $f(t)$ are given by*

$$ {}_aI_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s \tag{1}$$

and

$$ {}_a^C D_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s)\mathrm{d}s, \tag{2}$$

respectively, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition II.2. *[21] Let $a > 0$. The α-th order Hadamard fractional integral of $f(t)$ is*

$$ {}_a^H I_t^\alpha f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{\mathrm{d}s}{s}, \quad (3) $$

while its Hadamard-Caputo fractional derivative is given by

$$ {}_a^{HC} D_t^\alpha f(t) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} f'(s)\mathrm{d}s. \quad (4) $$

Definition II.3. *[22] The nonsingular fractional derivative of order α for $f(t)$ is defined by*

$$ {}_a^{NC} D_t^\alpha f(t) \triangleq \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right) f'(s)\mathrm{d}s, \quad (5) $$

where $M(\alpha)$ satisfies $M(0) = M(1) = 1$.

Remark II.1. *At first glance, Hadamard fractional calculus is not of a convolution form, since one cannot see the kernel function clearly. In [23], the authors introduced the logarithmic convolution for better revealing the characteristics of the Hadamard type.*

In addition, we see

$$ \begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{\mathrm{d}s}{s} \\ = & \frac{1}{\Gamma(\alpha)} \int_{\log a}^{\log t} (\log t - u)^{\alpha-1} f(\exp(u))\,\mathrm{d}u \\ = & \frac{1}{\Gamma(\alpha)} \int_k^t (t-s)^{\alpha-1} f(\exp(s))\,\mathrm{d}s, \end{aligned} \quad (6) $$

where $k = \log a$. We know that (3) convolutes power function $t^{\alpha-1}$ and $f(\exp(t))$. Though it can be regarded as a unique kind of convolution form, we may meet with some trouble in applications. Moreover, when $\alpha = 1$, the Hadamard fractional derivative operator degenerates to $t\frac{\mathrm{d}}{\mathrm{d}t}$, which doesn't coincide with the classical derivative.

In fact, the kernel functions are always used to reflect the decay rate. The kernel functions of the above definitions adopted three kinds of basic elementary functions, that are, power-law function, logarithmic function and exponential function. It is worth mentioning that here we use t^α to represent all power-law functions due to the fact that $\left(t^k\right)^\alpha = t^{k\alpha}$. We can change the value of α to present power-law functions of any order. These three kinds of functions are the basic functions scholars chosen for the definitions of fractional calculus. Moreover, some other kernel functions which are the combination of these three kinds functions are also applied, such as functions for tempered fractional operators [24] and stretched exponential functions [25].

In 2015, Tesla uploaded the data from their users about the degradation of Tesla Model S battery capacity. Two figures about the relationship between remaining battery capacity and mileage/ battery age are also plotted (see Figure 1 and 2).

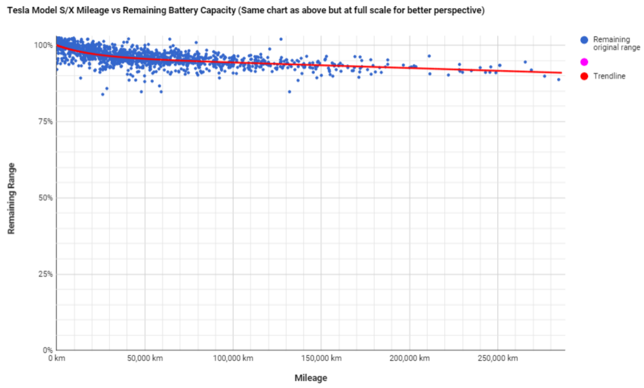

Figure 1: Tesla Model S/X Mileage VS Remaining Battery Capacity

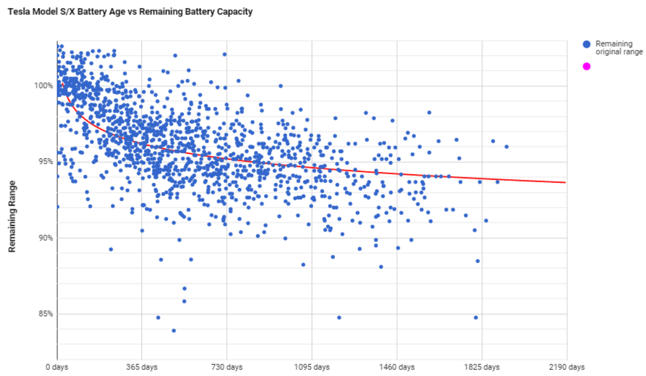

Figure 2: Tesla Model S/X Battery Age VS Remaining Battery Capacity

Remark II.2. *Figure 1 and 2 are the origin figures announced by Tesla, which cannot be modified for better readability. Unfortunately, due to the lack of the page width, the axes and the legends may be of a little unavoidable illegibility.*

It can be easily observed from these two figures that if we want to describe the decay rate of remaining battery capacity, the inverse power-law function decays too fast, while that of logarithmic function might decay too slow. However, to the best of our knowledge, no proper definition has been founded to describe the very slow decay phenomena yet. Therefore, we need a new kernel function for a proper definition!

III. Two new definitions with very-slow decay kernels

In this section, we provide two new kernel functions for fractional integral and derivative, which can well describe the very slow decay in some extent. Here, we choose the combination of logarithmic function and inverse power-law function, and thus we have two choices of kernel functions: $\frac{\log t}{t^\alpha}$ and $\left(\frac{\log t}{t}\right)^\alpha$. From

$$ \lim_{t\to\infty} \frac{\frac{\log t}{t^\alpha}}{t^\alpha} = 0, \qquad \lim_{t\to\infty} \frac{(\log t)^\alpha}{\frac{\log t}{t^\alpha}} = 0, $$

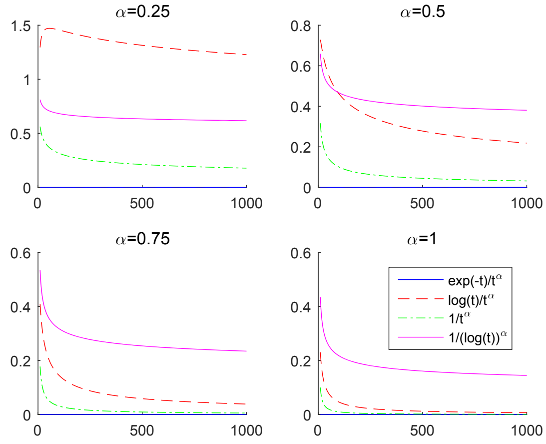

Figure 3: Comparison among different tails. x-axis: t

and

$$\lim_{t\to\infty} \frac{\left(\frac{\log t}{t}\right)^{\alpha}}{t^{\alpha}} = 0, \quad \lim_{t\to\infty} \frac{(\log t)^{\alpha}}{\left(\frac{\log t}{t}\right)^{\alpha}} = 0,$$

we know the decay rate of $\frac{\log t}{t^\alpha}$ and $\left(\frac{\log t}{t}\right)^{\alpha}$ are between logarithmic function and inverse power-law function. Since $\log t$ is the higher order infinitesimal of exponential functions and any power-law functions when t tends to infinity, the kernel function $\frac{1}{(\log t)^{\alpha}}$ can be used to describe the ultra-slow decay rate. On the other hand, the inverse power-law function $\frac{1}{t^\alpha}$ are usually used for normal slow decay. Therefore, $\frac{\log t}{t^\alpha}$ and $\left(\frac{\log t}{t}\right)^{\alpha}$ can be used to characterize those very slow decay phenomena.

A. New definition with kernel $\frac{\log t}{t^\alpha}$

Figure 3 shows the decay rates of four kinds of kernel functions, which implies that the decay rate of $\frac{\log t}{t^\alpha}$ is between the inverse power-law kernel and the Hadamard kernel.

Proposition III.1. *The Laplace transform of* $\frac{\log t}{t^\alpha}$ *is*

$$\mathfrak{L}\left(\frac{\log t}{t^{\alpha}}\right)(s) = s^{\alpha-1}\Gamma(1-\alpha)\left(\psi(1-\alpha) - \log s\right), \tag{7}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *denotes the digamma function.*

On the above basis, we provide a new definition for fractional calculus as follows.

Definition III.1. *The* α*-th order fractional integral and Caputo-type fractional derivative of* $f(t)$ *with kernel* $\frac{\log t}{t^\alpha}$ *are defined by*

$$ {}_0\mathfrak{I}_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\log(t-s)}{(t-s)^{1-\alpha}} f(s)\mathrm{d}s \tag{8}$$

and

$$ {}_0^C\mathfrak{D}_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\log(t-s)}{(t-s)^{\alpha}} f'(s)\mathrm{d}s, \tag{9}$$

respectively.

With the help of the relationship between the Laplace transform of convolution in time domain and the product in frequency domain, i.e.,

$$\mathfrak{L}\left(f(t) * g(t)\right) = F(s)G(s), \tag{10}$$

we can provide some basic properties on the fractional operators and give the fractional integrals and derivatives for some special functions.

Proposition III.2. *The fractional integral and derivative defined by (8) and (9) have the following properties:*

$$\begin{aligned} &(i)\ {}_0\mathfrak{I}_t^{\beta}\,{}_0\mathfrak{I}_t^{\alpha} f(t) = {}_0\mathfrak{I}_t^{\alpha}\,{}_0\mathfrak{I}_t^{\beta} f(t) \neq {}_0\mathfrak{I}_t^{\alpha+\beta} f(t); \\ &(ii)\ {}_0^C\mathfrak{D}_t^{\beta}\,{}_0^C\mathfrak{D}_t^{\alpha} f(t) = {}_0^C\mathfrak{D}_t^{\alpha}\,{}_0^C\mathfrak{D}_t^{\beta} f(t) \neq {}_0^C\mathfrak{D}_t^{\alpha+\beta} f(t). \end{aligned}$$

Proposition III.3. *The fractional integrals and derivatives of constant, power function and exponential function can be obtained as follows:*

$$\begin{aligned} &(i)\ {}_0\mathfrak{I}_t^{\alpha} c = \frac{c}{\Gamma(\alpha+1)} t^{\alpha}\left(\log t - \frac{1}{\alpha}\right); \\ &(ii)\ {}_0^C\mathfrak{D}_t^{\alpha} c = 0; \\ &(iii)\ {}_0\mathfrak{I}_t^{\alpha} t^k = \frac{\Gamma(k+1)(\psi(\alpha) - \psi(k+\alpha+1))}{\Gamma(k+\alpha+1)} t^{k+\alpha} \\ &\qquad + \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} t^{k+\alpha}\log t; \\ &(iv)\ {}_0^C\mathfrak{D}_t^{\alpha} t^k = \frac{\Gamma(k+1)(\psi(1-\alpha) - \psi(k-\alpha+1))}{\Gamma(k-\alpha+1)} t^{k-\alpha} \\ &\qquad + \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}\log t; \\ &(v)\ {}_0\mathfrak{I}_t^{\alpha} \exp(t) = \frac{\psi(\alpha)(\Gamma(\alpha) - \Gamma(\alpha,t))}{\Gamma(\alpha)} \exp(t) \\ &\qquad - \mathfrak{L}^{-1}\left(\frac{\log s}{s^{\alpha}(s-1)}\right)(t); \\ &(vi)\ {}_0^C\mathfrak{D}_t^{\alpha} \exp(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}_0\mathfrak{I}_t^{1-\alpha} \exp(t)\right), \end{aligned}$$

where $\Gamma(\alpha,t)$ *denotes the upper incomplete Gamma function defined by*

$$\Gamma(\alpha,t) \triangleq \int_t^{\infty} u^{\alpha-1}\exp(-u)\mathrm{d}u.$$

Proof. Here we only give the proof of (iii). The rest can be obtained similarly. From Definition III.1, we know

$$ {}_0\mathfrak{I}_t^{\alpha} t^k = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\log(t-s)}{(t-s)^{1-\alpha}} s^k \mathrm{d}s. \tag{11}$$

Utilizing Laplace transform on (10), we get

$$\begin{aligned}
&\mathfrak{L}\left({}_0\mathfrak{I}_t^\alpha t^k\right)\\
=&\frac{1}{\Gamma(\alpha)}\int_0^\infty \exp(-st)\mathrm{d}t\int_0^t \frac{\log(t-u)}{(t-u)^{1-\alpha}}u^k\mathrm{d}u\\
=&\frac{1}{\Gamma(\alpha)}\int_0^\infty u^k\exp(-su)\mathrm{d}u\int_0^\infty \frac{\log(t)}{t^{1-\alpha}}\exp(-st)\mathrm{d}t\\
=&\frac{\Gamma(k+1)}{\Gamma(\alpha)s^{k+1}}\cdot s^{-\alpha}\Gamma(\alpha)\left(\psi(\alpha)-\log s\right)\\
=&\frac{\Gamma(k+1)}{s^{k+\alpha+1}}\left(\psi(\alpha)-\log s\right).
\end{aligned} \tag{12}$$

Then the inverse Laplace transform on (12) provides

$$\begin{aligned}
{}_0\mathfrak{I}_t^\alpha t^k =&\mathfrak{L}^{-1}\frac{\Gamma(k+1)}{s^{k+\alpha+1}}\left(\psi(\alpha)-\log s\right)\\
=&\frac{\Gamma(k+1)\psi(\alpha)}{\Gamma(k+\alpha+1)\Gamma(\alpha)}\psi(\alpha)t^{k+\alpha}\\
&-\Gamma(k+1)\mathfrak{L}^{-1}\left(\frac{\log s}{s}\cdot\frac{1}{s^{k+\alpha}}\right).
\end{aligned}$$

According to (10), we have

$$\begin{aligned}
&\mathfrak{L}^{-1}\left(\frac{\log s}{s}\cdot\frac{1}{s^{k+\alpha}}\right)\\
=&-\int_0^t(\log\tau+\gamma)\frac{(t-\tau)^{k+\alpha+1}}{\Gamma(k+\alpha)}\mathrm{d}\tau\\
=&-\frac{\log t-\gamma-\psi(k+\alpha+1)}{(k+\alpha)\Gamma(k+\alpha)}t^{k+\alpha}\\
&-\frac{\gamma}{(k+\alpha)\Gamma(k+\alpha)}t^{k+\alpha}\\
=&-\frac{t^{k+\alpha}}{\Gamma(k+\alpha+1)}\left(\log t-\psi(k+\alpha+1)\right),
\end{aligned}$$

where $\gamma=-\int_0^\infty\exp(-x)\log x\mathrm{d}x$ is the Euler's constant. Hence, (iii) holds. □

What's more, though we cannot get the analytical form on t-function of the fractional integral and derivative of $\exp(t)$ in (v) and (vi) in Proposition III.3, it's not a problem. The NILT framework, constructing in [26], together with Prony technique [27] can be applied to deal with it. Figure 4 and 5 show the approximation of $\mathfrak{L}^{-1}\left(\frac{\log s}{s^\alpha(s-1)}\right)(t)$ and its Bode plot in the frequency domain. It is found that the impulse response in time domain and the magnitude in frequency domain are perfectly fitted while the phase in frequency domain is not fitted very well. However, it is enough for our practical usage. In addition, the approximation function of order 3 is

$$G(z^{-1})=\frac{0.9942z^2-1.988z+0.9936}{z^3-2.999z^2+2.998z-0.9992}. \tag{13}$$

B. New definition with kernel $\left(\frac{\log t}{t}\right)^\alpha$

Though the new definitions of fractional integral and Caputo-type fractional derivative have many mathematical properties, a drawback occurs, that is, when $\alpha=1$, it cannot degenerate to the classical integral and derivative. Luckily, the other kernel $\left(\frac{\log t}{t}\right)^\alpha$ can fill this gap in certain extent. The definitions of fractional integral and Caputo-type fractional derivative with kernel function $\left(\frac{\log t}{t}\right)^\alpha$ are as follows.

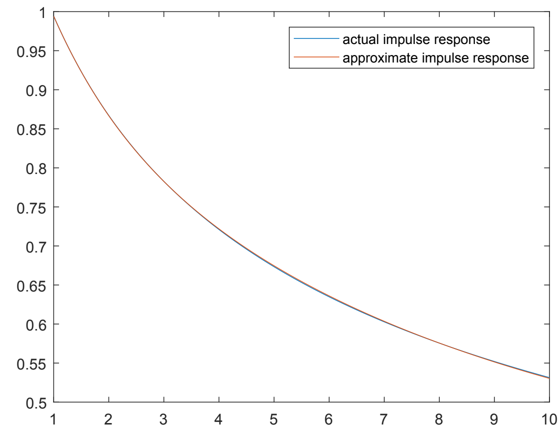

Figure 4: NILT α=0.4, order=3, t=10s, Ts=0.001s

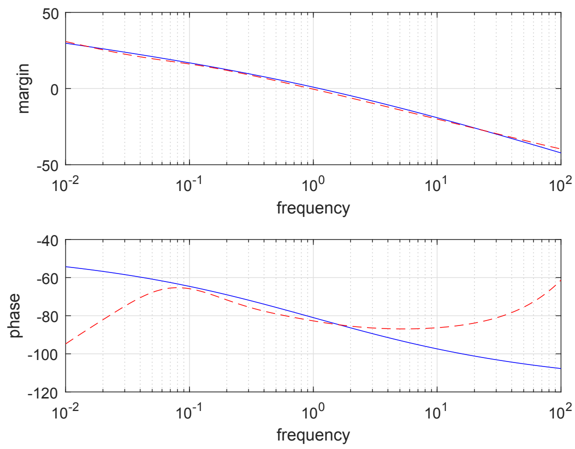

Figure 5: Bode diagram: α=0.4, order=3, Ss=0.01

Definition III.2. *The α-th order fractional integral and derivative of $f(t)$ with kernel $\left(\frac{\log t}{t}\right)^\alpha$ are defined by*

$${}_0\mathcal{I}_t^\alpha f(t)\triangleq\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{\log(t-s)}{t-s}\right)^{1-\alpha}f(s)\mathrm{d}s \tag{14}$$

and

$${}_0^C\mathcal{D}_t^\alpha f(t)\triangleq\frac{1}{\Gamma(1-\alpha)}\int_0^t\left(\frac{\log(t-s)}{t-s}\right)^{\alpha}f'(s)\mathrm{d}s, \tag{15}$$

respectively.

Furthermore, we present some properties for the fractional integral and derivative defined by (14) and (15).

Proposition III.4. *When $\alpha=1$, the following two equalities hold:*

$${}_0\mathcal{I}_t^\alpha f(t)=\int_0^t f(s)\mathrm{d}s,\quad {}_0^C\mathcal{D}_t^\alpha f(t)=\frac{\mathrm{d}f(t)}{\mathrm{d}t}.$$

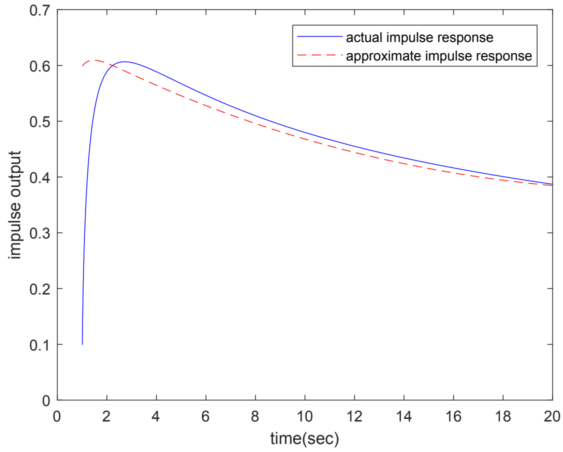

Figure 6: Comparison of impulse response for the actual and approximate kernel function

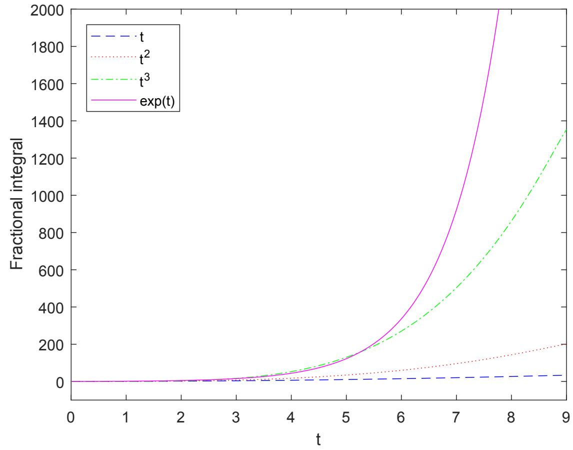

Figure 7: Fractional integrals of power functions t, t^2, t^3 and exponential function $\exp(t)$ under Definition (14)

In addition, the properties in Proposition III.2 also hold for Definition III.2.

Remark III.1. *The results in Proposition III.4 indicate that these definitions of fractional calculus are a kind of generation of classical integral and derivative.*

On one hand, the analytical s-function in frequency domain for kernel $\left(\frac{\log t}{t}\right)^{\alpha}$ is tough to obtain. On the other hand, even if we get the Laplace transform, it must be of infinite dimensions, which is impossible to implement in practical situations. Therefore, discretization method is considered and NILT and stmcb() command [28] are of great use. An approximation for $\left(\frac{\log t}{t}\right)^{0.5}$ is described by

$$G\left(z^{-1}\right) = \frac{0.1815z^2 - 0.3453z + 0.1638}{z^3 - 2.973z^2 + 2.945z - 0.9725}, \tag{16}$$

and the comparison between the impulse response of the actuality and approximation is shown in Figure 6. One can find that the approximate function can fit the tail well but the initial is not well fitted. However, we mainly focus on the tail of the kernel function to describe the decay and we can say the approximation is of good effect.

Next, we discuss the fractional integrals and derivatives of some special functions under Definition III.2. By employing numerical technique, we can provide the corresponding fractional integrals of power functions and exponential function, as shown in Figure 7, in which the numerical results for the fractional integrals of t, t^2, t^3 and $\exp(t)$ are presented. On the other hand, it's obvious that for some constant c, ${}_0^C\mathcal{D}_t^{\alpha} c = 0$. And Figure 8 provides the numerical results for the fractional derivatives of power functions $\frac{t^2}{2}$, $\frac{t^3}{3}$, $\frac{t^4}{4}$ and exponential function $\exp(t)$. Here in Figure 7 and 8, α is chosen to be 0.9.

Finally, we are to fit the Tesla data by using the two new kernel functions introduced in this paper. The trendlines are plotted as shown in Figure 9 and 10, which are corresponded

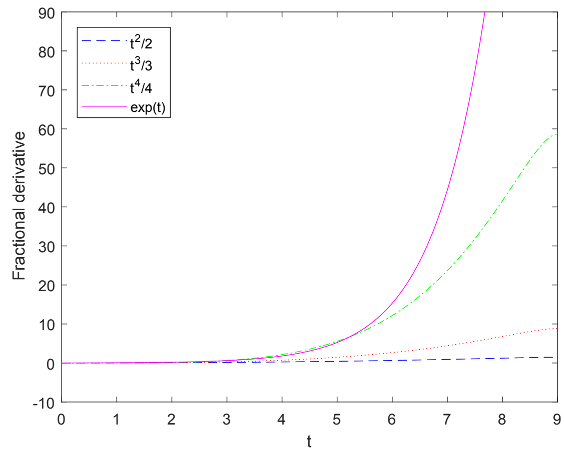

Figure 8: Fractional derivatives of power functions $\frac{t^2}{2}$, $\frac{t^3}{3}$, $\frac{t^4}{4}$ and exponential function $\exp(t)$ under Definition (15)

to Figure 1 and 2 respectively and these well fitting (especially the red dashed lines) indicate the effectiveness of the two kernel functions. Moreover, the optimal orders are also obtained as $\alpha = 0.123$ for $\frac{\log t}{t^{\alpha}}$ and $\alpha = 0.01526$ for $\left(\frac{\log t}{t}\right)^{\alpha}$ in Figure 9, while $\alpha = 0.2212$ and 0.01395 respectively in Figure 10.

IV. Conclusion

Two new definitions of fractional calculus are established in this work, which can well describe very slow decay phenomena. Some basic properties are derived with the aids of NILT, Prony and stmcb() technique. It's worth mentioning that this work is just a start in these types of fractional integrals and derivatives, which requires further investigations on their mathematical properties and on the analysis of systems governed by each kind fractional derivative.

In addition, these new kernels will be further used to test other phenomena or data sets for a better generalization.

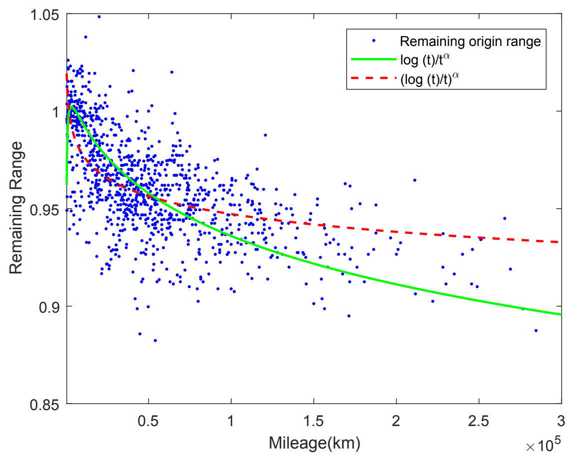

Figure 9: Fitting for data in Figure 1 by kernel functions $\frac{\log t}{t^{\alpha}}$ and $\left(\frac{\log t}{t}\right)^{\alpha}$

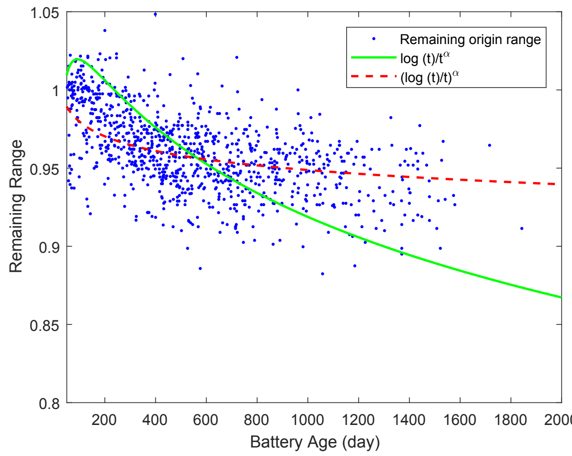

Figure 10: Fitting for data in Figure 2 by kernel functions $\frac{\log t}{t^{\alpha}}$ and $\left(\frac{\log t}{t}\right)^{\alpha}$

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science Limited, 2006.

[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, vol. 56. Mathematical in Science and Engineering, 1974.

[3] K. Oldham and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, vol. 111. Elsevier, 1974.

[4] I. Podlubny, *Fractional Differential Equations*, vol. 198. Academic Press, 1999.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*. World Scientific, 2000.

[6] J. Sabatier, J. Sabatier, J. Sabatier, and H. Allgemein, *Advances in Fractional Calculus*. Springer Netherlands, 2007.

[7] Z. Jiao, Y. Q. Chen, and I. Podlubny, *Distributed-Order Dynamic Systems*. Springer London, 2012.

[8] F. Ge, Y. Chen, and C. Kou, *Regional Analysis of Time-Fractional Diffusion Processes*. Springer, 2018.

[9] A. Cartea and D. del Castillo-Negrete, "Fluid limit of the continuous-time random walk with general Lévy jump distribution functions," *Phys. Rev. E*, vol. 76, p. 041105, Oct 2007.

[10] C. Zeng and Y. Q. Chen, "Optimal random search, fractional dynamics and fractional calculus," *Fractional Calculus and Applied Analysis*, vol. 17, no. 2, pp. 321–332, 2014.

[11] R. Gorenflo and F. Mainardi, *Fractional Diffusion Processes: Probability Distributions and Continuous Time Random Walk*, pp. 148–166. Springer Berlin Heidelberg, 2003.

[12] M. Naber, "Time fractional Schrödinger equation," *Journal of Mathematical Physics*, vol. 45, no. 8, pp. 3339–3352, 2004.

[13] D. Baleanu, A. K. Golmankhaneh, A. K. Golmankhaneh, and R. R. Nigmatullin, "Newtonian law with memory," *Nonlinear Dynamics*, vol. 60, no. 1, pp. 81–86, 2010.

[14] Y. Liang, S. Wang, W. Chen, Z. Zhou, and R. L. Magin, "A survey of models of ultraslow diffusion in heterogeneous materials," *Applied Mechanics Reviews*, vol. 71, pp. 040802–1–16, 2019.

[15] F. Cesarone, M. Caputo, and C. Cametti, "Memory formalism in the passive diffusion across highly heterogeneous systems," *Journal of Membrane Science*, vol. 250, no. 1, pp. 79–84, 2005.

[16] M. Caputo and C. Cametti, "Diffusion with memory in two cases of biological interest.," *Journal of Theoretical Biology*, vol. 254, no. 3, pp. 697–703, 2008.

[17] Y. Povstenko, "Fractional thermoelasticity," *Encyclopedia of Thermal Stresses*, vol. 219, pp. 1778–1787, 2014.

[18] G. M. Zaslavsky, "Chaos, fractional kinetics, and anomalous transport," *Physics Reports*, vol. 371, no. 6, pp. 461–580, 2002.

[19] R. L. Magin, *Fractional Calculus in Bioengineering*. Begell House Redding, 2006.

[20] E. C. D. Oliveira and J. A. T. Machado, "A review of definitions for fractional derivatives and integral," *Mathematical Problems in Engineering*, vol. 2014, pp. 1–6, 2014.

[21] Y. Y. Gambo, F. Jarad, D. Baleanu, and T. Abdeljawad, "On Caputo modification of the Hadamard fractional derivatives," *Advances in Difference Equations*, vol. 2014, no. 1, pp. 1–12, 2014.

[22] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progr. Fract. Differ. Appl*, vol. 1, no. 2, pp. 1–13, 2015.

[23] L. Ma and C. Li, "On the integral transforms for Hadamard fractional calculus.".

[24] C. Li, W. Deng, and L. Zhao, "Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations," *arXiv preprint arXiv:1501.00376*, 2015.

[25] R. Metzler and J. Klafter, "From stretched exponential to inverse power-law: fractional dynamics, cole–cole relaxation processes, and beyond," *Journal of Non-Crystalline Solids*, vol. 305, pp. 81–87, 2002.

[26] J. Abate and W. Whitt, "A unified framework for Numerically Inverting Laplace Transforms," *Informs Journal on Computing*, vol. 18, no. 4, pp. 408–421, 2006.

[27] S. Singh, *Application of Prony Analysis to Characterize Pulsed Corona Reactor Measurements*. PhD thesis, University of Wyoming, 2003.

[28] K. Steiglitz and L. McBride, "A technique for the identification of linear systems," *IEEE Transactions on Automatic Control*, vol. 10, no. 4, pp. 461–464, 1965.