Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Continuity of the Value Function for Stochastic Sparse Optimal Control

Kaito Ito * **Takuya Ikeda** ** **Kenji Kashima** ***

* *Kyoto University, Kyoto, Japan (e-mail: ito.kaito@bode.amp.i.kyoto-u.ac.jp)*
** *The University of Kitakyushu, Kitakyushu, Japan (e-mail: t-ikeda@kitakyu-u.ac.jp)*
*** *Kyoto University, Kyoto, Japan (e-mail: kk@i.kyoto-u.ac.jp)*

Abstract: In this paper, we investigate the continuity of the value function for a stochastic sparse optimal control. The most common method to solve stochastic optimal control problems is the dynamic programming. Specifically, if the value function is smooth, it satisfies the associated Hamilton-Jacobi-Bellman (HJB) equation. However, in general, the value function for our problem is not differentiable because of the nonsmoothness of the L^0 cost functional. Instead, we can expect that the value function is a viscosity solution to the HJB equation. This paper shows the continuity of our value function as a first step for showing that the value function is a viscosity solution.

Keywords: sparsity, optimal control, stochastic systems, value function, continuity

1. INTRODUCTION

This paper considers an L^0 optimal control problem for stochastic systems. The L^0 optimal control minimizes the length of the support of controls while realizing control objectives. The resulting control signal tends to take zero values on a set of positive measures. Due to this property, the L^0 optimal control is also referred to as *sparse optimal control*. Recently, this framework has been attracting much attention because we can stop actuators for long periods of time so that the control is environmentally friendly. As a typical example, in automobiles, this control method enables us to reduce CO and CO2 emissions and fuel consumption. We can raise many applications of the framework of the sparse optimal control problem, such as actuator placements (Stadler, 2009; Herzog et al., 2012; Kunisch et al., 2014), networked control systems (Nagahara et al., 2014; Ikeda and Kashima, 2018b), multi-period investments (Boyd et al., 2014), to name a few.

In general, L^0 optimization problems are challenging to solve computationally because of the discontinuity and the nonconvexity of the L^0 cost functional. A common method for handling this issue is to relax the original problem by replacing L^0 norm by L^1 norm (Donoho, 2006). Thanks to this relaxation, we can utilize various convex optimization methods to solve it. Interestingly, under some conditions, the optimal solution for the relaxed optimization problem coincides with the one for the original L^0 optimization problem. For *deterministic* control-affine systems, the equivalence between the L^0 optimality and the L^1 optimality has been shown in (Nagahara et al., 2015). Moreover, an equivalence theorem is also derived for deterministic general linear systems including infinite-dimensional systems (Ikeda and Kashima, 2018a). On the other hand, when it comes to *stochastic* systems, the sparse optimal control problem is not studied so much yet. In (Exarchos et al., 2018), a finite horizon optimal control problem with the L^1 cost functional for stochastic systems is dealt with and the authors propose sampling-based algorithm to solve the problem utilizing forward and backward stochastic differential equations. However, it is not obvious that the L^1 optimal control provides the L^0 optimal control for stochastic systems. For this reason, we tackle the analysis of the stochastic L^0 optimal control problem.

When we deal with stochastic optimal control problems, the most popular method to analyze them is the dynamic programming. In this method, the value function, which is the optimal value of the cost functional, plays an important role in characterizing the optimal control. If the value function is smooth, it satisfies the associated Hamilton-Jacobi-Bellman (HJB) equation. However, the value function is not differentiable, in general. In such a case, we can expect that it is a viscosity solution of the HJB equation (Fleming and Soner, 2006; Yong and Zhou, 1999).

In (Ikeda and Kashima, 2019), the dynamic programming approach is applied to a deterministic L^0 optimal control problem and it has been revealed that the value function is continuous and that it is a viscosity solution of the associated HJB equation. Furthermore, an equivalence theorem between the L^0 optimality and the L^1 optimality is shown via the uniqueness of the viscosity solution. It should be emphasized that the continuity of the value function plays a fundamental role in the results and makes it easier to discuss the viscosity solution. Note also that the continuity of the value function ensures that sufficiently small changes in the initial state and the initial time of the control system result in small changes in the optimal value. This property is not clear for L^0 optimal control problems due to the discontinuity of L^0 norm. In addition, the continuity is useful to show the stability in the sense of Lyapunov when the finite-horizon optimal control of interest is extended to model predictive control (Ikeda and Nagahara, 2016). Thus, the continuity of the value function is a fundamental and important property. Towards the stochastic counterparts of the results for deterministic systems, in this paper, we prove the continuity of the value function for the stochastic sparse optimal control. The main difficulty in the analysis of our value function is that the state of the systems is unbounded due to the stochastic noise. Therefore, we require more delicate analysis compared to the deterministic case.

Organization: This paper is organized as follows: In Section 2 we give mathematical preliminaries. In Section 3 we briefly review the related work (Ikeda and Kashima, 2019) and formulate our problem. In Section 4 we derive the continuity of the value function for stochastic systems. Some concluding remarks are given in Section 5.

2. MATHEMATICAL PRELIMINARIES

This section gives notation that will be used throughout the paper.

Let N, N_1, and N_2 be positive integers. For a matrix $M \in \mathbb{R}^{N_1 \times N_2}$, $M^\top$ denotes the transpose of M. For a matrix $M \in \mathbb{R}^{N \times N}$, $\mathrm{tr}(M)$ denotes the trace of M. Denote the Frobenius norm of $M \in \mathbb{R}^{N_1 \times N_2}$ by $\|M\|$, i.e., $\|M\| \triangleq \sqrt{\mathrm{tr}(M^\top M)}$. For a vector $a = [a_1, a_2, \ldots, a_N]^\top \in \mathbb{R}^N$, we denote the Euclidean norm by $\|a\| \triangleq (\sum_{i=1}^N a_i^2)^{1/2}$, the open ball with center at a and radius $r > 0$ by $B(a,r)$, i.e., $B(a,r) \triangleq \{x \in \mathbb{R}^N : \|x - a\| < r\}$, and the closed ball with center at a and radius $r > 0$ by $\bar{B}(a,r)$, i.e., $\bar{B}(a,r) \triangleq \{x \in \mathbb{R}^N : \|x - a\| \leq r\}$. We denote the inner product of $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^N$ by $a \cdot b$.

Let $T > 0$. For $p \in \{0, 1, \infty\}$, L^p denotes the set of all continuous-time signals $u(t) = [u_1(t), u_2(t), \ldots, u_N(t)]^\top \in \mathbb{R}^N$ over a time interval $[0,T]$ such that $\|u\|_p < \infty$, where $\|\cdot\|_p$, referred to as L^p *norm*, is defined by

$$\|u\|_0 \triangleq \sum_{j=1}^N \mu_L(\{t \in [0,T] : u_j(t) \neq 0\}),$$
$$\|u\|_1 \triangleq \sum_{j=1}^N \int_0^T |u_j(t)| dt,$$
$$\|u\|_\infty \triangleq \max_{j=1,2,\ldots,N} \operatorname*{ess\,sup}_{0 \leq t \leq T} |u_j(t)|,$$

with the Lebesgue measure μ_L on $\mathbb{R}$. The L^0 norm is also expressed by $\|u\|_0 = \int_0^T \psi_0(u(t)) dt$, where $\psi_0 : \mathbb{R}^N \to \mathbb{R}$ is a function that returns the number of nonzero components, i.e.,

$$\psi_0(a) \triangleq \sum_{j=1}^m |a_j|^0$$

with $0^0 = 0$.

Let $\alpha \in (0,1]$. A function $f : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ is called α-Hölder continuous if there exists a constant $L > 0$ such that, for all $x, y \in \mathbb{R}^{N_1}$,

$$\|f(x) - f(y)\| \leq L \|x - y\|^\alpha.$$

Especially when $\alpha = 1$, f is called Lipschitz continuous. f is called locally α-Hölder continuous if for any $x \in \mathbb{R}^{N_1}$, there exists a neighborhood U_x of x such that f restricted to U_x is α-Hölder continuous.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a natural filtration $\{\mathcal{F}_s\}_{s \geq t}$, and $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$. In this paper, we omit the subscript of stochastic processes when no confusion occur, e.g., $\{X_s\} = \{X_s\}_{s \geq t}$. For $A \in \mathcal{F}$, we denote the complement of A by $A^c \triangleq \{\omega \in \Omega : \omega \notin A\}$. The expected value of a random variable X restricted to A is denoted by $\mathbb{E}[X, A] \triangleq \int_A X(\omega) d\mathbb{P}(\omega) = \mathbb{E}[X \cdot \mathbb{1}_A]$ where $\mathbb{1}_A$ is the indicator function of A, i.e.,

$$\mathbb{1}_A(\omega) = \begin{cases} 0, & \omega \notin A \\ 1, & \omega \in A \end{cases}.$$

When X is a $\mathbb{R}$-valued random variable, it holds that

$$\mathbb{E}[X, A] \leq \mathbb{E}\left[|X|^2\right]^{1/2} \cdot (\mathbb{P}(A))^{1/2} \tag{1}$$

where Hölder's inequality is applied.

3. SPARSE OPTIMAL CONTROL PROBLEM

In this section, we briefly review the L^0 optimal control problem for deterministic systems based on the discussion in (Ikeda and Kashima, 2019). Next, we formulate the L^0 optimal control problem for stochastic systems.

3.1 Review of deterministic L^0 optimal control

Consider the following deterministic control system:

$$\begin{aligned} \dot{y}(s) &= f(y(s), u(s)), \quad s > t \\ y(t) &= x, \end{aligned} \tag{2}$$

where $y(s) \in \mathbb{R}^n$ is the state variable, $u(s) \in \mathbb{R}^m$ is the control variable, $t \geq 0$ is the initial time, and $x \in \mathbb{R}^n$ is the initial state. We assume the range of the control u is constrained in a compact set $\mathbb{U} \subset \mathbb{R}^m$ that contains $0 \in \mathbb{R}^m$, i.e., $u(s) \in \mathbb{U}$ for all s, and we denote the set of all such functions by $\mathcal{U}$, i.e.,

$$\mathcal{U} \triangleq \{u \in L^\infty : u(s) \in \mathbb{U} \text{ for all } s\}.$$

We fix a finite horizon $0 < T < \infty$. For given $(x,t) \in \mathbb{R}^n \times [0,T]$ and $u \in \mathcal{U}$, we denote by $y^{t,x,u}(s)$ the state at time s with the initial condition $y(t) = x$ and a control u.

For given $x \in \mathbb{R}^n$, $T > 0$, and $t \in [0,T]$, we consider the cost functional

$$J(x,t,u) \triangleq \int_t^T \psi_0(u(s)) ds + g(y^{t,x,u}(T)),$$

where the function $g : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost and the function $\psi_0 : \mathbb{R}^m \to \mathbb{R}$ returns the number of nonzero components, which is defined in Section 2. Note that the first term expresses the L^0 cost of the control input, and hence by minimizing J, we can expect that the resulting control is sparse. To sum up, the optimal control problem is formulated as follows:

Problem 1. Given $x \in \mathbb{R}^n$, $T > 0$, and $t \in [0,T]$, find a control input u on $[t,T]$ that solves

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J(x,t,u) \\ \text{subject to} \quad & \dot{y}(s) = f(y(s), u(s)), \\ & y(t) = x, \\ & u \in \mathcal{U}. \end{aligned}$$

Here, we assume the following conditions for the dynamics $f(y,u)$ and the terminal cost $g(y)$:

(A_1) $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ is continuous;
(A_2) f is Lipschitz continuous in the state variable, uniformly in the control variable, i.e., there exists a constant L such that

$$\|f(y,u) - f(z,u)\| \leq L\|y - z\| \tag{3}$$

for all $y, z \in \mathbb{R}^n$ and $u \in \mathbb{U}$;
(A_3) $g : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Assumptions (A_1) and (A_2) guarantee the existence and the uniqueness of a solution to the differential equation (2); see (Bardi and Capuzzo-Dolcetta, 2008). Assumption (A_3) is used to guarantee the continuity of the *value function*, which is defined by

$$V(x,t) \triangleq \inf_{u \in \mathcal{U}} J(x,t,u), \quad x \in \mathbb{R}^n, \quad t \in [0,T].$$

Remark 1. In (Ikeda and Kashima, 2019), the authors deal with slightly different framework. To be more precise, the cost functional is given by

$$\tilde{J}(x,t,u) \triangleq \int_0^t \psi_0(u(s))ds + g(y(T))$$

where $y(s)$ is a solution to

$$\begin{aligned} &\dot{y}(s) = f(y(s), u(s)), \quad s > 0 \\ &y(0) = x. \end{aligned} \tag{4}$$

The corresponding value function is defined by

$$\tilde{V}(x,t) \triangleq \inf_{u \in \mathcal{U}} \tilde{J}(x,t,u), \ x \in \mathbb{R}^n, \ t \in [0,\infty).$$

Noting that $V(x,t) = \tilde{V}(x, T-t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$, the results for $\tilde{V}$ in (Ikeda and Kashima, 2019) are also true for our value function V.

For the continuity of the value function, the following holds (Ikeda and Kashima, 2019, Theorem 1):

Theorem 1. Fix $T > 0$. Under assumptions (A_1), (A_2), and (A_3), the value function V is continuous on $\mathbb{R}^n \times [0,T]$. If in addition the terminal cost g is Lipschitz continuous, then V is locally Lipschitz continuous.

Moreover, it is revealed that V is a viscosity solution to the associated HJB equation:

$$\begin{cases} -v_t(x,t) + H(x, D_x v(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T), & (5) \\ v(x,T) = g(x) \quad \text{in } \mathbb{R}^n, & (6) \end{cases}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H(x,p) \triangleq \sup_{u \in \mathbb{U}} \{-f(x,u) \cdot p - \psi_0(u)\}, \tag{7}$$

v_t denotes the partial derivative with respect to the last variable and $D_x v$ denotes the gradient with respect to the first n variables. Thanks to this property, the equivalence between the L^0 optimality and the L^1 optimality and a sufficient and necessary condition for the L^0 optimality are also derived. Furthermore, the bang-off-bang property of the optimal control is shown. We emphasize again that the continuity of the value function V makes it easier to show that V is a viscosity solution. Hence, it is a crucial step in the study of the sparse optimal control to prove that V is continuous.

3.2 Stochastic L^0 optimal control

We next formulate the sparse optimal control problem for stochastic systems. Compared to the deterministic case, more delicate analysis is required for the stochastic case. Specifically, we have to vary not only control processes but also probability spaces in order to adopt the dynamic programming approach; for details see e.g., (Yong and Zhou, 1999; Nisio, 2014). We consider the following stochastic system where the state is governed by a stochastic differential equation valued in $\mathbb{R}^n$:

$$\begin{aligned} &dy_s = f(y_s, u_s)ds + \sigma(y_s, u_s)dw_s, \quad s > t \\ &y_t = x. \end{aligned} \tag{8}$$

Here, x is a deterministic initial state. The range of the control $\mathbb{U} \subset \mathbb{R}^m$ is a compact set that contains $0 \in \mathbb{R}^m$, and we fix a finite horizon $0 < T < \infty$.

As in the deterministic case, we are interested in the optimal control that minimizes the cost functional

$$J^{\mathsf{s}}(x,t,u) \triangleq \mathbb{E}\left[\int_t^T \psi_0(u_s)ds + g(y_T)\right]. \tag{9}$$

For each fixed $t \in [0,T)$, we denote by $\mathcal{U}^{\mathsf{s}}[t,T]$ the set of all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, \{w_s\}, \{u_s\})$ satisfying the following conditions:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space,
(ii) $\{w_s\}$ is a d-dimensional Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[t,T]$ (with $w_t = 0$ almost surely),
(iii) The control $\{u_s\}$ is a $\{\mathcal{F}_s^t\}_{s \geq t}$-progressively measurable and $\mathbb{U}$-valued process on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}_s^t$ is the σ-field generated by $\{w_r : t \leq r \leq s\}$,
(iv) Under $\{u_s\}$ and for any $x \in \mathbb{R}^n$, the equation (8) has a unique solution $\{y_s\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s \geq t}, \mathbb{P})$,
(v) $g(y_T)$ is $\mathcal{F}_T^t$-measurable and $\mathbb{E}[|g(y_T)|] < +\infty$ is satisfied.

For notational simplicity, we write $u \in \mathcal{U}^{\mathsf{s}}[t,T]$ instead of $(\Omega, \mathcal{F}, \mathbb{P}, \{w_s\}, \{u_s\}) \in \mathcal{U}^{\mathsf{s}}[t,T]$. Note that in (9) the expectation $\mathbb{E}$ is with respect to $\mathbb{P}$. For given $(x,t) \in \mathbb{R}^n \times [0,T]$ and $u \in \mathcal{U}^{\mathsf{s}}[t,T]$, we denote by $y_s^{t,x,u}$ the unique solution of (8).

Then, the main problem for the stochastic case is formulated as follows:

Problem 2. Given $x \in \mathbb{R}^n$, $T > 0$, and $t \in [0,T]$, find a 5-tuple $u \in \mathcal{U}^{\mathsf{s}}[t,T]$ that solves

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J^{\mathsf{s}}(x,t,u) \\ \text{subject to} \quad & dy_s = f(y_s,u_s)ds + \sigma(y_s,u_s)dw_s, \\ & y_t = x, \\ & u \in \mathcal{U}^{\mathsf{s}}[t,T]. \end{aligned}$$

We assume the following conditions for functions f, σ, g:

(B_1) There exist positive constants L, $\bar{M}$ and a nondecreasing function $\bar{m} \in C([0,+\infty))$ such that $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n \times d}$ satisfy the following condition:

$$\begin{aligned} &\|f(y,u) - f(z,v)\| + \|\sigma(y,u) - \sigma(z,v)\| \\ &\quad \leq L\|y - z\| + \bar{m}(\|u - v\|) \end{aligned} \tag{10}$$

for all $y, z \in \mathbb{R}^n$, $u, v \in \mathbb{U}$, where $\bar{m}(\cdot) \leq \bar{M}$ and $\bar{m}(0) = 0$;

(B_2) There exist constants $\hat{C} > 0$ and $p \geq 2$ such that $g : \mathbb{R}^n \to \mathbb{R}$ satisfies the following growth condition:

$$|g(y)| \leq \hat{C}(1 + \|y\|^p) \tag{11}$$

for all $y \in \mathbb{R}^n$;

(B_3) $g : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Note that assumption (B_1) ensures the existence and pathwise uniqueness of a solution to the stochastic differential equation (8) with any deterministic initial condition $y_t = x, (t, x) \in [0, T] \times \mathbb{R}^n$ and any progressively measurable process $\{u_s\}$; see (Nisio, 2014). In addition, under assumptions (B_1) and (B_2), the cost functional $J^{\mathsf{s}}(x, t, u)$ is finite for any $(x, t, u) \in \mathbb{R}^n \times [0, T] \times \mathcal{U}^{\mathsf{s}}[t, T]$; see Remark 4. Assumption (B_3) is introduced to show the continuity of the value function defined by

$$V^{\mathsf{s}}(x, t) \triangleq \inf_{u \in \mathcal{U}^{\mathsf{s}}[t,T]} J^{\mathsf{s}}(x, t, u), \quad x \in \mathbb{R}^n, \quad t \in [0, T],$$

which is illustrated in Theorem 2.

Remark 2. In Problem 2, we vary probability spaces. This problem formulation is called a *weak formulation*. This formulation is convenient for proving the dynamic programming principle. On the other hand, the problem where we fix a probability space and vary only control processes is referred to as a *strong formulation*. It is known that, under some conditions, the value function of the weak formulation coincides with the one of the strong formulation; see (Fleming and Soner, 2006).

4. CONTINUITY OF VALUE FUNCTION

In this section, we derive the continuity of the value function V^{s}. First, we supply a lemma to estimate p-th order moments of the state governed by the stochastic system (8) (Nisio, 2014, Theorem 1.2).

Lemma 1. Assume (B_1) and let $p \geq 2$ be given. Then there exists a positive constant K_p such that the following estimates hold:

(i) For any $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\mathbb{E}\left[\sup_{t \leq s \leq T} \|y_s^{t,x,u}\|^p\right] \leq K_p(1 + \|x\|^p), \ \forall u \in \mathcal{U}^{\mathsf{s}}[t, T]. \tag{12}$$

(ii) For any $x, z \in \mathbb{R}^n$ and $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{t \leq s \leq T} \|y_s^{t,x,u} - y_s^{t,z,u}\|^p\right] \leq K_p\|x - z\|^p, \ \forall u \in \mathcal{U}^{\mathsf{s}}[t, T]. \tag{13}$$

(iii) For any $0 \leq t_1 \leq t_2 \leq T$ and $x \in \mathbb{R}^n$,

$$\mathbb{E}\left[\sup_{t_2 \leq s \leq T} \|y_s^{t_1,x,u} - y_s^{t_2,x,u}\|^p\right] \leq K_p(1 + \|x\|^p)(t_2 - t_1)^{\frac{p}{2}}, \quad \forall u \in \mathcal{U}^{\mathsf{s}}[t_1, T]. \tag{14}$$

Remark 3. By applying Hölder's inequality, we can obtain the estimates for the first order moments, that is, (12), (13) and (14) also hold for $p = 1$.

Remark 4. The estimate (12) implies $\mathbb{E}[\|y_T^{t,x,u}\|^p] < +\infty$ for any $p \geq 2$. Note also that

$$\mathbb{E}\left[\int_t^T \psi_0(u_s)ds\right] \leq m(T - t)$$

where we used the boundedness of ψ_0. Hence, the growth condition in (B_2) ensures that the cost functional $J^{\mathsf{s}}(x, t, u)$ has a finite value for any $(x, t, u) \in \mathbb{R}^n \times [0, T] \times \mathcal{U}^{\mathsf{s}}[t, T]$.

Here, we are ready to state the main result:

Theorem 2. Fix $T > 0$. Under assumptions $(B_1), (B_2)$, and (B_3), the value function V^{s} is continuous on $\mathbb{R}^n \times [0, T]$. If in addition the terminal cost g is Lipschitz continuous, then $V^{\mathsf{s}}(x, t)$ is Lipschitz continuous in x uniformly in t, and locally 1/2-Hölder continuous in t for each x.

Proof. First, we show the continuity of $V^{\mathsf{s}}(x, t)$ in t. Let $0 \leq t_1 \leq t_2 \leq T$ and fix t_1. For given $\varepsilon > 0$ and $r > 0$, fix any $x \in B(0, r)$ and $u \in \mathcal{U}^s[t_1, T]$, and set

$$R_{\varepsilon,r} \triangleq \left(K_p(1 + r^p)\varepsilon^{-1}\right)^{1/p},$$

where $p \geq 2$ and $K_p > 0$ satisfy (11) and (12)–(14), respectively. For notational simplicity, we denote $y_s^{t,x,u}$ by $y(s, t)$. If necessary, we denote it by $y(s, t; \omega)$ for $\omega \in \Omega$ explicitly. Then, for any $x \in B(0, r)$ and $t \in [0, T]$, we have

$$\mathbb{P}\left(\sup_{t \leq s \leq T} \|y(s, t)\| > R_{\varepsilon,r}\right) \leq \mathbb{E}\left[\sup_{t \leq s \leq T} \|y(s, t)\|^p\right] R_{\varepsilon,r}^{-p} < \varepsilon, \tag{15}$$

where Chebyshev's inequality and (12) are applied.

Next, choose $\delta_0 = \delta_0(\varepsilon, r)$ satisfying for any $c, d \in \bar{B}(0, R_{\varepsilon,r})$,

$$|g(c) - g(d)| < \varepsilon \tag{16}$$

whenever $\|c - d\| < \delta_0$. In fact, such δ_0 exists since g is uniformly continuous on $\bar{B}(0, R_{\varepsilon,r})$ by assumption (B_3).

Fix any $t_2 \in [t_1, T]$. Then,

$$\begin{aligned}\mathbb{P}&\left(\sup_{t_2 \leq s \leq T} \|y(s, t_1) - y(s, t_2)\| > \delta_0\right) \\ &\leq \mathbb{E}\left[\sup_{t_2 \leq s \leq T} \|y(s, t_1) - y(s, t_2)\|^p\right] \delta_0^{-p} \\ &\leq K_p(1 + \|x\|^p)(t_2 - t_1)^{\frac{p}{2}} \delta_0^{-p}\end{aligned} \tag{17}$$

holds where (14) is applied. Now, we define $\Omega_i, \tilde{\Omega}$ by

$$\Omega_i \triangleq \left\{\omega \in \Omega : \sup_{t_i \leq s \leq T} \|y(s, t_i; \omega)\| \leq R_{\varepsilon,r}\right\}, \ i = 1, 2, \tag{18}$$

$$\tilde{\Omega} \triangleq \left\{\omega \in \Omega : \sup_{t_2 \leq s \leq T} \|y(s, t_1; \omega) - y(s, t_2; \omega)\| < \delta_0\right\}. \tag{19}$$

It follows from (15) and (17) that

$$\mathbb{P}(\Omega_i) > 1 - \varepsilon, \ i = 1, 2 \tag{20}$$

and

$$\mathbb{P}(\tilde{\Omega}) \geq 1 - \varepsilon \tag{21}$$

whenever $(t_2 - t_1)^{\frac{p}{2}} \leq \varepsilon\delta_0^p K_p^{-1}(1 + \|x\|^p)^{-1}$.

Now, let us evaluate

$$I(t_1, t_2) \triangleq \left|J^{\mathsf{s}}(x, t_1, u) - J^{\mathsf{s}}(x, t_2, u)\right|. \tag{22}$$

Here, we have

$$I(t_1,t_2) = \left|\mathbb{E}\left[\int_{t_1}^{t_2}\psi_0(u_s)ds + g(y(T,t_1)) - g(y(T,t_2))\right]\right| \le m(t_2-t_1) + \underbrace{\mathbb{E}\left[|g(y(T,t_1)) - g(y(T,t_2))|\right]}_{\triangleq I_2(t_1,t_2)}. \quad (23)$$

For the second term I_2, by splitting the sample space Ω and using (B_2), we obtain

$$\begin{aligned} I_2(t_1,t_2) &= \mathbb{E}\left[|g(y(T,t_1)) - g(y(T,t_2))|, \Omega_1\cap\Omega_2\cap\tilde{\Omega}\right] \\ &\quad + \mathbb{E}\left[|g(y(T,t_1)) - g(y(T,t_2))|, \Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c\right] \\ &\le \mathbb{E}\left[|g(y(T,t_1)) - g(y(T,t_2))|, \Omega_1\cap\Omega_2\cap\tilde{\Omega}\right] \\ &\quad + \hat{C}\mathbb{E}\left[2 + \|y(T,t_1)\|^p + \|y(T,t_2)\|^p, \Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c\right] \end{aligned} \quad (24)$$

where $\hat{C} > 0$ satisfies (11). Thanks to (16), (18) and (19),

$$\mathbb{E}\left[|g(y(T,t_1)) - g(y(T,t_2))|, \Omega_1\cap\Omega_2\cap\tilde{\Omega}\right] < \varepsilon. \quad (25)$$

For the second term of the right-hand-side of (24), we have

$$\begin{aligned} &\mathbb{E}\left[\|y(T,t_i)\|^p, \Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c\right] \\ &\le \left(\mathbb{E}\left[\|y(T,t_i)\|^{2p}\right]\right)^{1/2}\cdot\left(\mathbb{P}(\Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c)\right)^{1/2}, \\ &\le \left(\mathbb{E}\left[\|y(T,t_i)\|^{2p}\right]\right)^{1/2}\cdot\left(\mathbb{P}(\Omega_1^c) + \mathbb{P}(\Omega_2^c) + \mathbb{P}(\tilde{\Omega}^c)\right)^{1/2}, \\ &\qquad i = 1,2 \end{aligned} \quad (26)$$

where (1) is applied. Using (20), (21), (12) and noting that $\sqrt{1+r^{2p}} < 1 + r^p$, we get

$$\mathbb{E}\left[\|y(T,t_i)\|^p, \Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c\right] < \sqrt{K_{2p}}\cdot\sqrt{3\varepsilon}\,(1+r^p) \quad (27)$$

whenever

$$t_2 - t_1 < \left(\frac{\varepsilon}{K_p(1+r^p)}\right)^{\frac{2}{p}}\delta_0^2. \quad (28)$$

Moreover, under (28), it holds that

$$\hat{C}\mathbb{E}[2, \Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c] = 2\hat{C}\cdot\mathbb{P}(\Omega_1^c\cup\Omega_2^c\cup\tilde{\Omega}^c) < 6\hat{C}\varepsilon. \quad (29)$$

By (24), (25), (27) and (29), we can choose $c_0 > 0$ satisfying

$$I_2(t_1,t_2) < \begin{cases} c_0\sqrt{\varepsilon}(1+r^p), & \text{if } \varepsilon < 1 \\ c_0\varepsilon(1+r^p), & \text{if } \varepsilon \ge 1 \end{cases}$$

whenever (28) holds. This evaluation and (23) imply that, for any $\varepsilon > 0$ and $r > 0$, we can find $\delta_1 = \delta_1(\varepsilon, r) > 0$ such that, for all $u \in \mathcal{U}^{\mathsf{s}}[t_1,T]$ and $x \in B(0,r)$,

$$|J^{\mathsf{s}}(x,t_1,u) - J^{\mathsf{s}}(x,t_2,u)| < \varepsilon \quad (30)$$

whenever $t_2 - t_1 < \delta_1$. Note that, for any $x, z \in \mathbb{R}^n$ and $0 \le t_1 \le t_2 \le T$,

$$|V^{\mathsf{s}}(x,t_1) - V^{\mathsf{s}}(z,t_2)| \le \inf_{u\in\mathcal{U}^{\mathsf{s}}[t_1,T]} |J^{\mathsf{s}}(x,t_1,u) - J^{\mathsf{s}}(z,t_2,u)|. \quad (31)$$

Therefore, (30) implies the continuity of $V^{\mathsf{s}}(x,t)$ in t.

Next, we show that $V^{\mathsf{s}}(x,t)$ is continuous in x. Fix any $t \in [0,T]$, $u \in \mathcal{U}^s[t,T]$ and $z \in B(0,r)$. For simplicity, we denote $y_s^{t,x,u}$ by $y(s,x)$. For the same $R_{\varepsilon,r}$ and δ_0 as above, we define $\Omega_x, \tilde{\Omega}_x$ by

$$\Omega_x \triangleq \left\{\omega\in\Omega : \sup_{t\le s\le T}\|y(s,x;\omega)\| \le R_{\varepsilon,r}\right\} \text{ for } x \in B(0,r),$$
$$\tilde{\Omega}_x \triangleq \left\{\omega\in\Omega : \sup_{t\le s\le T}\|y(s,x;\omega) - y(s,z;\omega)\| < \delta_0\right\}.$$

It follows from the same discussion as above that, for any $x \in B(0,r)$,

$$\mathbb{P}(\Omega_x) > 1-\varepsilon,\ \mathbb{P}(\Omega_z) > 1-\varepsilon \quad (32)$$

and

$$\mathbb{P}(\tilde{\Omega}_x) \ge 1-\varepsilon \quad (33)$$

whenever

$$\|x - z\|^p \le \varepsilon\delta_0^p K_p^{-1}.$$

Thus, we can evaluate

$$\tilde{I}(x,z) \triangleq |J^{\mathsf{s}}(x,t,u) - J^{\mathsf{s}}(z,t,u)| \le \mathbb{E}\left[|g(y(T,x)) - g(y(T,z))|\right]$$

similarly to the above discussion, that is, we can choose $c_1 > 0$ satisfying

$$\tilde{I}(x,z) < \begin{cases} c_1\sqrt{\varepsilon}(1+r^p), & \text{if } \varepsilon < 1 \\ c_1\varepsilon(1+r^p), & \text{if } \varepsilon \ge 1 \end{cases}$$

whenever

$$x, z \in B(0,r) \text{ and } \|x - z\| < \left(\frac{\varepsilon}{K_p}\right)^{\frac{1}{p}}\delta_0.$$

This implies that, for any $\varepsilon > 0$ and $r > 0$, we can find $\delta_2 = \delta_2(\varepsilon, r) > 0$ such that, for all $t \in [0,T]$ and $u \in \mathcal{U}^{\mathsf{s}}[t,T]$,

$$|J^{\mathsf{s}}(x,t,u) - J^{\mathsf{s}}(z,t,u)| < \varepsilon \quad (34)$$

whenever $x, z \in B(0,r)$ and $\|x - z\| < \delta_2$. By using (31) again, we get the continuity of $V^{\mathsf{s}}(x,t)$ in x. The continuity of $V^{\mathsf{s}}(x,t)$ in t and in x leads to the continuity in (x,t) since

$$\begin{aligned} &|V^{\mathsf{s}}(x,t_1) - V^{\mathsf{s}}(z,t_2)| \\ &\le |V^{\mathsf{s}}(x,t_1) - V^{\mathsf{s}}(x,t_2) + V^{\mathsf{s}}(x,t_2) - V^{\mathsf{s}}(z,t_2)| \\ &\le |V^{\mathsf{s}}(x,t_1) - V^{\mathsf{s}}(x,t_2)| + |V^{\mathsf{s}}(x,t_2) - V^{\mathsf{s}}(z,t_2)| \end{aligned}$$

holds for any $x, z \in \mathbb{R}^n$ and $t_1, t_2 \in [0,T]$.

Next, let g be Lipschitz continuous. Fix any $(x,t) \in \mathbb{R}^n \times [0,T]$ and take any bounded neighborhood $D_{x,t}$ that contains (x,t). Take any $(w,s) \in D_{x,t}$ and $(z,\tau) \in D_{x,t}$ such that $\tau \le s$. For given $\varepsilon > 0$, we can take $\bar{u} \in \mathcal{U}^{\mathsf{s}}[\tau,T]$ satisfying

$$V^{\mathsf{s}}(z,\tau) + \varepsilon \ge J^{\mathsf{s}}(z,\tau,\bar{u})$$

by definition. Then we have

$$\begin{aligned} V^{\mathsf{s}}(w,s) - V^{\mathsf{s}}(z,\tau) &\le J^{\mathsf{s}}(w,s,\bar{u}) - J^{\mathsf{s}}(z,\tau,\bar{u}) + \varepsilon \\ &= \mathbb{E}\left[\int_\tau^s \psi_0(u_s)ds + g(y_T^{s,w,\bar{u}}) - g(y_T^{\tau,z,\bar{u}})\right] + \varepsilon \\ &\le m(s-\tau) + G\mathbb{E}\left[\|y_T^{s,w,\bar{u}} - y_T^{\tau,z,\bar{u}}\|\right] + \varepsilon \end{aligned} \quad (35)$$

where $G > 0$ is the Lipschitz constant of g. For the second term of the right-hand-side of (35), it holds that

$$\begin{aligned} &\mathbb{E}\left[\|y_T^{s,w,\bar{u}} - y_T^{\tau,z,\bar{u}}\|\right] \\ &= \mathbb{E}\left[\|y_T^{s,w,\bar{u}} - y_T^{\tau,w,\bar{u}} + y_T^{\tau,w,\bar{u}} - y_T^{\tau,z,\bar{u}}\|\right] \\ &\le K_1(1+\|w\|)(s-\tau)^{\frac{1}{2}} + K_1\|w - z\| \\ &\le K_1(1+\bar{r})(s-\tau)^{\frac{1}{2}} + K_1\|w - z\| \end{aligned}$$

where we apply (13) and (14), and define $\bar{r} \triangleq \sup\{\|w\| : (w,s) \in D_{x,t} \text{ for some } s\}$, which is finite from the boundedness of $D_{x,t}$. Therefore, we obtain

$$\begin{aligned} &V^{\mathsf{s}}(w,s) - V^{\mathsf{s}}(z,\tau) \\ &\le m|s-\tau| + G\left(K_1(1+\bar{r})|s-\tau|^{\frac{1}{2}} + K_1\|w-z\|\right) + \varepsilon \\ &\le \bar{G}|s-\tau|^{\frac{1}{2}} + GK_1\|w-z\| + \varepsilon \end{aligned}$$

where $\bar{G} > 0$ depends on $D_{x,t}$. Note that GK_1 does not depend on $D_{x,t}$. The similar discussion for $V^{\mathsf{s}}(z,\tau) - V^{\mathsf{s}}(w,s)$ holds. The arbitrariness of ε completes the proof. □

Remark 5. Note that the Lipschitz continuity of g shows the local Lipschitz continuity of the value function V in the deterministic case (Theorem 1). This property ensures that V is differentiable almost everywhere. On the other hand, we cannot expect the local Lipschitz continuity of the value function V^{s} in the stochastic case even under the Lipschitz continuity of g. This is essentially because $\int_0^t \sigma dw$ is only of order $t^{1/2}$.

5. CONCLUSION

In this paper, we formulated a stochastic L^0 optimal control problem and derived the continuity of the value function for this problem. The continuity is nontrivial due to the discontinuity of L^0 norm. The obtained continuity helps us to show that the value function is a viscosity solution of the associated HJB equation and to characterize the stochastic L^0 optimal control. This will be discussed in future work.

ACKNOWLEDGEMENTS

This work was supported in part by JSPS KAKENHI under Grant Number JP18H01461 and by JST, ACT-X under Grant Number JPMJAX1902.

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