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A New Extended State Observer with Low Sensitivity to High Frequency Noise and Low Gain Power

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Abstract: Linear Extended State Observer (LESO) with a large gain w_o leads to faster estimation error convergence. But a high gain LESO is sensitive to high frequency measurement noise and its digital implementation is rather complex. To address the above issues, for a given n-th order plant, a new $2n$-th order extended state observer is proposed. The proposed observer, while preserving the fast convergence property of LESO, has a lower gain power of 2 and is less sensitive to high frequency measurement noise. And the new observer can be easily tuned with gain w_o.

Keywords: Extended state observer, measurement noise, low gain power.

1. INTRODUCTION

Disturbance Observer is a class of observers which represent all uncertainties, including external disturbances and model uncertainties, as an augmented state(Chen et al., 2016). For these observes, the estimation error must decay fast and keep its value sufficiently small so that the estimated states can be used for the consequent control implementation. Extended State Observer (ESO), as one type of disturbance observer, not only estimating uncertainties but also the normal states, is the most essential component of Active Disturbance Rejection Control (ADRC) scheme. ADRC has been widely adopted in recent years due to its simplicity and satisfactory control performance (Huang and Xue, 2014). The successful industrial applications of ADRC include, but not limited to, multi-zone temperature regulation (Gao, 2015), air fuel ratio control in gasoline engines (Xue et al., 2015), permanent magnet synchronous motor control (Sira-Ramírez et al., 2014), super-heated steam temperature control (Wu et al., 2019), temperature control of a crystallizer (Liu et al., 2019). On the theoretical side, the performance recovery property of Extended High State Observer (EHGO) and the convergence of a general nonlinear ESO have been studied in Freidovich and Khalil (2008) and Guo and Zhao (2011) respectively. A detailed research about how to deal with disturbance and/or uncertainties via ESO has been presented in Chen et al. (2020).

Linear Extended State Observer (LESO), proposed in Gao (2003), has been widely used due to its fast convergence rate and simple tuning procedure. There exists, however, two problems of LESO when applied to real-world applications:

(1) There is a trade off between fast error convergence and high frequency noise sensitivity with a high gain w_o. Since the estimated states will be used to generate the control signal, estimated states with too much noise may even destabilize the system.
(2) For an n-th order LESO, its gain has the highest power of $n+1$. As the order of LESO goes higher, greater demands are placed on the digital controller due to rapidly growing value w_o^{n+1} ($w_o > 1$). For a fixed time step size implementation, the computation error will become much larger as the parameter value grows exponentially.

To solve the noise-amplification problem, many work has been done. In Madoński and Herman (2012), a disturbance observer consisting of an additional integral state and Generalized Proportional Integral Observer was proposed. In Zhao and Guo (2017), a nonlinear ESO based on fractional power functions was comprehensively investigated and simulation results showed a better noise tolerance compared to LESO. However, the complexity will inevitably increase and meanwhile the stability margin may be reduced in these methods owing to the introduction of nonlinear functions and additional parts.

In Astolfi and Marconi (2015), a novel high gain observer (HGO) design is proposed, which consists of $n-1$ segments and each segment is a second order observer. This observer solves the high gain-power problem of traditional HGO and achieves better high-frequency noise attenuation. Inspired by this idea, some novel observers design methods are presented in Khalil (2017b) and Teel (2016). Most recently, Wang and Kellett (2019) used an extended low-power state observer to achieve feedback-linearization for noise-free systems.

Inspired by Astolfi and Marconi (2015), this paper proposes a new ESO named Low Power Extended State

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Observer (LPESO). LPESO enhances the observer performance in presence of high frequency noise and relaxes the requirement for digital implementation. Similar to the observer-bandwidth analogy from Gao (2003), we simplify the parameter-procedure of LPESO based on the algorithm from Astolfi and Marconi (2015) to make it a one-parameter tuning observer. The convergence of LPESO is proved by perturbation theory. Frequency domain analysis of both LPESO and LESO with measurement noise is also presented. The main contributions of this paper are as follows:

(1) A new $2n$-th order Extended State Observer is proposed, which is less sensitive to high frequency noise than LESO and has gain with maximum power of 2.
(2) Simple parameters-selection method is provided which is similar to LESO.
(3) Fast exponential error-decay of LPESO is proved.
(4) Frequency domain analysis in terms of the impact of high frequency noise, for general LPESO is given and compared with LESO.

The rest of this paper are organized as follows: The basic design of LESO and LPESO are presented in section 2. Convergence of LPESO is studied in section 3. Frequency domain analysis of both LESO and LPESO are given in section 4. A parameter setting method and simulation results are presented in section 5. Conclusions are discussed in Section 6.

2. PROBLEM FORMULATION AND OBSERVER DESIGN

Consider a n-th order single-input-single-output system:

$$\begin{array}{lll} \dot{x}_i(t) & = x_{i+1}(t), & i = 1,2,...,n-1 \\ \dot{x}_n(t) & = f(x(t)) + d(t) + bu(t) & \\ y(t) & = x_1(t) + \nu(t) & \end{array} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in D \subset R^n$ is the state vector, $d(t) \in R$ is bounded external disturbance, b is a positive constant. $y(t) \in R$ is the measured output with bounded measurement noise $\nu(t) \in R, |\nu(t)| \leq N$. D is a domain containing the origin. $f(x(t))$ is an unknown function which is continuously differentiable in x over any compact set $D_0 \subset D$. $u(t)$ is continuously differentiable in t and bounded. In the rest of this paper, we will write these variables in a concise format without symbol t.

Remark 1. The assumption of bounded x and u can be removed when studying the stability of the closed-loop feedback system, but it is required on the convergence property of ESO.

Since $f(x)$ and d are both unknown, model-based control method might be unable to obtain a satisfactory performance. A question comes up immediately that, can we design an observer to estimate these unknown parts and compensate their impacts on the output by feedback control. This is the basic concept of ADRC and a well-performed ESO is required.

According to Gao (2003), ESO can be designed by augmenting the system with an additional state x_{n+1}, which represents the unknown part, $x_{n+1} = f(x) + (b - b_0)u$, where $b_0 > 0$ is a constant parameter. This leads to the following equivalent extended state model:

$$\begin{array}{lll} \dot{x}_i & = x_{i+1}, & i = 1,2,...,n-1 \\ \dot{x}_n & = x_{n+1} + b_0 u & \\ \dot{x}_{n+1} & = \dfrac{\partial f}{\partial x}\dot{x} + (b - b_0)\dfrac{\mathrm{d}u}{\mathrm{d}t} + \dot{d} \overset{def}{=} g(t,x) & \end{array} \tag{2}$$

where $g(t,x)$ is bounded with

$$|g(t,x)| \leq M, \quad \forall (t,x) \in [0,\infty) \times D$$

Based on (2), LESO and LPESO can be designed accordingly.

The LESO from Gao (2003) is:

$$\begin{array}{ll} \dot{z}_i = z_{i+1} + \alpha_i w_o^i (y - z_1), & i = 1,2,...,n-1 \\ \dot{z}_n = z_{n+1} + \alpha_n w_o^n (y - z_1) + b_0 u & \\ \dot{z}_{n+1} = \alpha_{n+1} w_o^{n+1} (y - z_1) & \end{array} \tag{3}$$

where oberver states $z = (z_1, ..., z_{n+1})^T$. $\alpha_i \ (i = 1,...,n+1)$ are *observer parameters* and w_o is called the *observer gain.*

LPESO designed for system (2) is given by:

$$\begin{array}{ll} \dot{\xi}_{i,1} = \xi_{i,2} + k_{i,1} w_o e_i, & i = 1,2,...,n-2 \\ \dot{\xi}_{i,2} = \xi_{i+1,2} + k_{i,2} w_o^2 e_i, & i = 1,2,...,n-2 \\ \dot{\xi}_{n-1,1} = \xi_{n-1,2} + k_{n-1,1} w_o e_{n-1} & \\ \dot{\xi}_{n-1,2} = \xi_{n,2} + k_{n-1,2} w_o^2 e_{n-1} + b_0 u & \\ \dot{\xi}_{n,1} = \xi_{n,2} + k_{n,1} w_o e_n + b_0 u & \\ \dot{\xi}_{n,2} = k_{n,2} w_o^2 e_n & \end{array} \tag{4}$$

where $e_1 = y - \xi_{1,1}$ and $e_i = \xi_{i-1,2} - \xi_{i,1}, i = 2,3,...,n$. $K_i = (k_{i,1}, k_{i,2})^T, i = 1,2,...,n$ are *observer parameters* and their tuning rule will be discussed in Section 5. Extracting $n+1$ states as the approximation of x by

$$\hat{x} = L\xi$$

where $\xi\,(\xi_{1,1}, \xi_{1,2}, \cdots, \xi_{2n,1}, \xi_{2n,2})^T \in R^{2n}$ and L is a block diagonal matrix defined as

$$L = diag\left([1\ \ 0], \cdots, [1\ \ 0], I_2\right)_{(n+1)\times 2n}$$

As can be seen from equation (4), the gain w_o has a maximum power of 2. On the other hand, from (3), the gain w_o from LESO has maximum order of $n+1$. Since w_o usually satisfies $w_o \gg 1$, this improvement will increase the accuracy when applied for numerical implementation for a fixed step size.

Because the LESO has the analogous structure as HGO, the convergence bound of LESO can be stated in the following theorem (Khalil, 2017a).

Theorem 1. For system (2) and LESO (3), if $\alpha_i, i = 1,2,...,n+1$ are chosen such that

$$s^{n+1} + \alpha_1 s^n + ... + \alpha_n s + \alpha_{n+1}$$

is Hurwitz polynomial, the estimation error $\tilde{x}_i = z_i - x_i$ satisfies inequality

$$|\tilde{x}_i| \leq max\left\{c_1 w_o^{i-1} e^{-c_2 w_o t}\,\|\tilde{x}(0)\|, c_3\frac{M}{w_o^{n-i+2}} + c_4 w_o^{i-1} N\right\}$$

for $(i = 1,2...,n+1)$ where $c_i, i = 1,2,3,4$ are positive constants. M is a positive constant corresponding to the modeling error.

From Theorem 1, if $N = 0$, *i.e.* there is no measurement noise, by increasing w_o, the effect of initial state and modelling mismatch will decrease exponentially with time. In the next section, we will show that the newly proposed LPESO possesses the same property.

3. CONVERGENCE OF LPESO

Before presenting the main result, some matrices need to be defined. Let $A, B, C, D, J, B_{2n}, \bar{K}$ be

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = (1 \ 0) \quad D = \begin{bmatrix} w_o & 0 \\ 0 & w_o^2 \end{bmatrix}$$

$$J = \begin{bmatrix} E_1 & N & 0 & \cdots & 0 \\ Q_2 & E_2 & N & \ddots & \vdots \\ 0 & Q_i & E_i & N & 0 \\ \vdots & \ddots & \ddots & \ddots & N \\ 0 & \cdots & 0 & Q_n & E_n \end{bmatrix}, B_{2n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \bar{K} = \begin{pmatrix} k_{1,1} \\ k_{1,2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where E_i, Q_i, and N are matrices defined as

$$E_i = A - K_i C, \quad Q_i = \begin{bmatrix} 0 & k_{i,1} \\ 0 & k_{i,2} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem 2. For system (2) and LPESO (4), if matrix J is a Hurwitz matrix, the estimation error $\tilde{x}_i = \hat{x}_i - x_i$ satisfies

$$|\tilde{x}_i| \le max \left\{ c_1 w_o^{i-1} e^{-c_2 w_o t} \|\tilde{x}(0)\|, c_3 \frac{M}{w_o^{n-i+2}} + c_4 w_o^{i-1} N \right\}$$

for $(i = 1, 2..., n+1)$ where $c_i, i = 1, 2, 3, 4$ are positive constants. M is a positive constant corresponding to the modeling error.

Proof. Changing variables by $\tilde{\xi}_i = \xi_i - (x_i, x_{i+1})^T$ with $\xi_i = (\xi_{i,1}, \xi_{i,2})^T$ will transform the system into

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= (A - DK_1C)\tilde{\xi}_1 + N\tilde{\xi}_2 + DK_1\nu(t) \\ \dot{\tilde{\xi}}_i &= (A - DK_iC)\tilde{\xi}_i + N\tilde{\xi}_{i+1} + DK_iB^T\tilde{\xi}_{i-1} \\ & \quad i = 2, \cdots, n-1 \\ \dot{\tilde{\xi}}_n &= (A - DK_nC)\xi_n + DK_nB^T\tilde{\xi}_{n-1} - Bg(t,x) \end{aligned}$$

Scaling the estimation error by

$$\eta_i = w_o^{2-i} D^{-1} \tilde{\xi}_i$$

Through some calculation, equation (5) is obtained.

$$\begin{aligned} \dot{\eta}_1 &= w_o(A - K_1C))\eta_1 + w_oN\eta_2 + w_oI_2K_1\nu(t) \\ \dot{\eta}_i &= w_o(A - K_iC)\eta_i + w_oN\eta_{i+1} + w_oQ_i\eta_{i-1} \\ & \quad i = 2, \cdots, n-1 \\ \dot{\eta}_n &= w_o(A - K_nC)\eta_n + w_oQ_n\eta_{n-1} - w_o^{-n}Bg(t,x) \end{aligned} \tag{5}$$

The state space equation will be

$$\dot{\eta} = w_oJ\eta + w_o\bar{K}v(t) - w_o^{-n}B_{2n}g(t,x) \tag{6}$$

Consider equation (7) as the target system

$$\dot{\eta} = w_oJ\eta \tag{7}$$

Suppose J is Hurwitz matrix, by Lyapunov converse theorem (Khalil, 2002), there exists a positive definite quadratic function $V(\eta) = \eta^T P\eta$,where P is the solution of Lyapunov equation

$$PJ + J^TP = -I$$

Thus, the derivative of V along the trajectory (6) will be

$$\begin{aligned} \dot{V}(\eta) &= -w_o\|\eta\|^2 + 2\eta^T \left[Pw_o\bar{K}\nu - Pw_o^{-n}B_{2n}g(t,x)\right] \\ &\le -w_o\|\eta\|^2 + 2Nw_o\left\|P\bar{K}\right\|\|\eta\| + 2Mw_o^{-n}\|PB_{2n}\|\|\eta\| \\ &\le -w_o(1-\theta)\|\eta\|^2 \\ &\quad - \|\eta\|\left(\theta w_o\|\eta\| - 2\lambda_1Nw_o - 2\lambda_2Mw_o^{-n}\right) \\ &\le -w_o(1-\theta)\|\eta\|^2, \forall\|\eta\| \ge \frac{2N\lambda_1}{\theta} + \frac{2M\lambda_2}{\theta w_o^{n+1}} \end{aligned} \tag{8}$$

for some positive constant $\theta \in (0,1)$.

$$\dot{V}(\eta) \le -w_o(1-\theta)\frac{V(\eta)}{\lambda_{max}}, \forall\|\eta\| \ge \frac{2N\lambda_1}{\theta} + \frac{2M\lambda_2}{\theta w_o^{n+1}}$$

By the comparison lemma (Khalil, 2002), $V(\eta)$ satisfies

$$V(\eta) \le e^{-\frac{w_o(1-\theta)}{\lambda_{max}(P)}t} V(\eta(0))$$

The definite function $V(\eta)$ also satisfies inequality

$$\lambda_{min}(P)\|\eta\|^2 \le V(\eta) \le \lambda_{max}(P)\|\eta\|^2$$

Hence, η satisfies the inequality

$$\begin{aligned} \|\eta\| &\le \frac{1}{2}c_1e^{-c_2w_ot}\|\eta(0)\| & ,\forall t \le T_2 \\ \|\eta\| &\le \frac{2N\lambda_1}{\theta} + \frac{2M\lambda_2}{\theta w_o^{n+1}} & ,\forall t > T_2 \end{aligned} \tag{9}$$

for some $T_2 > 0$. According to the definition of η

$$\begin{aligned} \|\eta(0)\| &\le \|2\tilde{x}(0)\| \\ |\tilde{x}_i| &\le w_o^{i-1}\|\eta\| \end{aligned} \tag{10}$$

Substituting inequalities (10) into (9) obtains

$$|\tilde{x}_i| \le max \left\{ c_1 w_o^{i-1} e^{-c_2 w_o t} \|\tilde{x}(0)\|, c_3 \frac{M}{w_o^{n-i+2}} + c_4 w_o^{i-1} N \right\}$$

The proof is completed.

Remark 2. Theorem 2 shows that similar to LESO, the LPESO proposed in this paper also has the exponentially error-decay property. The decay speed could be arbitrary large by choosing a large w_o.

In system model (1), the measurement noise $\nu(t) \le N$ is a bounded noise. However, in many applications, the measurement noise is better modeled a colored noise with high frequency components. In the next section, we will study the noise attenuation property of LESO and LPESO in high frequency.

4. FREQUENCY DOMAIN ANALYSIS

Scaling variables by $\epsilon_i = w_o^{n+1-i}\tilde{x}_i$ leads to a rewritten error dynamics (11) for LESO (3) and system (1).

$$\dot{\epsilon} = w_oF\epsilon + \Gamma w_o^{n+1}\nu(t) - B_{n+1}g(t,x) \tag{11}$$

where F, Γ, B_{n+1} are matrices defined as

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -\alpha_{n+1} & 0 & \cdots & 0 \end{bmatrix}, \Gamma = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n+1} \end{pmatrix}, B_{n+1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Substituting $g(t,x) = 0$ into equation (11) and (6) gives

$$\begin{aligned} \dot{\epsilon} &= w_oF\epsilon + \Gamma w_o^{n+1}\nu(t) \\ \dot{\eta} &= w_oJ\eta + w_o\bar{K}\nu(t) \end{aligned} \tag{12}$$

Consider the estimation errors of LESO and LPESO as $\tilde{x}_i = z_i - x_i = \frac{\epsilon_i}{w_o^{n+1-i}}, \tilde{x}_i = \hat{x}_i - x_i = \frac{\eta_i}{w_o^{i-1}}, 1 \le i \le n+1$ as the output, and $\nu(t)$ as the input. We will calculate the transfer functions $G_i(s) = \frac{\mathcal{L}[\tilde{x}_i(t)]}{\mathcal{L}[\nu(t)]}$ for LESO $T_i(s) = \frac{\mathcal{L}[\tilde{x}_i(t)]}{\mathcal{L}[\nu(t)]}$ for LPESO, respectively.

For LESO, transfer functions between ν and $\tilde{x}_i$ are

$$G_i(s) = \frac{\mathcal{L}[\tilde{x}_i(t)]}{\mathcal{L}[\nu(t)]} = \frac{1}{w_o^{n+1-i}}C_i(sI - w_oF)^{-1}\Gamma w_o^{n+1}$$

where $C_i \in R^{n+1}$ is the vector with $C_i(i) = 1$ and 0s at other places. $\mathcal{L}$ is the *Laplace operator*. Since our main concern is about the high frequency characteristics, only the highest order of nominator and denominator matters. In particular, the highest order of nominator of $(sI - w_o F)^{-1}$ is located at the diagonal. Let $p^i(s)$ represents a ith order monic polynomial of s. Taking $F_s = (sI - w_o F)$ will get:

$$F_s^{-1} = \begin{bmatrix} F_{s,11} & \cdots & F_{s,1(n+1)} \\ \vdots & \ddots & \vdots \\ F_{s,(n+1)1} & \cdots & F_{s,(n+1)(n+1)} \end{bmatrix} \frac{1}{p^{n+1}(s)}$$

$$F_{s,ii} = \begin{vmatrix} s+\alpha_1 w_o & -w_o & 0 & | & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & | & \ddots & \ddots & \ddots \\ \alpha_{i-1} w_o & 0 & s & | & 0 & 0 & \cdots \\ - & - & - & | & - & - & - \\ \alpha_{i+1} w_o & & 0 & | & s & -w_o & 0 \\ \vdots & & & | & & s & -w_o \\ \alpha_{n+1} w_o & \cdots & \cdots & | & & 0 & s \end{vmatrix} \tag{13}$$

Then, it follows that

$$\begin{aligned} F_{s,ii} &= s^{n+1-i}\left\{ s\left[s(\cdot) + \cdots\right] + \alpha_{i-1}(-1)^{2i-2}(w_o)^{i-1} \right\} \\ &= s^n + \beta_1 w_o s^{n-1} + \cdots + \beta_{i-1}(w_o)s^{n+1-i} = p^n(s) \end{aligned}$$

Hence, the transfer functions $G_i(s)$ can be approximated as equation (14) when noise frequency satisfies $\omega \gg w_o$.

$$G_i(s) = w_o^i \frac{\alpha_i p^n(s)}{p^{n+1}(s)} \approx \frac{\alpha_i w_o^i}{s} \quad , i = 1, \cdots, n+1 \tag{14}$$

For LPESO, since the $\bar{K}$ has only two nonzero elements $k_{1,1}$, and $k_{1,2}$, we only need to calculate the fist two columns of $J_s^{-1} = (sI - w_o J)^{-1}$.

$$T_i(s) = \frac{\mathcal{L}[\tilde{x}_i(t)]}{\mathcal{L}[\nu(t)]} = -C_i L_i w_o^i (sI - w_o J)^{-1} \bar{K}$$

for $1 \le i \le n+1$, where L_i is the ith row of L.

$$J_s^{-1} = \begin{bmatrix} J_{s,11} & J_{s,12} & \cdots & * \\ \vdots & \vdots & \cdots & * \\ J_{s,(2n)1} & J_{s,(2n)2} & \cdots & * \end{bmatrix} \frac{1}{p^{2n}(s)}$$

It is easy to verify that

$$\begin{aligned} J_{s,11} &= -\frac{s}{w_o} J_{s,12} = p^{2n-1}(s) \\ J_{s,(2i-1)2} &= -\frac{s + w_o k_{1,1}}{w_o k_{1,2}} J_{s,(2i-1)1}, \quad 2 \le i \le n \\ J_{s,(2n)2} &= -\frac{s + w_o k_{1,1}}{w_o k_{1,2}} J_{s,(2n)1} \end{aligned}$$

Hence, we only need calculate $J_{s,(2i-1)1}$. By the property

$$\begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix} = |\mathcal{A}| \left|\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right|$$

There is

$$\begin{aligned} J_{s,(2n)1} &= -w_o k_{1,2} \prod_{i=2}^{n} (w_o k_{i,2} s) = -w_o^n s^{n-1} \prod_{i=1}^{n} k_{i,2} \\ J_{s,(2i-1)1} &= (-1)^{2i} w_o k_{1,2} \left(|\mathcal{A}| \left|\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right|\right) \end{aligned}$$

where $\mathcal{A} = diag(A_2, \cdots, A_{i-1}, A')$, with

$$A_j = \begin{bmatrix} -w_o k_{j,1} & s + w_o k_{j,1} \\ -w_o k_{j,2} & w_o k_{j,2} \end{bmatrix}, A' = \begin{bmatrix} -w_o k_{i,1} & -w_o \\ -w_o k_{i,2} & s \end{bmatrix}$$

$$|\mathcal{D}| = \begin{vmatrix} sI - w_o E_{i+1} & -w_o N & \cdots & 0 \\ -w_o Q_{i+2} & sI - w_o E_{i+2} & \ddots & \vdots \\ \ddots & \ddots & \ddots & -w_o N \\ 0 & \cdots & -w_o Q_n & sI - w_o E_n \end{vmatrix} = p^{2(n-1)}(s)$$

$\mathcal{C} \in R^{(2n-2i)\times(2i-2)}$ is a zero matrix with $\mathcal{C}(1, 2i-2) = -w_o k_{i+1,1}$, $\mathcal{C}(2, 2i-2) = -w_o k_{i+1,2}$. $\mathcal{B} \in R^{(2i-2)\times(2n-2i)}$ is a zero matrix with $\mathcal{B}(2i-2, 1) = -w_o$.

$$\mathcal{C}\mathcal{A}^{-1}\mathcal{B} = \begin{bmatrix} w_o^2 k_{i+1,1} \dfrac{p^{i-2}(s)}{p^{i-1}(s)} & 0 & * & 0 \\ w_o^2 k_{i+1,2} \dfrac{p^{i-2}(s)}{p^{i-1}(s)} & 0 & * & \vdots \\ \vdots & & * & * & 0 \\ 0 & & \cdots & * & 0 \end{bmatrix}$$

Hence, $\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}$ is a block diagonal matrix. By the Bauer and Fike Theorem (Horn and Johnson, 2012), when $|s|$ is sufficiently large,

$$J_{s,(2i-1)1} \approx -w_o^i p^{i-1}(s) \left(\prod_{j=1}^{i-1} k_{j,2}\right) k_{i,1} p^{2(n-i)}(s)$$

and the transfer functions approximates equation (15)

$$\begin{aligned} T_1(s) &\approx \frac{k_{1,1} w_o}{s} \\ T_i(s) &\approx \left(\prod_{j=1}^{i-1} k_{j,2}\right) k_{i,1} w_o^{2i-1} \frac{1}{s^i}, \quad 2 \le i \le n \\ T_{n+1}(s) &\approx \left(\prod_{j=1}^{n} k_{j,2}\right) w_o^{2n} \frac{1}{s^n} \end{aligned} \tag{15}$$

Table 1 shows the comparison between LPESO and LESO designed for nth order system.

Table 1. Approximate $T_i(s)$(LPESO) and $G_i(s)$(LESO) for high frequency noise

Order	LPESO	LESO
1	$\frac{k_{1,1} w_o}{s}$	$\frac{\alpha_1 w_o}{s}$
$i = 2, \cdots, n$	$\frac{k_{i,1} w_o^{2i-1} \prod_{j=1}^{i-1} k_{j,2}}{s^i}$	$\frac{\alpha_i w_o^i}{s}$
n+1	$\frac{w_o^{2n} \prod_{j=1}^{n} k_{j,2}}{s^n}$	$\frac{\alpha_{n+1} w_o^{n+1}}{s}$

Remark 3. From table 1, the LESO is a 1-st order low pass filter while LPESO is an i-th order low pass filter. It is natural to conclude that if frequency is sufficiently high, LPESO achieves better noise attenuation than LESO. This result coincides with the results in Astolfi and Marconi (2015) ($|T_i(s)| \le \frac{c_i}{s^i}$), but is more detailed.

5. PARAMETER-SELECTION AND SIMULATION RESULTS

In this section, a parameter selection rule of LPESO is introduced first. Then we will use the bode-plot of $|T_{n+1}(s)|$ and $|G_{n+1}(s)|$ to verify the accuracy of results in Section 4, which shows that the approximation error is quite small. At the end of the section, a simulation of LPESO and LESO based ADRC is presented to illustrate the advantage of LPESO.

5.1 Parameter-selection

Since the matrix J is rather complex when n is large, the computation of $K = \left(K_1^T, \cdots, K_n^T\right)^T$ will impose unnecessary burdens on engineers. Inspired by bandwidth-tuning method proposed in Gao (2003), by selecting K appropriately, characteristic polynomial of J could satisfy

$$|sI - J| = (s+1)^{2n} \tag{16}$$

Thus, all elements of K and Γ will be deterministic for each n. Based on the algorithm in Astolfi and Marconi (2015), here gives the parameter selection rule of the $2n$ order LPESO

$$k_{i,1} = 2 \quad ,1 \le i \le n$$
$$k_{i,2} = \frac{n+1-i}{i} \quad ,1 \le i \le n$$

Once K is determined, we only need to tune w_o to follow the rule-of-thumb: Select an appropriately large w_o so that the estimation error decay fast enough while the control signal should not be too noisy at the same time.

5.2 Bode-plot of LESO and LPESO

Following the parameter-selection method in Section 5.1 and Gao (2003) respectively, parameters of a 6-th order LPESO and a 4-th order LESO designed for nominal 3-rd order plant (17) are given in (18) with bandwidth (gain) $w_o = 50, b_0 = 1$.

$$\dddot{x}_1 = u, \quad y = x_1 + \nu(t) \tag{17}$$

$$K = [2,3,2,1,2,1/3]^T, \quad \Gamma = [4,6,4,1]^T \tag{18}$$

Fig. 1 shows the magnitude bode-plot of $T_{n+1}(s)$ and $G_{n+1}(s)$. It is seen that LPESO has a better high frequency noise attenuation than LESO.

As shown in table (2), the approximation error, which is between the actual magnitude and approximate magnitude obtained from table (1), is rather small.

Table 2. $|T_{n+1}(\omega)|$ and $|G_{n+1}(\omega)|$ obtained through MATLAB function and table (1), respectively. ("appro"=approximate)

fequency(rad/s)	Magnitude(dB)			
	LPESO		LESO	
ω	actual	appro	actual	appro
300	54.6	55.2	85.9	86.4
1000	23.7	23.9	75.8	75.9
10000	-36.1	-36.1	55.9	55.9

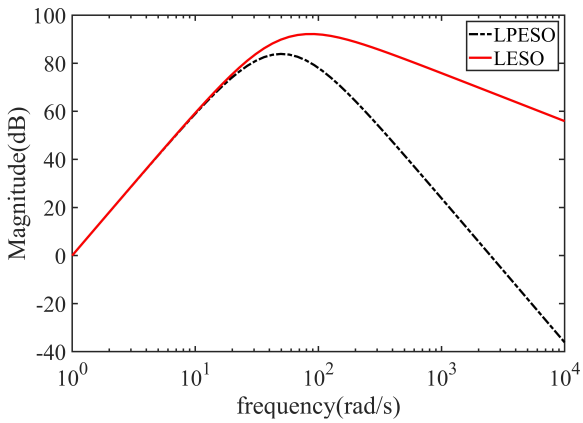

Fig. 1. Bode magnitude plot of $T_{n+1}(s)$ and $G_{n+1}(s)$.

5.3 LPESO based motion control

We use the motion control testbed from Gao (2003) to compare the performance of LESO and LPESO, the system is modelled as:

$$\ddot{y} = (-1.41\dot{y} + 23.2T_d) + 23.2u$$

where y is the output position, u is the control voltage sent to the power amplifier that drives the motor, T_d is the torque disturbance. The control requirements are: (1)$|u| < 3.5$, (2) the control signal should be smooth with noise level limited to ± 100 mv. (3) There is a step torque disturbance from $T_d = 0$ to $T_d = -2.32$ at $t = 2s$ which should be rejected with fast transient.

Let

$$x_1 = y,\ x_2 = \dot{y},\ x_3 = (b_0 - 23.2)u - 1.41\dot{y} + 23.2T_d$$

Take $\bar{x}_i = z_i$ for LESO and $\bar{x}_i = Ł_i\xi$ for LPESO as the estimation of $x_i\,(i = 1,2,3)$, respectively.

According to Gao (2003), feedback law is

$$u = \left[-w_c^2(\bar{x}_1 - r) - 2w_c\bar{x}_2 - \bar{x}_3\right]/b_0$$

where r is the reference signal.

With sufficiently small estimation error, the closed loop system can be approximated as

$$\ddot{y} \approx -w_c^2(y - r) - 2w_c\dot{y}$$

Larger w_c leads to faster transient and bigger control signal. To attenuate the effect of peaking phenomenon (Khalil, 2017b), we use a saturation function (19) to limit the amplitude of u .

$$sat(u) = \begin{cases} -3.5 & u < -3.5 \\ u & |u| \le 3.5 \\ 3.5 & u > 3.5 \end{cases} \tag{19}$$

The parameters of two controllers are chosen as

$$w_o^{LPESO} = 80, K = [2,2,2,1/2]^T, w_c = 20, b_0 = 40$$
$$w_o^{LESO} = 40, \Gamma = [3,3,1]^T, w_c = 20, b_0 = 40$$

In the simulation, output measurement is corrupted by white noise through a high-pass filter $\frac{s}{s+500}$ and noise power is set to 1×10^{-9}. The sampling time is 0.001s.

From Fig. 2, it can be seen that the control signal from LPESO based strategy is less noisy and achieves better disturbance rejection performance. Table 3 shows the ∞-

Norm of estimation error $\tilde{x}_i, i = 1, 2, 3$, which verifies the results in Section 4.

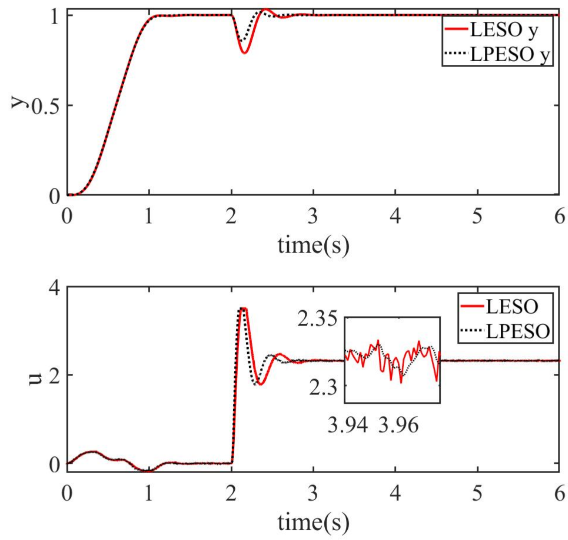

Fig. 2. Comparison of output response y and control signal u under LESO and LPESO strategy.

Table 3. ∞-Norm of estimation error during 4s to 6s (steady state), $\|x_i(t)\|_\infty = \max_{t\in[4\ 6]} |x_i(t)|$

Estimation error	LPESO	LESO
$\|x_1(t) - \hat{x}_1(t)\|_\infty$	0.0021	0.0015
$\|x_2(t) - \hat{x}_2(t)\|_\infty$	0.0515	0.0623
$\|x_3(t) - \hat{x}_3(t)\|_\infty$	0.5550	0.8338

6. CONCLUSIONS

A new low power extended state observer has been proposed. It possesses the same fast convergence as the standard LESO while having a lower gain power, and better high frequency measurement noise attenuation. A parameters-selection method is presented which makes LPESO a one-parameter tuning observer. Convergence analysis by Lyapunov methods is presented to verify the arbitrary fast convergence property. The high frequency noise attenuation feature is verified through frequency domain analysis for a general $2n$ order LPESO. Since LPESO has better noise attenuation ability than LESO, a bigger w_0 can be selected which results in a better disturbance rejection performance.

REFERENCES

Astolfi, D. and Marconi, L. (2015). A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11), 3059–3064.

Chen, S., Bai, W., Hu, Y., Huang, Y., and Gao, Z. (2020). On the conceptualization of total disturbance and its profound implications. *Science China Information Sciences*, 63(2).

Chen, W., Yang, J., Guo, L., and Li, S. (2016). Disturbance-Observer-Based Control and Related Methods - An Overview. *IEEE Transactions on Industrial Electronics*, 63(2), 1083–1095.

Freidovich, L.B. and Khalil, H.K. (2008). Performance recovery of feedback-linearization-based designs. *IEEE Transactions on Automatic Control*, 53(10), 2324–2334.

Gao, Z. (2003). Scaling and bandwidth-parameterization based controller tuning. *Proceedings of the 2003 American Control Conference, 2003.*, 6, 4989–4996.

Gao, Z. (2015). Active Disturbance Rejection Control From an Enduring Idea to an Emerging Technology. *Communications in Information and Systems*, 15(3), 361–421.

Guo, B.Z. and Zhao, Z.L. (2011). On the convergence of an extended state observer for nonlinear systems with uncertainty. *Systems and Control Letters*, 60(6), 420–430.

Horn, R.A. and Johnson, C.R. (2012). *Matrix analysis*. Cambridge university press.

Huang, Y. and Xue, W. (2014). Active disturbance rejection control: Methodology and theoretical analysis. *ISA Transactions*, 53(4), 963–976.

Khalil, H.H. (2017a). *High-gain observers in nonlinear feedback control*, volume 31. SIAM.

Khalil, H.K. (2002). Nonlinear systems. *Upper Saddle River*.

Khalil, H.K. (2017b). Cascade high-gain observers in output feedback control. *Automatica*, 80, 110–118.

Liu, T., Hao, S., and Li, D. (2019). Predictor-Based Disturbance Rejection Control for Sampled Systems with Input Delay. *IEEE Transactions on Control Systems Technology*, 27(2), 772–780.

Madoński, R. and Herman, P. (2012). Method of sensor noise attenuation in high-gain observers - Experimental verification on two laboratory systems. *2012 IEEE International Symposium on Robotic and Sensors Environments, ROSE 2012 - Proceedings*, 121–126.

Sira-Ramírez, H., Linares-Flores, J., García-Rodríguez, C., and Contreras-Ordaz, M.A. (2014). On the control of the permanent magnet synchronous motor: An active disturbance rejection control approach. *IEEE Transactions on Control Systems Technology*, 22(5), 2056–2063.

Teel, A.R. (2016). Further variants of the Astolfi/Marconi high-gain observer. *Proceedings of the American Control Conference*, 2016-July, 993–998.

Wang, L. and Kellett, C.M. (2019). Robust Output Feedback Stabilization of Multivariable Invertible Nonlinear Systems: A Feedback Linearization-Based Method. *arXiv e-prints*, arXiv:1910.01903.

Wu, Z., He, T., Li, D., Xue, Y., Sun, L., and Sun, L. (2019). Superheated steam temperature control based on modified active disturbance rejection control. *Control Engineering Practice*, 83(January 2018), 83–97.

Xue, W., Bai, W., Yang, S., Song, K., Huang, Y., and Xie, H. (2015). ADRC with Adaptive Extended State Observer and its Application to Air-Fuel Ratio Control in Gasoline Engines. *IEEE Transactions on Industrial Electronics*, 62(9), 5847–5857.

Zhao, Z.L. and Guo, B.Z. (2017). A nonlinear extended state observer based on fractional power functions. *Automatica*, 81(April), 286–296.