Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Boundary stabilization and disturbance rejection for a time fractional order diffusion-wave equation ⋆

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Abstract: In this paper, we study the boundary stabilization and disturbance rejection for an unstable time fractional diffusion-wave equation involving Caputo time fractional derivative. When there is no boundary external disturbance, both state feedback control and output feedback control via boundary actuation are proposed by the classical backstepping method. It is proved that the state feedback makes the closed-loop system Mittag-Leffler stable while the output feedback makes the closed-loop system asymptotically stable. When there is boundary external disturbance, we propose a disturbance estimator which is constructed by two infinite dimensional auxiliary systems to recover the external disturbance. The resulting closed-loop system is Mittag-Leffler stable and the states of all subsystem involved are uniformly bounded. As a byproduct, we solve rigorously completely the two longtime unsolved problems raised in [Nonlinear Dynam., 38(2004), 339-354] where all the results are only verified by simulations.

Keywords: Disturbance rejection, fractional diffusion-wave equation, Mittag-Leffler stability.

1. INTRODUCTION

Pioneering work on boundary stabilization for fractional partial differential equations (PDEs) can be tracked back to the 2004 paper Liang et al. (2004), where time fractional diffusion-wave equation is studied and the system is described by

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) = u_{xx}(x,t), \ \ x \in (0,1), \ t \geq 0, \\ u(0,t) = 0, \quad u_x(1,t) = U(t), \ t \geq 0, \\ u(x,0) = w_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{1}$$

which could be considered as a "special" cable, possibly made with special smart materials, fixed at one end, and stabilized by a boundary controller at the other end. In Liang et al. (2004), the boundary controller for (1) is designed $U(t) = -ku_t(1,t)$ with $k > 0$, which is a classical negative velocity feedback control law based on the passive principle for the wave equation. The performance and properties of a fractional order boundary controller given by $U(t) = -k_0^C D_t^{\mu} u(1,t)$ with $k > 0$ for system (1) are verified. However, we notice that the effectiveness of the above results are only illustrated by numerical simulations without rigorous proof due to the lack of effective mathematical tool. For system (1) with $\alpha = 2$ and $U(t) = -k_0^C D_t^{\mu} u(1,t), k > 0$ in Mbodje and Gerard (1995), the asymptotical stability is rigorously proved by the LaSalle's invariance principle. The study of stabilization of fractional time derivative PDEs catches more and more attention. Recently, both the state feedback and output feedback for unstable time fractional reaction diffusion equations is developed in Zhou and Guo (2018) by utilizing the Riesz basis method and backstepping method. The output stabilization of fractional reaction diffusion with spatially-varying diffusion coefficient is developed in Chen et al. (2018). The Mittag-Leffler convergent observer based feedback control of a coupled semilinear subdiffusion systems is expolored in Ge et al. (2018). Mittag-Leffler stabilization for unstable fractional hyperbolic system can be founded in Lv et al. (2019). However, the key techniques obtaining the stability in Zhou and Guo (2018); Chen et al. (2018); Ge et al. (2018); Lv et al. (2019) are based on the fractional Lyapunov method firstly established in Li et al. (2009) where the concept of Mittag-Leffler stability is introduced and on a useful fractional derivative inequality: ${}_0^C D_t^{\alpha} x^2(t) \leq 2x(t){}_0^C D_t^{\alpha} x(t)$ for $\alpha \in (0,1)$, which is not known until 2014 in Aguila-Camacho et al. (2014). For the fractional Halanay inequalities with time-varying delay, we refer to He et al. (2018). However, when

⋆ This work was partially supported by the National Natural Science Foundation of China (Nos.61803386,61903087,61873260) and the Natural Science Foundation of Guangdong Province (Nos. 2018A030310357 and 2018A1660005).

$\alpha \in (1,2)$, the inequality ${}_0^C D_t^\alpha x^2(t) \le 2x(t){}_0^C D_t^\alpha x(t)$ fails to hold, which can be seen from a simple counter-example. Taking $\alpha = 1.5$ and $x(t) = t$, then ${}_0^C D_t^\alpha x^2(t) = \frac{2}{\Gamma(1.5)} t^{1.5}$ and $x(t){}_0^C D_t^\alpha x(t) \equiv 0$, which implies ${}_0^C D_t^\alpha x^2(t) > 2x(t){}_0^C D_t^\alpha x(t) = 0$ for $t > 0$. Therefore, the method used in Zhou and Guo (2018); Chen et al. (2018); Lv et al. (2019) is not applicable for fractional wave equation.

In many situations, system operated in the environment with uncertainties may suffer from the external disturbance. To maintain the stability of system, the designed control law must be required to be robust against the disturbance or uncertainty in some extent. There are many effective control methods to handle the uncertainty, such as adaptive control, sliding model control, active disturbance rejection control (ADRC). The investigation of the ADRC for fractional PDEs with fractional derivative order $\alpha \in (0,1)$ is also available in our recent work Zhou et al. (2019) but ADRC for the case where $\alpha \in (1,2)$ has not yet studied. Noting the disturbance rejection for fractional PDEs with $\alpha \in (1,2)$ is verified in Liang et al. (2004) by numerical simulations without mathematical proof, in this paper we will provide a rigorous proof.

The paper is organized as follows: Section 2 is devoted to the problem formation and prelimaries. In Section 3 and 4, we propose a state feedback controller and an output feedback controller, respectively. Finally, the disturbance rejection control scheme is designed in Section 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

The paper Liang et al. (2004) left our two open problems:

Problem I: Can we give an appropriate control law U to stabilize system (1) and rigorously prove the closed-loop system being asymptotical stable?

The boundary $u_x(1,t) = U(t)$ of system (1) is replaced by $u_x(1,t) = U(t) + n(t)$.

Problem II: Can we propose an output feedback control law to stabilize system (1) by rejecting the noise $n(t)$? Also, provide a rigorous proof to the stability of the closed-loop system.

To answer the above problems, in this paper, we consider the more general unstable time fractional diffusion-wave equation with Neumann boundary control governed by

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \lambda(x) u(x,t), \quad x \in (0,1), \\ u(0,t) = 0, \ t \ge 0, \\ u_x(1,t) = U(t), \ t \ge 0, \\ u(x,0) = w_0(x), \ u_t(x,0) = u_1(x), \ 0 \le x \le 1, \end{cases} \tag{2}$$

where $\alpha \in (1,2)$ is the order of the fractional derivative, $u(x,t)$ is the displacement of wave propagation, $\lambda(x)$ is a continuous function on $[0,1]$. $U(t)$ is the input (control). ${}_0^C D_t^\alpha u(x,t)$ stands for the Caputo derivative with respect to time variable t, defined as ${}_0^C D_t^\alpha u(x,t) = I^{2-\alpha}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]$, where $I^{2-\alpha}$ is the Riemann-Liouville fractional integral operator given by

$$I^{2-\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u(x,s) \mathrm{d}s.$$

It is well known from the definition of the Caputo derivative that $\lim_{\alpha \to 2^-} {}_0^C D_t^\alpha w(x,t) = \frac{\partial^2 w(x,t)}{\partial t^2}$. To analyze the stability, we recall two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined as $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}$, with $\alpha, \beta > 0$, where $\Gamma(\cdot)$ is the gamma function. Taking $\beta = 1$, we define one-parameter Mittag-Leffler function $E_\alpha(z) := E_{\alpha,1}(z)$. It is easy to see that $E_1(x) = e^x$, $E_2(-x^2) = \cos(x)$ and $E_{2,2}(-x^2) = \frac{\sin(x)}{x}$ for all $x \in \mathbb{R}$.

System (2) without control input is unstable, which can be seen from the following example. Let $\lambda(x) = \frac{\pi^2}{2}$. Let the initial value be $u_0(x) = \sin(\frac{\pi}{2}x)$, $u_1(x) = \sin(\frac{\pi}{2}x)$. Then, system (2) without control admits a solution given by

$$u(x,t) = E_\alpha\Big(\frac{\pi^2}{4} t^\alpha\Big) \sin\Big(\frac{\pi}{2} x\Big) + t E_{\alpha,2}\Big(\frac{\pi^2}{4} t^\alpha\Big) \sin\Big(\frac{\pi}{2} x\Big),$$

which indicates that both $\|u(\cdot,t)\|_{L^2(0,1)} \to \infty$ and $\|u_t(\cdot,t)\|_{L^2(0,1)} \to \infty$ as $t \to \infty$. From this, we can see that an appropriate control must be posed at the control end to ensure the asymptotical stability of the closed-loop system of (2).

The first objective of the paper is to design a control law $U(t)$ to stabilize (2) and to present a rigourous mathematical proof. Obviously, once this objective is achieved, just by taking $\lambda(x) = 0$, we would give a complete answer for problem I.

The following lemmas play a key role in the proof of the stability of the closed-loop system.

Lemma 2.1. (Podlubny, 1999, Theorem 1.6) Let $\alpha \in (0,2)$ and $\beta \in \mathbb{R}$. For $\frac{\pi}{2}\alpha < \mu < \min\{\pi, \pi\alpha\}$, then there exists a constant $M = M(\alpha,\beta,\mu)$ such that

$$|E_{\alpha,\beta}(z)| \le \frac{M}{1+|z|}, \quad \text{for all } z \in \mathbb{C} \text{ with } \mu \le |\arg(z)| \le \pi,$$

where $E_{\alpha,\beta}(z)$ is a Mittag-Leffler function with double parameters.

Lemma 2.2. Let $\alpha \in (0,2)$ and $\beta \in \mathbb{R}$. For $\frac{\pi}{2}\alpha < \mu < \min\{\pi, \pi\alpha\}$, then there exists a constant $M = M(\alpha,\mu)$ such that for all $z \in \mathbb{C}$ with $\mu \le |\arg(z)| \le \pi$

$$|E_{\alpha,\alpha-1}(z)| \le \frac{M}{1+|z|^2}, \quad |E_{\alpha,\alpha}(z)| \le \frac{M}{1+|z|^2}, \tag{3}$$

where $E_{\alpha,\beta}(z)$ is a Mittag-Leffler function with double parameters.

Remark 2.1. The importance of Lemma 2.2 lies in that Lemma 2.2 gives a more precise asymptotic behavior estimation than Lemma 2.1 as $z \to \infty$. This precise estimation will be used in the proof of Theorem 4.1. Moreover, from the proof of Lemma 2.2, constant M depends on α, i.e., $M = M(\alpha)$. It is worth stressing that $M(\alpha) \to +\infty$ as $\alpha \to 2^-$. Otherwise, $M(2) < +\infty$, since $E_{2,1}(-x) = \cos\sqrt{x}$ for all $x \ge 0$, we get

$$|\cos\sqrt{x}| = |E_{2,1}(-x)| \le \frac{M(2)}{1+|x|^2}. \tag{4}$$

Taking $x = (2n\pi)^2$ in (4) and letting $n \to +\infty$, we obtain $1 \le 0$, which is a contradiction. Also, from $E_{2,2}(-x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$, we can see that $\frac{1+x^2}{\sqrt{x}} \sin\sqrt{x} \le M(2)$. Taking $x = (2n\pi + \pi/2)^2$ and letting $n \to +\infty$ result in $+\infty < M(2) < +\infty$. This results in a contradiction again.

3. BOUNDARY STABILIZATION WITH STATE FEEDBACK SCHEME

In this section, we apply the backstepping approach to design a state feedback stabilizing controller for system (2). Motivated by Smyshlyaev and Krstic (2010); Zhou and Guo (2018), we introduce a transformation $u \to w$:

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y, \tag{5}$$

to map system (2) into the following equivalent system:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t), \ \ x \in (0,1), \ t \geq 0, \\ w(0,t) = 0, \quad w_x(1,t) = 0, \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{6}$$

which is asymptotically stable. Now we determine the kernel function $k(x,y)$ of (5). For this purpose, taking Caputo's fractional derivative for (5) and using the first equation of (2), through performing the integration by parts, we obtain

$$\begin{aligned} &{}_0^C D_t^\alpha w(x,t) = {}_0^C D_t^\alpha u(x,t) - \big(k(x,x)u_x(x,t) \\ &-k(x,0)u_x(0,t) - [k_y(x,x)u(x,t) - k_y(x,0)u(0,t)]\big) \\ &- \int_0^x (k_{yy}(x,y) + \lambda(y)k(x,y))u(y,t)\mathrm{d}y \end{aligned} \tag{7}$$

and

$$\begin{aligned} &w_{xx}(x,t) = u_{xx}(x,t) - \frac{d}{dx}(k(x,x))u(x,t) - k(x,x) \\ &\times u_x(x,t) - k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)\mathrm{d}y. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (6), it follows that the kernel function $k(x,y)$ should satisfy the following partial differential equation:

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \lambda(y)k(x,y), \\ k(x,0) = 0, \ \ k(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x \lambda(y)\mathrm{d}y. \end{cases} \tag{9}$$

By (Smyshlyaev and Krstic, 2010, Theorem 2.1), the PDE (9) has a unique solution $k \in C^2(\bar{\mathcal{F}})$ where $\mathcal{F} := \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$. Especially, if $\lambda(x) = \lambda$ is a constant, then its unique explicit solution is given by $k(x,y) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}$ where $I_1(x)$ is a first-order modified Bessel function of the first kind given by $I_1(x) = \sum_{n=0}^\infty \frac{x^{2n+1}}{2^{2n+1}n!(n+1)!}$. To find the inverse of transform (5), suppose

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\mathrm{d}y. \tag{10}$$

Analogously, one can get that the kernel function $l(x,y)$ satisfies the following partial differential equation:

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda(x)l(x,y), \\ l(x,0) = 0, \ \ l(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x \lambda(y)\mathrm{d}y, \end{cases} \tag{11}$$

which has a unique solution $l \in C^2(\bar{\mathcal{F}})$.

Next, we state and prove a Lemma to show the asymptotic stability of system (6).

Lemma 3.1. Let $\alpha \in (1,2)$. For any initial value $(w_0, w_1) \in [L^2(0,1)]^2$, system (6) admits a unique solution $w(\cdot,t) \in C(0,\infty; L^2(0,1))$ and there exists a constant $M > 0$ such that

$$\|w(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{M}{1+t^{2\alpha-2}}\|(w_0,w_1)\|^2_{[L^2(0,1)]^2}. \tag{12}$$

Moreover, assuming that $(w_0,w_1) \in [H^2(0,1)]^2$, then there exists a constant $M' > 0$ such that

$$\|(w(\cdot,t), w_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M'}{1+t^{2\alpha-2}}\|(w_0,w_1)\|^2_{[H^2(0,1)]^2}.$$

Proof. Define the operator $A : D(A)(\subset L^2(0,1)) \to L^2(0,1)$ as follows: $[Af](x) = f''(x)$ with $D(A) = \{f \in H^2(0,1) | \ f(0) = 0, f'(1) = 0\}$. A simple computation shows that A is self-adjoint in $L^2(0,1)$ with the eigen-pairs $\{\mu_j, e_j(x)\}$ given by

$$\mu_j = -\big(j+\frac{1}{2}\big)^2\pi^2, \ e_j(x) = \sqrt{2}\sin\big(j+\frac{1}{2}\big)\pi x, \ j \in \mathbb{N}. \tag{13}$$

Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, the solution of (6) can be expressed as

$$w(x,t) = \sum_{j=0}^\infty \varphi_j(t)e_j(x) \tag{14}$$

with the initial value given by $w_0(x) = \sum_{j=0}^\infty a_je_j(x), w_1(x) = \sum_{j=0}^\infty b_je_j(x)$, where the coefficients a_j, b_j can be computed by $a_j = \int_0^1 e_j(x)w_0(x)\mathrm{d}x$, $\quad b_j = \int_0^1 e_j(x)w_1(x)\mathrm{d}x$. Since $w_0, w_1 \in L^2(0,1)$, we know that $\{a_j\}, \{b_j\} \in l^2$. Moreover, $\|\{a_j\}\|_{l^2} = \|w_0\|_{L^2(0,1)}$ and $\|\{b_j\}\|_{l^2} = \|w_1\|_{L^2(0,1)}$. It is seen that $\varphi_j \in C(0,\infty;\mathbb{R})$ in (14) should satisfy the following linear fractional differential equation:

$${}_0^C D_t^\alpha \varphi_j(t) = \mu_j\varphi_j(t), \varphi_j(0) = a_j, \varphi_j'(0) = b_j, \ j \in \mathbb{N}, \tag{15}$$

The solution of (15) is found to be $\varphi_j(t) = a_jE_\alpha(\mu_jt^\alpha) + b_jtE_{\alpha,2}(\mu_jt^\alpha)$. Thus, the solution of (6) is finally given by

$$w(x,t) = \sum_{j=0}^\infty \big[a_jE_\alpha(\mu_jt^\alpha) + b_jtE_{\alpha,2}(\mu_jt^\alpha)\big]e_j(x). \tag{16}$$

From Lemma 2.1, there exists a consant $C_1 > 0$ such that for all $t \geq 0$,

$$E_\alpha(\mu_jt^\alpha) \leq \frac{C_1}{1-\mu_jt^\alpha}, \quad tE_{\alpha,2}(\mu_jt^\alpha) \leq \frac{C_1t}{1-\mu_jt^\alpha}. \tag{17}$$

Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, it follows from (16) and (17) that

$$\begin{aligned} &\|w(\cdot,t)\|^2_{L^2(0,1)} = \sum_{j=0}^\infty \big[a_jE_\alpha(\mu_jt^\alpha) + b_jtE_{\alpha,2}(\mu_jt^\alpha)\big]^2 \\ &\leq 2\frac{C_1^2(1+t^2)}{(1-\mu_0t^\alpha)^2}\sum_{j=0}^\infty [a_j^2 + b_j^2] \\ &\leq \frac{M}{1+t^{2\alpha-2}}\|(w_0,w_1)\|^2_{[L^2(0,1)]^2}. \end{aligned} \tag{18}$$

In (18), we used the fact that $\frac{C_1^2(1+t^2)}{(1-\mu_0t^\alpha)^2} \leq \frac{M}{1+t^{2\alpha-2}}$ for some $M > 0$ due to $\lim_{t\to\infty} 2\frac{C_1^2(1+t^2)}{(1-\mu_0t^\alpha)^2}(1+t^{2\alpha-2}) = 2C_1^2/\mu_0^2$.

Next, we suppose that $(w_0,w_1) \in [H^2(0,1)]^2$, then $\{j^2a_j\}, \{j^2b_j\} \in l^2$. From (16), we get $w_t(x,t)$ and the rest of the proof is similar to the asymptotical estimation of $w(x,t)$. The detail is omitted. □

Remark 3.1. From Lemma 3.1, for the stabilization problem of system (1) considered in Liang et al. (2004) the controller can be taken as $U(t) \equiv 0$ if there is no disturbance.

That is, nothing is done, the system (1) is automatically asymptotically stable. It is seen that lemma 3.1 provides a rigorous mathematical proof to the stability of (1) raised in Liang et al. (2004).

Remark 3.2. It is worth emphasizing that the asymptotical stability of (6) does not hold when fractional order takes $\alpha = 2$. To see this, when $\alpha = 2$, system (6) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ w(0,t) = 0, \quad w_x(1,t) = 0, \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{19}$$

which is a classical wave equation. Since Mittag-Leffler function has the properties: $E_2(-x^2) = \cos(x), E_{2,2}(-x^2) = \frac{\sin(x)}{x}, \forall x \geq 0$ from (16), we have that the solution of (19) is explicitly solved by

$$w(x,t) = \sum_{j=0}^{\infty} \left[a_j E_\alpha(\mu_j t^\alpha) + b_j t E_{\alpha,2}(\mu_j t^\alpha)\right] e_j(x)$$

$$= \sum_{j=0}^{\infty} \left[a_j \cos\left(j + \frac{1}{2}\right)\pi t + b_j \frac{\sin\left(j+\frac{1}{2}\right)\pi t}{(j+\frac{1}{2})\pi}\right] e_j(x)$$

This means that in the fractional case, the effect of the natural frequency of the system dies out with the passage of time, which displays a damping feature and is different from the integer-order case $\alpha = 2$ in (19).

We propose the following state feedback control law:

$$U(t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y. \tag{20}$$

under which, the closed-loop system of (2) becomes

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \\ u(0,t) = 0, \ t \geq 0, \\ u_x(1,t) = k(1,1)u(1,t) + \displaystyle\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{21}$$

Theorem 3.1. For any initial value $(u_0, u_1) \in [H^2(0,1)]^2$, the closed-loop system (21) admits a unique solution $(u, u_t) \in C(0,\infty; [L^2(0,1)]^2)$. Moreover, there exist two positive constants $M > 0$ such that

$$\|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M}{1+t^{2\alpha-2}} \|(u_0,u_1)\|^2_{[H^2(0,1)]^2}.$$

By the invertibility of transformation (5), Theorem 3.1 can be proved by Lemma 3.1 and the detail is omitted. □

Remark 3.3. Theorem 3.1 implies that (21) is Mittag-Leffler stable. Indeed, for any $\beta \in (0,1)$, we have $\Gamma(1-\beta) \geq \Gamma(1) = 1$ and $\frac{\Gamma(1-\beta)M}{\Gamma(1-\beta)+\Gamma(1-\beta)t^{2\alpha-2}} \leq \frac{\Gamma(1-\beta)M}{1+\Gamma(1-\beta)t^{2\alpha-2}}$. By the well-known inequality of Mittag-Leffler function for $x \geq 0$, $\frac{1}{1+\Gamma(1-\alpha)x} \leq E_\alpha(-x) \leq \frac{1}{1+\Gamma(1+\alpha)^{-1}x}$,it follows that for $M' = \Gamma(1-\beta)M\|(u_0,u_1)\|^2_{[H^2(0,1)]^2}$,

$$\|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq M' E_\beta(-t^{2\alpha-2}).$$

which, jointly with the definition of Mittag-Leffler stability Li et al. (2009), gives Mittag-Leffler stability of (21). It is worth noting that Mittag-Leffler stability implies the asymptotical stability and not conversely.

4. OBSERVER AND CONTROLLER DESIGN FOR (2)

We propose the following observer for system (2):

$$\begin{cases} {}_0^C D_t^\alpha \widehat{u}(x,t) = \widehat{u}_{xx}(x,t) + \lambda(x)\widehat{u}(x,t) \\ \qquad +p_1(x)[\widehat{u}(1,t) - u(1,t)], \\ \widehat{u}(0,t) = 0, \ t \geq 0, \\ \widehat{u}_x(1,t) = p_0[\widehat{u}(1,t) - u(1,t)] + U(t), \ t \geq 0, \\ \widehat{u}(x,0) = \widehat{u}_0(x), \ \widehat{u}_t(x,0) = \widehat{u}_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{22}$$

where $x \in [0,1]$, $t \geq 0$ and $\widehat{u}_0(x), \widehat{u}_1(x) \in H^2(0,1)$ are the initial states. $p_1(x)$ and p_0 are the observer gains that should be designed to make the observer error $\widehat{u}(x,t) - u(x,t)$ achieve to zero in certain norm sense as $t \to \infty$.

Let $\widetilde{u}(x,t) = \widehat{u}(x,t) - u(x,t)$, it is straightforward to verify that $\widetilde{u}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{u}(x,t) = \widetilde{u}_{xx}(x,t) + \lambda(x)\widetilde{u}(x,t) + p_1(x)\widetilde{u}(1,t), \\ \widetilde{u}(0,t) = 0, \quad \widetilde{u}_x(1,t) = p_0\widetilde{u}(1,t), \ t \geq 0. \end{cases} \tag{23}$$

To seek the observer gains $p_1(x)$ and p_0, we introduce the transformation:

$$\widetilde{u}(x,t) = z(x,t) - \int_x^1 p(x,y)z(y,t)\mathrm{d}y, \tag{24}$$

that converts (23) into the following system:

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \\ z(0,t) = z_x(1,t) = 0, \end{cases} \tag{25}$$

which, by (12) and Remark 3.3, is Mittag-Leffler stable. With a length compuation, we have that $p(x,y)$ satisfies the following partial differential equation:

$$\begin{cases} p_{yy}(x,y) - p_{xx}(x,y) = \lambda(x)p(x,y), \\ p(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x \lambda(\xi)\mathrm{d}\xi, \quad p(0,y) = 0, \end{cases} \tag{26}$$

which has a unique solution, and the observer gains should be chosen as $p_1(x) = p_y(x,1), \quad p_0 = p(1,1)$.

With the transformation (24), from Lemma 3.1, we obtain immediately the following convergence for observer (22).

Lemma 4.1. For any control input $U \in L^2_{loc}(0,\infty)$ and initial state $(u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [L^2(0,1)]^4$, the closed-loop system (23) admits a unique solution and there exists a constant M depending only on the initial state $\widetilde{u} \in C(0,\infty; L^2(0,1))$

$$\|\widetilde{u}(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{M}{1+t^{2\alpha-2}}. \tag{27}$$

Moreover, assuming that $(u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [H^2(0,1)]^4$, then there exists a constant $M' > 0$ such that

$$\|(\widetilde{u}(\cdot,t), \widetilde{u}_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M'}{1+t^{2\alpha-2}}. \tag{28}$$

Since by observer (22), we obtain an approximate $\widehat{u}(x,t)$ of the state $u(x,t)$, a natural output feedback control, inspired by state feedback (20), should be designed as follows

$$U(t) = k(1,1)\widehat{u}(1,t) + \int_0^1 k_x(1,y)\widehat{u}(y,t)\mathrm{d}y. \tag{29}$$

This is an observer-based control law that just replaces $\widehat{u}(x,t)$ with $u(x,t)$. We consider the closed-loop system consisting of systems (2), (22) and output feedback law (29) in the space $[L^2(0,1)]^4$.

Theorem 4.1. For any given initial value $(u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [H^2(0,1)]^4$, the closed-loop system (2), (22) and (29) admits a unique solution $(u, u_t, \widehat{u}, \widehat{u}_t) \in C(0,\infty; [L^2(0,1)]^4)$. Moreover, the solution of the closed-loop system satisfies

$$\lim_{t\to\infty} \|(u(\cdot,t), u_t(\cdot,t), \widehat{u}(\cdot,t), \widehat{u}_t(\cdot,t))\|_{[L^2(0,1)]^4} = 0. \quad (30)$$

The above theorem can be estabished by carefully estimating the explicit expression of solution obtained by Riesz basis approach.

Remark 4.1. Similarly to Remark 3.2, we shall emphasize that Theorem 4.1 is true only for $\alpha \in (1,2)$ and fractional order α in the closed-loop system cannot take $\alpha = 2$.

5. A DISTURBANCE REJECTION SCHEME

In this section, as discussed in Liang et al. (2004), we assume that a disturbance force $n(t)$ is added at the same end where the boundary control signal enters, that is, we consider the stabilization and disturbance rejection problem of the system described by

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t),\ t \ge 0 \\ u(0,t) = 0,\ \ u_x(1,t) = U(t) + n(t),\ t \ge 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \quad (31)$$

where $x \in (0,1)$, $\alpha \in (1,2)$ is the order of the fractional derivative, $u(x,t)$ is the displacement of wave propagation, $\lambda(x) \in C[0,1]$. $U(t)$ is the input (control) and $n(t)$ is the boundary disturbance.

The second objective of the paper is to seek a control law $U(t)$ to stabilize (2) and to reject the disturbance n. Also, present a rigourous mathematical proof. Obviously, once this objective is achieved, by taking $\lambda(x) = 0$, we would give a complete answer for problem II.

To achieve Mittag-Leffler stability to system (31) and to reject the disturbance $n(t)$, we make the following assumption on $n(t)$.

Assumption 4.1 The disturbance n satisfies $n \in L^\infty(0,\infty)$ and ${}_0^C D_t^\alpha n(t) \in L^\infty(0,\infty)$.

The examples of such kinds of disturbances satisfying Assumption 4.1 include all finite sum of harmonic disturbances like $n(t) = a_j \sin(\omega_j t)$ with unknown amplitude a_j and unknown frequency ω_j. To see this, for any given frequency ω, we prove that $\sup_{t\ge0} |{}_0^C D_t^\alpha \sin(\omega t)| < +\infty$ and $\sup_{t\ge0} |{}_0^C D_t^\alpha \cos(\omega t)| < +\infty$. Indeed, by the Caputo's derivative definition, we get $|{}_0^C D_t^\alpha \sin(\omega t)| = \frac{\omega^2}{\Gamma(2-\alpha)} \left| \sin\omega t \int_0^t s^{1-\alpha}\cos\omega s \mathrm{d}s - \cos\omega t \int_0^t s^{1-\alpha} \sin\omega s \mathrm{d}s \right|$It follows from (Gorenflo et al., 2014, Page 284, A.4.11) that $\int_0^\infty s^{1-\alpha}\cos\omega s \mathrm{d}s = \frac{\Gamma(2-\alpha)}{\omega^\alpha} \sin\frac{\alpha\pi}{2}$, $\int_0^\infty s^{1-\alpha}\sin\omega s \mathrm{d}s = -\frac{\Gamma(2-\alpha)}{\omega^\alpha}\cos\frac{\alpha\pi}{2}$ hold for $\alpha \in (1,2)$, which infers that $\int_0^t s^{1-\alpha}\cos\omega s\mathrm{d}s$ and $\int_0^t s^{1-\alpha}\sin\omega s \mathrm{d}s$ are bounded on $[0,\infty)$. By the boundedness of $\sin\omega t$, $\cos\omega t$, we know that $\sup_{t\ge0} |{}_0^C D_t^\alpha \sin(\omega t)| < +\infty$ and thus ${}_0^C D_t^\alpha \sin(\omega t) \in L^\infty(0,\infty)$. Similarly, ${}_0^C D_t^\alpha \cos(\omega t) \in L^\infty(0,\infty)$.

Since we have designed the stabilizing control law for system without disturbance force $n(t)$ in Section 3 and Section 4, it is a naturally idea that we should find a disturbance estimator to estimate the disturbance force $n(t)$ and cancel/compensate the disturbance in the closed-loop system.

To estimate the disturbance, following Feng and Guo (2017), we propose a disturbance estimator as follows:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)u(x,t),\ \ x \in (0,1), \\ v(0,t) = 0, \quad v_x(1,t) = U(t),\ t \ge 0, \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t),\ \ x \in (0,1),\ t \ge 0, \\ z(0,t) = 0, \quad z(1,t) = v(1,t) - u(1,t),\ t \ge 0, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ 0 \le x \le 1, \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x),\ 0 \le x \le 1, \end{cases} \quad (32)$$

which is an infinite-dimensional with the state consisting of the functions v, z, defined on $(0,1)$. It is seen that system (32) is completely determined by the displacement $u(x,t)$ and input $U(t)$. In other words, system (32) is a completely known system. Since the disturbance estimator (32) proposed here looks complicated, now let us explain why we make such a construction.

Firstly, a "v"-part of system (32) is to channel the disturbance from original system to a Mittag-Leffler stable system. Indeed, let $\widehat{v}(x,t) = v(x,t) - u(x,t)$, it is easy to verify that $\widehat{v}(x,t)$ satisfies the following time fractional wave equation:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{v}(x,t) = \widehat{v}_{xx}(x,t),\ \ x \in (0,1),\ t \ge 0, \\ \widehat{v}(0,t) = 0, \quad \widehat{v}_x(1,t) = -n(t),\ t \ge 0, \\ \widehat{v}(x,0) = \widehat{v}_0(x),\ \widehat{v}_t(x,0) = \widehat{v}_1(x),\ 0 \le x \le 1, \end{cases} \quad (33)$$

Lemma 5.1. Let Assumption 4.1 hold and $\alpha \in (1,2)$. For any initial value $(\widehat{v}_0(x), \widehat{v}_1(x)) \in [L^2(0,1)]^2$, then there exists a unique solution to (33) such that $\widehat{v} \in C(0,\infty; L^2(0,1))$ satisfying $\sup_{t\ge0} \|\widehat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. Moreover, suppose $\lim_{t\to\infty} n(t) = 0$ and $\lim_{t\to\infty} {}_0^C D_t^\alpha n(t) = 0$, then $\lim_{t\to\infty} \|\widehat{v}(\cdot,t)\|_{L^2(0,1)} = 0$. Further, assuming that $(\widehat{v}_0(x), \widehat{v}_1(x)) \in [H^2(0,1)]^2$, then $\sup_{t\ge0} \|(\widehat{v}(\cdot,t), \widehat{v}_t(\cdot,t))\|_{L^2(0,1)} < +\infty$.

Secondly, a "z"-part of system (32) is to estimates the disturbance $d(t)$. Actually, let $\widetilde{z}(x,t) = z(x,t) - \widehat{v}(x,t)$, then we can verify that $\widetilde{z}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{z}(x,t) = \widetilde{z}_{xx}(x,t),\ \ x \in (0,1),\ t \ge 0, \\ \widetilde{z}(0,t) = 0, \quad \widetilde{z}(1,t) = 0,\ t \ge 0, \\ \widetilde{z}(x,0) = \widetilde{z}_0(x),\ \widetilde{z}_t(x,0) = \widetilde{z}_1(x),\ 0 \le x \le 1. \end{cases} \quad (34)$$

Lemma 5.2. For any initial value $(\widetilde{z}_0, \widetilde{z}_1) \in [L^2(0,1)]^2$, system (34) admits a unique solution $\widetilde{z}(\cdot,t) \in C(0,\infty; L^2(0,1))$ and there exists a constant $M > 0$ such that

$$\|\widetilde{z}(\cdot,t)\|^2_{[L^2(0,1)]^2} \le \frac{M}{1 + t^{2\alpha-2}} \|(\widetilde{z}_0, \widetilde{z}_1)\|^2_{[L^2(0,1)]^2}. \quad (35)$$

Moreover, assuming that $(w_0, w_1) \in [H^2(0,1)]^2$, then there exists a constant $M' > 0$ such that

$$\|(\widetilde{z}(\cdot,t), \widetilde{z}_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \le \frac{M'}{1 + t^{2\alpha-2}} \|(\widetilde{z}_0, \widetilde{z}_1)\|^2_{[H^2(0,1)]^2}.$$

By (20), we propose the following control law

$$U(t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y - (-z_x(1,t)).$$

The purpose of this control law is essentially just cancelling the disturbance by its estimation. The closed-loop system is governed by

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \\ u(0,t) = 0,\ u_x(1,t) = k(1,1)u(1,t) \\ +\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + z_x(1,t) + n(t), \\ {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)u(x,t), \\ v(0,t) = 0,\ v_x(1,t) = k(1,1)u(1,t) \\ +\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + z_x(1,t), \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \quad x \in (0,1), \\ z(0,t) = 0,\ z(1,t) = v(1,t) - u(1,t). \end{cases} \tag{36}$$

Theorem 5.1. Suppose that Assumption 4.1 holds. For any initial value $(u_0, v_0, z_0, u_1, v_1, z_1) \in [L^2(0,1)]^6$, there exists a unique solution $(u, v, z) \in C(0, \infty; [L^2(0,1)]^3)$ to (36) and there exist two positive constants $M > 0$ such that

$$\|u(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{M}{1+t^{2\alpha-2}} \|(u_0, v_0, z_0, u_1, v_1, z_1)\|^2_{[L^2(0,1)]^6}. \tag{37}$$

and $\sup_{t\geq 0} \|(v(\cdot,t), z(\cdot,t))\|_{[L^2(0,1)]^2} < +\infty$. Moreover, assuming that $(u_0, v_0, z_0, u_1, v_1, z_1) \in [H^2(0,1)]^6$, then there exist two constants $M' > 0$ such that

$$\begin{aligned} &\|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \\ &\leq \frac{M'}{1+t^{2\alpha-2}} \|(u_0, v_0, z_0, u_1, v_1, z_1)\|^2_{[H^2(0,1)]^6}. \end{aligned} \tag{38}$$

and $\sup_{t\geq 0} \|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t))\|_{[L^2(0,1)]^4} < +\infty$.

Remark 5.1. As discussed in Remark 3.3, from (37) and (38), we can see that the state (u, u_t) is Mittag-Leffler stable.

Remark 5.2. As dicussed in Remark 3.2 and Remark 4.1, when $\alpha = 2$ in the closed-loop system, the result of Theorem 5.1 is not valid, that is, we cannot take $\alpha = 2$.

At the end of this section, we point out that Theorem 5.1 provides a disturbance estimator for system considered in Liang et al. (2004) which reads as

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t), \quad x \in (0,1),\ t \geq 0, \\ u(0,t) = 0,\ u_x(1,t) = U(t) + n(t),\ t \geq 0, \\ u(x,0) = w_0(x),\ u_t(x,0) = u_1(x),\ 0 \leq x \leq 1. \end{cases} \tag{39}$$

The disturbance estimator for (39) is given by

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t), \quad x \in (0,1),\ t \geq 0, \\ v(0,t) = 0, \quad v_x(1,t) = U(t),\ t \geq 0, \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \quad x \in (0,1),\ t \geq 0, \\ z(0,t) = 0, \quad z(1,t) = v(1,t) - u(1,t),\ t \geq 0, \end{cases} \tag{40}$$

which is completely determined by output $u(1,t)$ and input $U(t)$ and thus is an output-based disturbance estimator. Theorem 5.1 also provides a new output feedback control law and a rigorous mathematical proof for the stability of the closed-loop system.

6. CONCLUDING REMARKS

In this paper, the boundary stabilization of an unstable time fractional diffusion-wave equation involving Caputo time fractional derivative with or without noise disturbance are considered. When no noise disturbance is involved, we achieve the Mittag-Leffler stability by state feedback and asymptotical stability by output feedback. When there is noise disturbance flowing the boundary, we derive that the u-part of the resulting closed-loop system is asymptotically sable while the (v, z)-part is bounded. As a byproduct, the paper solves rigorously completely the two longtime unsolved problems raised in [*Nonlinear Dynam.*, 38(2004), 339-354].

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