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Stabilization of Stochastic Fluctuations in Hyperbolic Systems ⋆

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Abstract
We consider steady states of physical systems that are described by hyperbolic balance laws. We derive control policies that damp small perturbations over time. Uncertainties in model parameters are taken into account. Theoretical results are illustrated by stabilizing a viscoplastic material.

Keywords: Feedback control, hyperbolic balance laws, networks, stochastic stability, Lyapunov functions, stochastic Galerkin

1. INTRODUCTION

Hyperbolic balance laws can be used to model flow dynamics on networks. For example, isothermal Euler and shallow water equations form 2×2 hyperbolic systems that describe the temporal and spatial evolution of gas and water flow. Also viscoplastic materials can be described by hyperbolic balance laws. Boundary control of such systems is subject of current research, see Bastin and Coron (2016).

An underlying tool for the study of these problems are Lyapunov functions that yield upper bounds for the deviation from steady states. Exponential decay of a continuous Lyapunov function under so-called *dissipative* boundary conditions has been proven by Coron et al. (2008b,a); Coron and Bastin (2015). In particular, analytical results have been presented in the case of gas flow by Gugat et al. (2012) and water flow by Gugat and Leugering (2003); Leugering and Schmidt (2002); Gugat et al. (2018). Explicit decay rates for linearized balance laws with possibly large source term are presented in Gerster and Herty (2019). Also explicit decay rates for numerical schemes have been established by Banda and Herty (2013); Schillen and Göttlich (2017); Gerster and Herty (2019); Baumgärtner et al. (2020). If the destabilizing effect of the source term is sufficiently large, the system cannot be controlled by boundary feedback. Unstable systems are presented by Bastin and Coron (2011); Gugat and Gerster (2019).

We represent stochastic perturbations by piecewise orthogonal polynomials, known as generalized polynomial chaos (gPC) expansions, which were introduced by Wiener (1938); Cameron and Martin (1947). Expansions of the stochastic input are substituted into the governing equations and they are projected to obtain deterministic evolution equations for its coefficients. This paper introduces a Lyapunov stability analysis for the system of gPC coefficients.

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2. FEEDBACK CONTROL

We consider a network with n arcs as illustrated in Figure 1. The dynamics $y_j \in \mathbb{R}^2$ on each arc are described by strictly hyperbolic linear balance laws

$$\begin{aligned} &\partial_t y_j(t,x) + \bar{A}_j \partial_x y_j(t,x) = -\bar{S}_j y_j(t,x) \\ &\text{for} \quad y_j \in (\rho_j, q_j)^{\mathrm{T}} \quad \text{and} \quad j = 1,\dots,n. \end{aligned} \tag{1}$$

We introduce the assumption [1] $\bar{\lambda}_j^- < 0 < \bar{\lambda}_j^+$ for the eigenvalue decomposition

$$\bar{A}_j = \bar{T}_j \bar{\Lambda}_j \bar{T}_j^{-1} \quad \text{with} \quad \bar{\Lambda}_j := \operatorname{diag}\{\bar{\lambda}_j^+, \bar{\lambda}_j^-\}$$

and we introduce the notation

$$\boldsymbol{y} := \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{q} \end{pmatrix} \quad \text{with} \quad \boldsymbol{\rho} := \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_n \end{pmatrix}, \quad \boldsymbol{q} := \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

to equip the dynamics (1) with boundary conditions. The resulting boundary value problem (BVP) reads as

$$\partial_t \boldsymbol{y}(t,x) + \bar{A} \partial_x \boldsymbol{y}(t,x) = -\bar{S} \boldsymbol{y}(t,x), \tag{2}$$

$$\begin{pmatrix} \boldsymbol{\rho}(t,0) \\ \boldsymbol{q}(t,L) \end{pmatrix} = K \begin{pmatrix} \boldsymbol{\rho}(t,L) \\ \boldsymbol{q}(t,0) \end{pmatrix}. \tag{3}$$

We assume the system is close to a **steady state** $\bar{\boldsymbol{y}}(x)$ satisfying $\partial_t \bar{\boldsymbol{y}}(x) = \partial_t \bar{\boldsymbol{y}}(t,x) = 0$ and we introduce **perturbations**

$$\Delta \boldsymbol{y}(t,x) := \boldsymbol{y}(t,x) - \bar{\boldsymbol{y}}(x).$$

Since the balance law (2) and boundary conditions (3) are *linear*, the perturbations satisfy also the boundary value problem (2), (3).

[1] For simplicity, we assume constant eigenvalues and source terms. For an extension to non-uniform systems we refer to the results by Bastin and Coron (2016); Gerster and Herty (2019).

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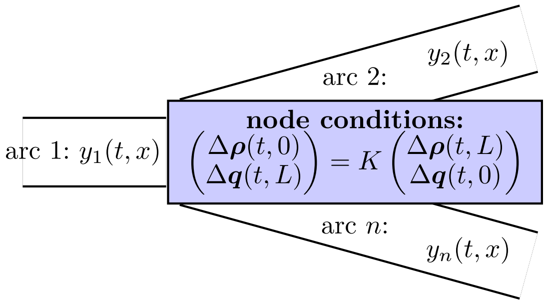

Figure 1. Network with n arcs

We will show how to specify boundary conditions such that the deviations decay exponentially fast with a decay rate $\mu > 0$ in the sense

$$\big\|\Delta\boldsymbol{y}(t,\cdot)\big\| \leq c\, e^{-\mu t}\big\|\Delta\boldsymbol{y}(0,\cdot)\big\|$$

for all $t \in \mathbb{R}_0^+$ and $c > 0$. It is convenient to rewrite the BVP (2), (3) in Riemann invariants

$$\bar{\mathcal{R}}_j(t,x) \coloneqq \begin{pmatrix}\bar{\mathcal{R}}_j^+(t,x)\\ \bar{\mathcal{R}}_j^-(t,x)\end{pmatrix} \coloneqq \bar{T}_j^{-1}\Delta y_j(t,x).$$

The source term reads as

$$\bar{C}_j \coloneqq \bar{T}_j^{-1}\bar{S}_j\bar{T}_j.$$

Then, the balance law (1) is equivalent to

$$\begin{aligned} &\partial_t\bar{\mathcal{R}}_j(t,x) + \bar{\Lambda}_j\partial_x\bar{\mathcal{R}}_j(t,x) = -\bar{C}_j\bar{\mathcal{R}}_j(t,x)\\ &\text{for} \qquad \bar{\Lambda}_j = \operatorname{diag}\Big\{\bar{\lambda}_j^+,\bar{\lambda}_j^-\Big\}\\ &\text{and} \qquad \bar{\mathcal{R}}_j = \operatorname{diag}\Big\{\bar{\mathcal{R}}_j^+,\bar{\mathcal{R}}_j^-\Big\}. \end{aligned} \tag{4}$$

We introduce the notation

$$\begin{aligned} \bar{\mathcal{R}} &\coloneqq \big(\bar{\mathcal{R}}^+,\bar{\mathcal{R}}^-\big)^{\mathrm{T}}, \;\; \bar{\mathcal{R}}^\pm \coloneqq \big(\bar{\mathcal{R}}_1^\pm,\ldots,\bar{\mathcal{R}}_n^\pm\big)^{\mathrm{T}},\\ \bar{\Lambda} &\coloneqq \big(\bar{\lambda}^+,\bar{\lambda}^-\big)^{\mathrm{T}}, \quad \bar{\lambda}^\pm \coloneqq \big(\bar{\lambda}_1^\pm,\ldots,\bar{\lambda}_n^\pm\big)^{\mathrm{T}} \end{aligned}$$

to express the BVP (2), (3) with initial values $\bar{\mathcal{I}}(x)$ conveniently as

$$\begin{aligned} &\partial_t\bar{\mathcal{R}}(t,x) + \bar{\Lambda}\partial_x\bar{\mathcal{R}}(t,x) = -\bar{C}\bar{\mathcal{R}}(t,x),\\ &\begin{pmatrix}\bar{\mathcal{R}}^+(t,0)\\ \bar{\mathcal{R}}^-(t,L)\end{pmatrix} = \bar{G}\begin{pmatrix}\bar{\mathcal{R}}^+(t,L)\\ \bar{\mathcal{R}}^-(t,0)\end{pmatrix},\\ &\bar{\mathcal{R}}(0,x) = \bar{\mathcal{I}}(x). \end{aligned} \tag{5}$$

Finding L^2-solutions [2] to this **initial boundary value problem (IBVP)** is a well-posed problem (Bastin and Coron, 2016, Th. A.4).

3. STOCHASTIC GALERKIN

We equip the IBVP with an M-dimensional random variable $\xi \sim \mathbb{P}$ with probability measure $\mathbb{P}$. The parameterized IBVP reads as

$$\begin{aligned} \partial_t\bar{\mathcal{R}}(t,x;\xi) + \bar{\Lambda}(\xi)\partial_x\bar{\mathcal{R}}(t,x;\xi)&\\ = -\bar{C}(\xi)\bar{\mathcal{R}}(t,x;\xi)&,\\ \begin{pmatrix}\bar{\mathcal{R}}^+(t,0;\xi)\\ \bar{\mathcal{R}}^-(t,L;\xi)\end{pmatrix} = \bar{G}\begin{pmatrix}\bar{\mathcal{R}}^+(t,L;\xi)\\ \bar{\mathcal{R}}^-(t,0;\xi)\end{pmatrix}&,\\ \bar{\mathcal{R}}(0,x;\xi) = \bar{\mathcal{I}}(x;\xi)&. \end{aligned} \tag{6}$$

[2] The interested reader finds a precise definition in (Bastin and Coron, 2016, Def. A.3). The only necessary restriction are square-integrable initial values.

We assume the randomly perturbed eigenvalues remain separated, i.e.

$$\bar{\Lambda}^-(\xi) < 0 < \bar{\Lambda}^+(\xi) \quad \text{for all} \quad \xi \sim \mathbb{P},$$

and we assume the solution $y(t,x;\cdot)$ is square-integrable such that it belongs to the $\mathbb{L}^2$-space with inner product

$$\big\langle g(\cdot),h(\cdot)\big\rangle_{\mathbb{P}} \coloneqq \int g(\xi)h(\xi)\,\mathrm{d}\mathbb{P}.$$

A **generalized polynomial chaos** (gPC) is a set of orthogonal subspaces

$$\begin{aligned} &\hat{\mathcal{S}}_k \subseteq \mathbb{L}^2(\Omega,\mathbb{P}) \quad \text{with}\\ &\mathcal{S}_K \coloneqq \bigoplus_{k=0}^{K}\hat{\mathcal{S}}_k \;\to\; \mathbb{L}^2(\Omega,\mathbb{P}) \;\; \text{for} \;\; K\to\infty. \end{aligned}$$

We refer to an orthogonal basis of $\mathcal{S}_K$ as a gPC basis $\{\phi_k(\xi)\}_{k=0}^K$ with germ $\xi \sim \mathbb{P}$. A common choice are **Legendre polynomials** with uniformly distributed germ $\xi \sim \mathcal{U}(-1,1)$, which are recursively defined by

$$\begin{aligned} &\phi_0(\xi) = 1, \quad \phi_1(\xi) = \xi,\\ &\phi_{k+1}(\xi) = \frac{2k+1}{k+1}\xi\phi_k(\xi) - \frac{k}{k+1}\phi_{k-1}(\xi). \end{aligned}$$

We use the multi-index $\boldsymbol{k} \coloneqq (k_1,\ldots,k_M) \in \mathbb{K}$ and an index set $\mathbb{K} \subseteq \mathbb{N}_0^M$ to approximate the solution $\bar{\mathcal{R}}(t,x;\cdot)$ by

$$\begin{aligned} \mathcal{G}_K[\bar{\mathcal{R}}](t,x;\xi) &\coloneqq \sum_{\boldsymbol{k}\in\mathbb{K}}\hat{\boldsymbol{\mathcal{R}}}_{\boldsymbol{k}}(t,x)\phi_{\boldsymbol{k}}(\xi),\\ \phi_{\boldsymbol{k}}(\xi) &\coloneqq \phi_{k_1}(\xi_1)\cdot\ldots\cdot\phi_{k_M}(\xi_M). \end{aligned}$$

A common choice is the **sparse basis**

$$\mathbb{K}_{\mathrm{S}} \coloneqq \big\{\boldsymbol{k}\in\mathbb{N}_0^M \;\big|\; \|\boldsymbol{k}\|_1 \leq K\big\}$$

with $|\mathbb{K}_{\mathrm{S}}| = (M+K)!(M!K!)^{-1}$, where the number of one-dimensional gPC bases is denoted as $K \in \mathbb{N}_0$. For simplicity we use the one-dimensional notation and we write

$$\mathcal{G}_K[\bar{\mathcal{R}}](t,x;\xi) \coloneqq \sum_{k=0}^{K}\hat{\boldsymbol{\mathcal{R}}}_k(t,x)\phi_k(\xi).$$

The **Galerkin product** for two square-integrable random variables $y(\xi), z(\xi) \in \mathbb{L}^2(\Omega,\mathbb{P})$ is defined as

$$\begin{aligned} &\hat{\mathcal{G}}_K[y,z](\xi) \coloneqq \sum_{k=0}^{K}(\hat{y}*\hat{z})_k\phi_k(\xi) \quad \text{with}\\ &(\hat{y}*\hat{z})_k \coloneqq \sum_{i,j=0}^{K}\hat{y}_i\hat{z}_j\big\langle\phi_i\phi_j,\phi_k\big\rangle_{\mathbb{P}}. \end{aligned}$$

We express it by the symmetric matrix

$$\begin{aligned} \mathcal{P}(\hat{y}) &\coloneqq \sum_{k=0}^{K}\hat{y}_k\mathcal{M}_k \quad \text{with}\\ \mathcal{M}_k &\coloneqq \Big(\langle\phi_k,\phi_i\phi_j\rangle_{\mathbb{P}}\Big)_{i,j=0,\ldots,K}. \end{aligned} \tag{7}$$

Then, we have $\hat{y}*\hat{z} = \mathcal{P}(\hat{y})\hat{z}$. A stochastic Galerkin formulation for the random system (6) is derived by

$$\begin{aligned} \sum_{k=0}^{K}\Big\langle\partial_t\hat{\boldsymbol{\mathcal{R}}}_k(t,x)\phi_k(\cdot) + \bar{\Lambda}(\cdot)\partial_x\hat{\boldsymbol{\mathcal{R}}}_k(t,x)\phi_k(\cdot)&\\ + \bar{C}(\cdot)\hat{\boldsymbol{\mathcal{R}}}_k(t,x)\phi_k(\cdot),\;\phi_j(\cdot)\Big\rangle_{\mathbb{P}} = 0&, \end{aligned}$$

which leads to the **stochastic Galerkin formulation**

$$\partial_t \hat{\mathcal{R}}(t,x) + \hat{\mathcal{A}}\partial_x \hat{\mathcal{R}}(t,x) = -\hat{\mathcal{S}}\hat{\mathcal{R}}(t,x) \quad \text{for}$$

$$\hat{\mathcal{A}} := \begin{pmatrix} \hat{\mathcal{A}}^+ & \\ & \hat{\mathcal{A}}^- \end{pmatrix}, \quad \hat{\mathcal{S}} := \begin{pmatrix} \hat{\mathcal{S}}_{1,1} & \hat{\mathcal{S}}_{1,2} \\ \hat{\mathcal{S}}_{2,1} & \hat{\mathcal{S}}_{2,2} \end{pmatrix}$$

$$\text{with} \qquad (\hat{\mathcal{A}}^{\pm})_{i,j} := \Big\langle \bar{\Lambda}^{\pm}(\cdot), \phi_i(\cdot)\phi_j(\cdot) \Big\rangle_{\mathbb{P}}$$

$$\text{and} \qquad (\hat{\mathcal{S}}_{k,\ell})_{i,j} := \Big\langle \bar{C}_{k,\ell}(\cdot), \phi_i(\cdot), \phi_j(\cdot) \Big\rangle_{\mathbb{P}}.$$

Furthermore, the matrix $\hat{\mathcal{A}}^+$ is strictly positive definite and $\hat{\mathcal{A}}^-$ is strictly negative definite, see (Sonday et al., 2011, Th. 2). Due to

$$\hat{y}^{\mathrm{T}}\hat{\mathcal{A}}^{\pm}\hat{y} = \pm \int \left(\sqrt{|\bar{\Lambda}^{\pm}(\xi)|} \sum_{k=0}^{K} \hat{y}_k \phi_k(\xi) \right)^2 \mathrm{d}\mathbb{P}$$

we have $\hat{y}^{\mathrm{T}}\hat{\mathcal{A}}^+\hat{y} > 0$ and $\hat{y}^{\mathrm{T}}\hat{\mathcal{A}}^-\hat{y} < 0$, respectively for all $\hat{y} \in \mathbb{R}^{K+1} \setminus \{(0,\ldots,0)^{\mathrm{T}}\}$ and basis functions ϕ_k. Using the orthonormal eigenvalue decomposition

$$\hat{\mathcal{A}}^{\pm} = \hat{\mathcal{T}}^{\pm}\hat{\mathcal{D}}^{\pm}(\hat{\mathcal{T}}^{\pm})^{\mathrm{T}}$$

with $\hat{\mathcal{D}} := \mathrm{diag}\{\hat{\mathcal{D}}^+, \hat{\mathcal{D}}^-\}$ and $\hat{\mathcal{D}}^- < 0 < \hat{\mathcal{D}}^+$, we diagonalize the stochastic Galerkin formulation. The resulting IBVP with projected initial and boundary values (3) reads

$$\partial_t \hat{\boldsymbol{\zeta}}(t,x) + \hat{\mathcal{D}}\partial_x \hat{\boldsymbol{\zeta}}(t,x) = -\hat{\mathcal{B}}\hat{\boldsymbol{\zeta}}(t,x),$$

$$\begin{pmatrix} \hat{\boldsymbol{\zeta}}^+(t,0) \\ \hat{\boldsymbol{\zeta}}^-(t,L) \end{pmatrix} = \underbrace{\begin{pmatrix} G_{1,1}\mathbb{1} & G_{1,2}\mathbb{1} \\ G_{2,1}\mathbb{1} & G_{2,2}\mathbb{1} \end{pmatrix}}_{=:\hat{\mathcal{G}}} \begin{pmatrix} \hat{\boldsymbol{\zeta}}^+(t,L) \\ \hat{\boldsymbol{\zeta}}^-(t,0) \end{pmatrix},$$

$$\hat{\boldsymbol{\zeta}}(0,x) = \hat{\mathcal{I}}(x).$$

Finding L^2-solutions to this augmented systems is also a well-posed problem (Bastin and Coron, 2016, Th. A.4).

4. LYAPUNOV STABILITY ANALYSIS

We are interested in boundary conditions such that the random system (6) is exponentially stable according to the following definition.

Definition 1. The random IBVP (6) is exponentially stable for the $\mathbb{L}^2(\Omega,\mathbb{P})$-norm if there exist a positive constant $c > 0$ and a positive decay rate $\mu > 0$ such that for every initial values $\bar{\mathcal{R}}(0,\cdot,\xi) \in L^2\big((0,L);\mathbb{R}^{2n}\big)$ the L^2-solution satisfies

$$\begin{aligned} &\mathbb{E}\Big[\big\|\bar{\mathcal{R}}(t,\cdot;\xi)\big\|^2\Big] \\ &\leq c\, e^{-\mu t}\mathbb{E}\Big[\big\|\bar{\mathcal{R}}(0,\cdot;\xi)\big\|^2\Big]. \end{aligned} \tag{8}$$

Note that also the variance and the mean of deviations decay exponentially fast according to this definition, since the mean squared error satisfies

$$\begin{aligned} &\mathbb{E}\Big[\big\|\bar{\mathcal{R}}(t,x;\xi)\big\|^2\Big] \\ &= \mathrm{Var}\Big[\big\|\bar{\mathcal{R}}(t,x;\xi)\big\|\Big] + \Big\|\mathbb{E}\big[\bar{\mathcal{R}}(t,x;\xi)\big]\Big\|^2. \end{aligned}$$

Similarly to the results by Ahbe et al. (2019) we stabilize a relaxation that is based on the gPC approximation $\bar{\mathcal{R}}(t,x;\xi) \approx \mathcal{G}_K[\bar{\mathcal{R}}](t,x;\xi)$. An overview on approximation errors can be found in Mühlpfordt et al. (2017) and an analysis of the limit $K \to 0$ is found in Gerster et al. (2020). As basic tool we use a Lyapunov function, which is introduced in the following definition.

Definition 2. Let positive constants $\hat{\mu} > 0$ and $h_k^+, h_k^- > 0$ for $k = 0,\ldots,K$ be given. Define the weights

$$w_k^+(x) := \frac{h_k^+}{\hat{\mathcal{D}}_{\boldsymbol{k}}^+}\exp\Big(-\frac{\hat{\mu}\,x}{\hat{\mathcal{D}}_{\boldsymbol{k}}^+}\Big),$$

$$w_k^-(x) := \frac{h_k^-}{|\hat{\mathcal{D}}_{\boldsymbol{k}}^-|}\exp\Big(\frac{\hat{\mu}(L-x)}{\hat{\mathcal{D}}_{\boldsymbol{k}}^-}\Big),$$

$$W^{\pm}(x) := \mathrm{diag}\Big\{\frac{w_0^{\pm}(x)}{w_{\min}^{\pm}(x)}, \ldots, \frac{w_K^{\pm}(x)}{w_{\min}^{\pm}(x)}\Big\},$$

$$W(x) := \mathrm{diag}\Big\{W^+(x), W^-(x)\Big\}$$

$$\text{for} \quad w_{\min}^{\pm}(x) := \min_{k=0,\ldots,K}\big\{w_k^{\pm}(x)\big\}.$$

This yields for each fixed point in space the weighted inner product $\langle a,b\rangle_{W(x)} := a^{\mathrm{T}}W(x)b$ and the **Lyapunov function**

$$\mathcal{L}(t) := \int_0^L \Big\|\hat{\boldsymbol{\zeta}}(t,x)\Big\|_{W(x)}^2 \mathrm{d}x.$$

The following theorem states sufficient conditions for exponential stability.

Theorem 3. Assume there are positive values $\hat{\mu}, h_k^+, h_k^- > 0$ that satisfy the inequalities

$$0 < \hat{\mu} - 2 \max_{\substack{x\in[0,L],\\ i,j=0,\ldots,K}} \left\{ \sqrt{\frac{w_i^{\pm}(x)}{w_j^{\pm}(x)}}\, \big\|\hat{\mathcal{B}}\big\| \right\} =: \mu,$$

$$1 > e^{\hat{\mu}\frac{L}{2\lambda_{\min}}}\big\|\mathcal{D}\hat{\mathcal{G}}\mathcal{D}^{-1}\big\|$$

$$\text{for} \quad \lambda_{\min} := \min_{k=0,\ldots,K}\Big\{\big|\hat{\mathcal{D}}_{\boldsymbol{k}}^{\pm}\big|\Big\},$$

$$\mathcal{D} := \mathrm{diag}\Big\{h_0^+,\ldots,h_K^+,\ h_0^-,\ldots,h_K^-\Big\}.$$

Then, the Lyapunov function satisfies

$$\mathcal{L}(t) \leq e^{-\mu t}\mathcal{L}(0).$$

and the gPC approximation is bounded by

$$\mathbb{E}\Big[\big\|\mathcal{G}_K[\bar{\mathcal{R}}](t,\cdot;\xi)\big\|^2\Big] \leq \mathcal{L}(t).$$

Proof. The estimate $\mathcal{L}(t) \leq e^{-\mu t}\mathcal{L}(0)$ is proven by Gerster and Herty (2019). We define the symmetric and strictly positive definite matrices

$$\Xi^{\pm}(x) := \hat{\mathcal{T}}^{\pm}W^{\pm}(x)(\hat{\mathcal{T}}^{\pm})^{\mathrm{T}}.$$

Then, we obtain

$$\big\|\hat{\mathcal{R}}^{\pm}\big\|_{\Xi^{\pm}(x)}^2 \geq \sigma_{\min}\big\{\Xi^{\pm}(x)\big\}\big\|\hat{\mathcal{R}}^{\pm}\big\|^2 = \big\|\hat{\mathcal{R}}^{\pm}\big\|^2$$

$$\begin{aligned} \implies \quad \mathcal{L}(t) &= \int \hat{\boldsymbol{\zeta}}(t,x)^{\mathrm{T}}W(x)\hat{\boldsymbol{\zeta}}(t,x)\,\mathrm{d}x \\ &= \int \big\|\hat{\mathcal{R}}^+(t,x)\big\|_{\Xi^+(x)}^2 + \big\|\hat{\mathcal{R}}^-(t,x)\big\|_{\Xi^-(x)}^2\,\mathrm{d}x \\ &\geq \int \big\|\hat{\mathcal{R}}(t,x)\big\|^2\,\mathrm{d}x \\ &= \mathbb{E}\Big[\big\|\mathcal{G}_K[\bar{\mathcal{R}}](t,x;\xi)\big\|^2\Big]. \end{aligned}$$

Note that positive values $\hat{\mu}, h_k^+, h_k^- > 0$, which satisfy the sufficient conditions for exponential stability in Theorem 3, may not exist. Bastin and Coron (2011); Gugat and Gerster (2019) presented conditions, when certain systems cannot be stabilized.

5. STABILIZATION OF A VISCOPLASTIC MATERIAL

There are several models for viscoplastic materials that describe a displacement $u(t,x)$ and stress $\sigma(t,x)$ at time $t \geq 0$ and position $x \in [0, L]$. An overview on various models can be found e.g. in Simo and Hughes (2016). To illustrate our approach, we consider a very simplified one-dimensional model. The total strain

$$\epsilon(t,x) = \epsilon^e(t,x) + \epsilon^p(t,x)$$

is decomposed into an elastic part ϵ^e and plastic part ϵ^p. The elastic relationship is described by

$$\sigma(t,x) = E\big(\epsilon(t,x) - \epsilon^p(t,x)\big)$$

with a constant $E > 0$. There is flexibility to relate the stress and total strain by a yield condition. The plastic part can be viewed as a function of the stress, which we denote as $\bar{\epsilon}^p(\sigma)$. The displacement velocity $v(t,x) := \partial_t u(t,x)$ and stress are described by the balance law

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ \sigma \end{pmatrix} + \begin{pmatrix} & -1 \\ -E & \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\epsilon}^p(\sigma) \end{pmatrix}.$$

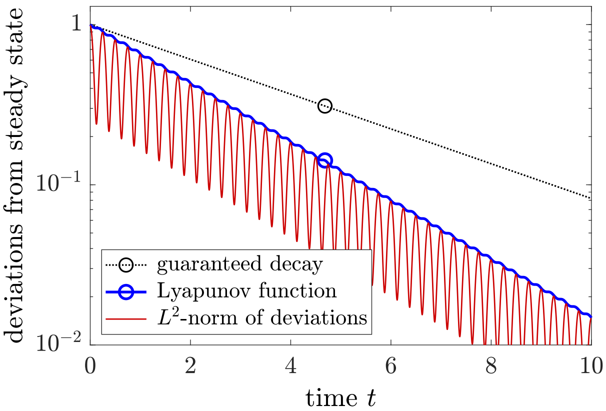

Figure 2. Deterministic exponential stability

We linearize the source term. Then, deviations at steady state $\Delta y(t,x) = y(t,x) - \bar{y}(x)$ are described by the linear system

$$\partial_t \Delta y(t,x) + \bar{A}\partial_x \Delta y(t,x) = -\bar{S}\Delta y(t,x).$$

We diagonalize the Jacobian $\bar{A} = \bar{T}\bar{\Lambda}\bar{T}^{-1}$ to obtain the Riemann invariants $\bar{\mathcal{R}} = \bar{T}^{-1}\Delta y$. For the system

$$\bar{\mathcal{R}}_t + \bar{\Lambda}\bar{\mathcal{R}}_x = -\bar{C}\bar{\mathcal{R}}$$

we prescribe the linear feedback boundary conditions

$$\begin{pmatrix} \bar{\mathcal{R}}^+(t,0) \\ \bar{\mathcal{R}}^-(t,L) \end{pmatrix} = \begin{pmatrix} & \kappa_0 \\ \kappa_1 & \end{pmatrix}\begin{pmatrix} \bar{\mathcal{R}}^+(t,L) \\ \bar{\mathcal{R}}^-(t,0) \end{pmatrix}. \tag{9}$$

These boundary conditions, reformulated in terms of the displacement velocity, read

$$\frac{v(t,0) - \bar{v}(0)}{\sigma(t,0) - \bar{\sigma}(t,0)} = \frac{1-\kappa_0}{\sqrt{E} + \kappa_0\sqrt{E}},$$
$$\frac{v(t,L) - \bar{v}(L)}{\sigma(t,L) - \bar{\sigma}(t,L)} = \frac{\kappa_1 - 1}{\sqrt{E} + \sqrt{E}\kappa_1}.$$

We choose $\mathcal{D} = \operatorname{diag}\{1,1\}$. Boundary conditions $\kappa_0, \kappa_1 \in \mathbb{R}$ should be specified such that there exists a parameter $\hat{\mu} \in \mathbb{R}$ that satisfies the inequalities of Theorem 3, i.e.

$$0 < \mu,$$
$$1 < e^{\hat{\mu}\frac{L}{2\lambda_{\min}}}\left\|\mathcal{D}\bar{G}\mathcal{D}^{-1}\right\|$$
$$= e^{\hat{\mu}\frac{L}{2\lambda_{\min}}}\max\left\{|\kappa_0|, |\kappa_1|\right\}.$$

The deviations Δy are bounded by

$$\int \left\|\Delta y(t,x)\right\|^2 \mathrm{d}x = \frac{\mathcal{L}(t)}{c} \leq \frac{\mathcal{L}(0)}{c}e^{-\mu t} \quad \text{for}$$
$$c := \min_{x\in[0,L]}\left\{\sigma_{\min}\left[\bar{T}^{-1}(x)^{\mathrm{T}}W(x)\bar{T}^{-1}(x)\right]\right\}.$$

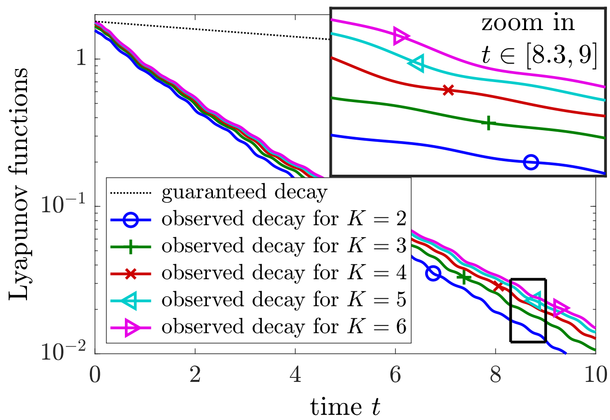

Figure 3. Stochastic exponential stability

Figure 2 illustrates the deterministic case with the values $\sqrt{E} = 2$, $\partial_\sigma \bar{\epsilon}^p(\bar{\sigma}) = 0$, $\kappa_0 = \kappa_1 = 0.9$ and $L = 1$. The L^2-norm is bounded by the Lyapunov function. It decays exponentially fast.

We consider in Figure 3 a uniformly distributed stress $\partial_\sigma \bar{\epsilon}^p(\bar{\sigma}) \sim \mathcal{U}(-0.1, 0.1)$ and $\sqrt{E} \sim \mathcal{U}(1,3)$. The Lyapunov function, defined in Definition 2, is shown for serveral gPC bases. In all simulations the sparse basis is used. We observe for varies truncations $K = 2, \ldots, 6$ a decay which illustrates the stochastic stability.

6. SUMMARY

We have introduced a general framework to stabilize systems of random linearized balance laws. A Lyapunov function has been introduced to stabilize the system of coefficients for truncated polynomial chaos expansions.

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