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Extended State based Kalman Filter for Uncertain Systems with Bias ⋆

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Abstract: This paper addresses the state estimation for a class of stochastic systems with both uncertain dynamics and measurement bias. By using the idea of uncertainty/disturbance estimation, an extended state based Kalman filter algorithm is developed to estimate the original state, the uncertain dynamics and the measurement bias. Furthermore, a necessary and sufficient condition for the observability of augmented system is presented. Also, the stability of the proposed algorithm is analyzed. It is shown that the proposed filter can achieve unbiased estimation of measurement bias, such that the influence of measurement bias is eliminated. Finally, a simulation study is provided to illustrate the effectiveness of proposed method.

Keywords: Extended state observer, Kalman filter, Uncertain estimation, Measurement bias.

1. INTRODUCTION

State estimation plays a central role in many control engineering problems. As is well known, the classical Luenberger observer (Luenberger, 1966) is a popular method for the state reconstruction of linear systems with exact model information. And for the linear systems with white noise, Kalman filter (KF) provides an optimal estimation in the mean square sense (Kalman, 1959). However, model uncertainties and disturbances, which are ubiquitous in practice, are not considered in the original KF. In the past years, various state estimation methods have been proposed for systems with model uncertainties. Generally speaking, these model uncertainties can be divided into sensor errors and process errors.

In practice, sensor errors are often modeled more accurately as the sum of a white-noise component and a strongly correlated component. The correlated component can, for example, be random constant bias (Zanetti and Bishop, 2012). And a common technique to deal with this case is to augment the state vector of the original problem by adding additional component to represent the unknown bias. In an attempt to reduce the computation cost, Friedland (1969) proposed the two-stage or separate-bias estimation to decouple the augmented filter into two parallel reduced-order filters. This idea have also been expanded for systems with bias modeled by first-order Markov process and systems with nonlinear dynamics (Keller and Darouach, 1997; Zhang et al., 2014). Although the estimator with bias observation has drawn much research attention, the observability of the augmented system with bias, which is an important condition to guarantee the stability of an estimator, has not yet been adequately investigated, to the best of our knowledge.

On the other hand, the process errors of a dynamical system also contain a strongly correlated component. Therefore, similar to the sensor errors, the process errors should be modeled as the the sum of a white-noise component and a strongly correlated uncertain dynamics. In order to deal with the influence of uncertain dynamics or external disturbances in actual systems, a number of observer design method have been proposed. Extended state observer (ESO) (Han, 1995; Xue et al., 2016; Chen et al., 2020) was proposed to estimate both the original state and the extended state lumping the unknown internal dynamics and external disturbances. And the convergence of the nonlinear ESO proposed in Han (1995) has been proved recently in Zhao and Guo (2018). Moreover the ESO based control methods have been successfully used in several industrial sectors (Sira-Ramírez et al., 2014; Zhu et al., 2014; Qiu et al., 2014; Zheng and Gao, 2018). Additionally, disturbance observer (DOB) (Schrijver and van Dijk, 2002; Hu et al., 2014) and nonlinear disturbance observer (NDOB) (Yang et al., 2013; Li et al., 2014; Chen et al., 2016) are popular methods of estimating the external disturbance for linear systems and nonlinear systems, respectively. Treating the modeling uncertainties and external disturbances as a lumped term, uncertainty and disturbance estimator (UDE) based control has been shown to be effective in estimating model uncertainty and external disturbance (Chen et al., 2016; Sun et al., 2016). For the sake of attenuating the influence of measurement noise, the pioneer work in Bai et al. (2018) and Zhang et al. (2018) has started the work of combining ESO and KF/KBF

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algorithms to estimate states of some nonlinear systems without considering measurement bias. Although some estimation methods for systems with uncertain dynamics have been studied thoroughly from different perspectives, little of these literatures considered the situation that both the uncertain dynamics and measurement bias present.

Summarizing the above discussions, this paper will focus on the state estimation problem for a class of stochastic systems with both uncertain dynamics and measurement bias which is more common in practice. By using the idea of extended state based Kalman filter (ESKF) in Bai et al. (2018), a filter to estimate the original state, the uncertain dynamics and the measurement bias will be developed. The main contribution of this paper is threefold:

i) The ESKF algorithm is constructed for a class of stochastic systems with both uncertain dynamics and measurement bias;

ii) A necessary and sufficient condition for the observability of augmented system is presented, based on which the stability of ESKF is analyzed;

iii) The bias estimation of the proposed filter is also prove to be convergent in mean square sense, such that the influence of bias can be eliminated.

The remainder of the paper is organized as follows: The problem formulation is given in Section 2. Section 3 introduces the design method of ESKF in detail. Follow on, Section 4 analyzes the performance of ESKF. After that, an illustrative example is presented in Section 5, and finally some concluding remarks are given in Section 6.

Notation: Throughout this paper, the notations used are fairly standard. The sub-index k to denote each variable at the k-th time instant. $\mathbb{R}^n$ represents the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ stands for the space of real $m\times n$-matrices; $\mathbb{C}$ denotes the complex number set. $\mathbf{I}_m$ stands for the identity matrix of size m, and $\mathbf{0}_{m\times n}$ stands for the zero matrix of m rows and n columns and the sub-index will be occasionally removed for notational convenience if no confusion is expected. For a vector or matrix X, X^T denotes its transpose; $\mathrm{rank}\,(X)$ denotes its rank; $\|X\|$ denotes its spectral norm. For a square matrix X, $\mathrm{tr}\,(X)$ denotes its trace; $\lambda_{\max}(X)$ represents its maximal eigenvalue. For a invertible matrix X, X^{-1} denotes its inverse matrix. For symmetric matrices X and Y, $X \succeq Y$ (or $X \succ Y$) denotes $X-Y$ is a positive semidefinite (or positive definite) matrix. For a stochastic vector X, $\mathbb{E}\{X\}$ denotes its mathematical expectation, and $\mathrm{var}\,(X)$ denotes its covariance matrix.

2. PROBLEM FORMULATION

Consider the following stochastic system with uncertain dynamics and unknown measurement bias:

$$\begin{cases} x_{k+1} = Ax_k + B\xi_k(x_k) + \omega_{k+1}, \\ \quad y_k = Cx_k + Db + \nu_k, \end{cases} \quad k = 0, 1, \cdots, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state to be estimated, $y_k \in \mathbb{R}^m$ is the measurement contaminated by unknown bias and noise, $\xi_k \in \mathbb{R}^p$ is the uncertain dynamics, $b \in \mathbb{R}^q$ is the measurement bias, $\omega_k \in \mathbb{R}^n$ and $\nu_k \in \mathbb{R}^m$ are zero mean white process noise with covariance matrices $Q \succ \mathbf{0}$ and $R \succ \mathbf{0}$, respectively. x_0, ω_k and ν_k are assumed independent. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times q}$ are all known matrices. In addition, we assume A is invertible.

We aim to develop a state estimation algorithm to reconstruct the system state, despite of the above uncertainties. To ensure the well-posedness of the state estimation problem, some assumptions on the system structure and uncertain dynamics are introduced. Firstly, it is natural to assume that the nominal system without bias and uncertain dynamics is observable.

Assumption 1. The pair (A, C) is observable.

Different from the fast changing process noise, we assume ξ_k is slow time-varying, as show in the following assumptions.

Assumption 2. The change of uncertain dynamics at each step is bounded, i.e.,

$$\mathbb{E}\left\{\delta_k\delta_k^T\right\} \preceq Q_\delta, \quad \delta_k \triangleq \xi_k(x_k) - \xi_{k-1}(x_{k-1}), \tag{2}$$

where $Q_\delta \succeq \mathbf{0}$ is known matrix.

Obviously, compared with the completely irregular process noise and measurement noise, the uncertain dynamics $\xi_k(x_k)$ and unknown bias b are likely to be estimated. Therefore, it is an intuitive and simple idea to treat $\xi_k(x_k)$ and b as augmented states being estimated and attenuated. As a result, system (1) can be equivalently transformed to

$$\begin{cases} \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \\ b_{k+1} \end{bmatrix} = \begin{bmatrix} A & B & \mathbf{0}_{n\times q} \\ \mathbf{0}_{p\times n} & \mathbf{I}_p & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times p} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} x_k \\ \xi_k \\ b_k \end{bmatrix} + \begin{bmatrix} \omega_{k+1} \\ \delta_{k+1} \\ \mathbf{0}_{q\times 1} \end{bmatrix}, \\ \qquad y_k = [C \ \mathbf{0}_{m\times p} \ D] \begin{bmatrix} x_k \\ \xi_k \\ b_k \end{bmatrix} + \nu_k, \quad k = 0, 1, \cdots. \end{cases} \tag{3}$$

For notational convenience, denote

$$\begin{aligned} X_k &\triangleq \left[x_k^T \ \xi_k^T \ b_k^T\right]^T, \quad \Delta_k \triangleq \left[\omega_k^T \ \delta_k^T \ \mathbf{0}_{1\times q}\right]^T, \\ \bar{A} &\triangleq \begin{bmatrix} A & B & \mathbf{0}_{n\times q} \\ \mathbf{0}_{p\times n} & \mathbf{I}_p & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times p} & \mathbf{I}_q \end{bmatrix}, \quad \bar{C} \triangleq [C \ \mathbf{0}_{m\times p} \ D]. \end{aligned} \tag{4}$$

As a result, system (3) can be rewritten as

$$\begin{cases} X_{k+1} = \bar{A}X_k + \Delta_{k+1}, \\ \quad y_k = \bar{C}X_k + \nu_k, \end{cases} \quad k = 0, 1, \cdots. \tag{5}$$

Formally, system (5) appears to be structurally similar to the standard Kalman filter model. Nevertheless, it is important to point out that due to the presence of δ_k, the process error $\Delta_k(\cdot)$ in system (5) is highly correlated with the system state, which is far beyond the white noise hypothesis required by standard Kalman filter. Thus, how to deal with this uncertain correlation is a fundamental problem to be solved. In the next section, this problem will be discussed in detail.

3. FILTER DESIGN

Based on the augmented system (5), we consider the following filtering structure:

$$\hat{X}_k = \bar{X}_k + K_k\left(y_k - \bar{C}\bar{X}_k\right), \tag{6a}$$

$$\bar{X}_{k+1} = \bar{A}\hat{X}_k, \tag{6b}$$

where $\bar{X}_k$ and $\hat{X}_k$ are the state prediction and state update at the k-th moment, respectively. And K_k is the filter gain to be designed, such that filter (6) can retain some basic features similar to the standard Kalman filter.

One of the most fundamental properties of an estimator is that the true estimation errors should be consistent with their predicted statistics. Owing to this, the following definition is introduced.

Definition 1. (Julier and Uhlmann, 1997) Consider a random vector x. Further, let $\hat{x}$ be an estimate of x and P an estimate of the corresponding error covariance. Then, the pair $(\hat{x}, P)$ is said to be consistent if

$$\mathbb{E}\left\{(\hat{x}-x)(\hat{x}-x)^T\right\} \preceq P. \tag{7}$$

Consistency implies that the estimated error covariance P be an upper bound of the true error covariance. This property becomes even more important for the filter with unknown cross correlation between process error and system state. Based on the concept of consistency, the following theorem provides a design principle for filter gain K_k.

Theorem 1. Consider the system (1) with Assumption 2 and the filtering structure (6) with initialization $\bar{X}_0 = \mathbb{E}\{X_0\}$ and $\bar{P}_0 = \mathrm{var}(X_0)$. For any positive sequence η_k, the pairs $(\bar{X}_k, \bar{P}_k)$ and $(\hat{X}_k, P_k)$ are both consistent, where $\bar{P}_k$ and P_k are recursively calculated through

$$P_k = \left(\mathbf{I} - K_k\bar{C}\right)\bar{P}_k\left(\mathbf{I} - K_k\bar{C}\right)^T + K_kRK_k^T, \tag{8a}$$
$$\bar{P}_{k+1} = (1+\eta_{k+1})\bar{A}P_k\bar{A}^T + \bar{Q}_{k+1}, \tag{8b}$$

and

$$\bar{Q}_{k+1} = \begin{bmatrix} Q & \mathbf{0}_{n\times p} & \mathbf{0}_{n\times q} \\ \mathbf{0}_{p\times n} & \dfrac{1+\eta_{k+1}}{\eta_{k+1}}Q_\delta & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times p} & \mathbf{0}_{p\times p} \end{bmatrix}. \tag{9}$$

Proof. The proof can be achieved by mathematical induction. According to the filter structure (6), the estimation error $\bar{e}_k \triangleq \bar{X}_k - X_k$ and $e_k \triangleq \hat{X}_k - X_k$ satisfy

$$e_k = \left(\mathbf{I} - K_k\bar{C}\right)\bar{e}_k + K_k\nu_k, \tag{10a}$$
$$\bar{e}_{k+1} = \bar{A}e_k - \Delta_{k+1}. \tag{10b}$$

Suppose that at the k-th moment, $\mathbb{E}\left\{\bar{e}_k\bar{e}_k^T\right\} \preceq \bar{P}_k$. Since $\mathbb{E}\left\{\bar{e}_k\nu_k^T\right\} = \mathbf{0}$, according to (8a) and (10a), it is immediate to see that $\mathbb{E}\left\{e_ke_k^T\right\} \preceq P_k$. On the other hand, due to $\mathbb{E}\left\{e_k\omega_{k+1}^T\right\} = \mathbf{0}$, by (10b), the mean square estimation error of $\bar{X}_{k+1}$ satisfies

$$\mathbb{E}\left\{\bar{e}_{k+1}\bar{e}_{k+1}^T\right\} = \bar{A}\mathbb{E}\left\{e_ke_k^T\right\}\bar{A}^T + \begin{bmatrix} Q & \mathbf{0}_{n\times p} & \mathbf{0}_{n\times q} \\ \mathbf{0}_{p\times n} & \mathbb{E}\left\{\delta_{k+1}\delta_{k+1}^T\right\} - \mathbb{E}\left\{\bar{A}e_k\delta_{k+1}^T + \delta_{k+1}e_k^T\bar{A}^T\right\} & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times p} & \mathbf{0}_{p\times p} \end{bmatrix}.$$

Then, according to the Youngs inequality for the matrices case, for any positive scalar η_{k+1}, there is

$$\mathbb{E}\left\{\bar{A}e_k\delta_{k+1}^T + \delta_{k+1}e_k^T\bar{A}^T\right\} \leq \eta_{k+1}\bar{A}\mathbb{E}\left\{e_ke_k^T\right\}\bar{A}^T + \frac{1}{\eta_{k+1}}\mathbb{E}\left\{\delta_{k+1}\delta_{k+1}^T\right\}.$$

Therefore, according to Assumption 2, (8b) and (9) imply $\mathbb{E}\left\{\bar{e}_{k+1}\bar{e}_{k+1}^T\right\} \preceq \bar{P}_{k+1}$. The proof can be concluded by noting that, with the initialization of $\bar{X}_0$ and $\bar{P}_0$, the consistency holds at time instant $k = 0$. □

Remark 1. Different from the standard Kalman filter, since the coupling issue between the δ_{k+1} and system state in prediction step, equation (8b) adds a scaling term to handle the cross term between e_k and δ_{k+1} with the assistance of Young's inequality. And based on the consistency of $(\hat{X}_k, P_k)$, the filter gain is derived from the following optimization problem:

$$K_k^* = \underset{K_k}{\mathrm{argmin}} P_k = \bar{P}_k\bar{C}^T\left(\bar{C}\bar{P}_k\bar{C}^T + R\right)^{-1}. \tag{11}$$

Remark 2. In (8b), a new parameter η_k has been introduced. It can be selected as a positive constant for simplicity or derived from the following optimization problem:

$$\eta_k^* = \underset{\eta_k}{\mathrm{argmin}}\ \mathrm{tr}\left(\bar{P}_k\right) = \sqrt{\frac{\mathrm{tr}\left(Q_\delta\right)}{\mathrm{tr}\left(\bar{A}P_{k-1}\bar{A}^T\right)}}. \tag{12}$$

Summarizing the above results, a complete solution to the problem of state estimation for system (1) or (5) is provided in Algorithm 1.

Algorithm 1 Extended State based Kalman Filter (ESKF)

1: **if** $k = 0$ **then**
2: Initializes $\bar{X}_0 = \mathbb{E}\{X_0\}$, $\bar{P}_0 = \mathrm{var}(X_0)$.
3: **end if**
4: Updates the state estimation and approximated covariance

$$K_k = \bar{P}_k\bar{C}^T\left(\bar{C}\bar{P}_k\bar{C}^T + R\right)^{-1},$$
$$\hat{X}_k = \bar{X}_k + K_k\left(y_k - \bar{C}\bar{X}_k\right),$$
$$P_k = \left(\mathbf{I} - K_k\bar{C}\right)\bar{P}_k\left(\mathbf{I} - K_k\bar{C}\right)^T + K_kRK_k^T.$$

5: Predicts the state prediction and approximated covariance

$$\bar{X}_{k+1} = \bar{A}\hat{X}_k,$$
$$\bar{Q}_{k+1} \triangleq \begin{bmatrix} Q & \mathbf{0}_{n\times p} & \mathbf{0}_{n\times q} \\ \mathbf{0}_{p\times n} & \dfrac{1+\eta_{k+1}}{\eta_{k+1}}Q_\delta & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times n} & \mathbf{0}_{q\times p} & \mathbf{0}_{p\times p} \end{bmatrix},$$
$$\bar{P}_{k+1} = (1+\eta_{k+1})\bar{A}P_k\bar{A}^T + \bar{Q}_{k+1}.$$

4. FILTER PERFORMANCE ANALYSIS

Since system (1) has both nonlinear uncertain dynamics, unknown bias and stochastic noise, how to ensure the stability of the proposed algorithm should be another significant issue to be studied. Firstly, a necessary and sufficient condition for the observability of the pair $\left(\bar{A}, \bar{C}\right)$ is given, which is also an fundamental condition to ensure the stability of ESKF.

Lemma 1. Under Assumption 1, the pair $\left(\bar{A}, \bar{C}\right)$ is observable if and only if

$$\mathrm{rank}\left(\begin{bmatrix} A - \mathbf{I}_n & B & \mathbf{0}_{n\times q} \\ C & \mathbf{0}_{m\times p} & D \end{bmatrix}\right) = n + p + q. \tag{13}$$

Proof. The observability of the pair $\left(\bar{A}, \bar{C}\right)$ is equivalent to

$$\mathrm{rank}\left(\begin{bmatrix} \bar{A} - s\mathbf{I}_{n+p+q} \\ \bar{C} \end{bmatrix}\right) = n + p + q, \quad \forall s \in \mathbb{C}. \tag{14}$$

(*Necessity*) In (14), taking $s = 1$ leads to condition (13).

(*Sufficiency*) According to Assumption 1, the pair (A, C) is observable, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix} A - s\mathbf{I}_n \\ C \end{bmatrix}\right) = n, \quad \forall s \in \mathbb{C}.$$

Therefore, while $s \neq 1$, by the definitions of $\bar{A}$ and $\bar{C}$ in (4), equation (14) holds. As for the situation $s = 1$, (14) is directly deduced from condition (13). $\square$

Next, the boundedness of the estimation error of ESKF in mean square sense is discussed.

Theorem 2. Considering the system (1) with Assumptions 1-2 and (13), and the Algorithm 1 with η_k satisfies $\inf_{k\geq 0} \eta_k \triangleq \underline{\eta} > 0$ and $\sup_{k\geq 0} \eta_k \triangleq \overline{\eta} < \infty$, there is

$$\sup_{k\geq 0} \mathbb{E}\left\{\left(\hat{X}_k - X_k\right)\left(\hat{X}_k - X_k\right)^T\right\} \prec \infty. \tag{15}$$

Proof. Due to the consistency of Algorithm 1 in Theorem 1, we just need to prove $\sup_{k\geq 0} P_k \prec \infty$. In fact, P_k in equation (8) can be see as the minimum estimation error covariance of the follow system:

$$\begin{cases} \tilde{X}_{k+1} = \sqrt{1+\eta_{k+1}}\bar{A}\tilde{X}_k + \tilde{\omega}_{k+1}, \\ \tilde{y}_k = \bar{C}\tilde{X}_k + \tilde{\nu}_k, \end{cases} \quad k = 1, 2, \cdots, \tag{16}$$

where $\tilde{\omega}_k$ and $\tilde{\nu}_k$ are white noise with covariance matrices $\bar{Q}_k$ and R, respectively. According to Lemma 1, the pair $\left(\bar{A}, \bar{C}\right)$ is observable. Thus there exist $\alpha > 0$, $\beta > 0$, and a positive integer N, such that

$$\alpha\mathbf{I} \leq \sum_{i=0}^{N} (\bar{A}^{-i})^T \bar{C}^T R^{-1} \bar{C} \bar{A}^{-i} \leq \beta\mathbf{I}.$$

Then by the upper boundedness of η_k, it is easy to verify that system (16) is uniformly completely observable, i.e., there is $\bar{\alpha} = \frac{\alpha}{(1+\overline{\eta})^N}$, for any $k \geq N$, such that

$$\bar{\alpha}\mathbf{I} \leq \sum_{i=k-N}^{k} \Phi_{i,k}^T \bar{C}^T R^{-1} \bar{C} \Phi_{i,k} \leq \beta\mathbf{I},$$

where

$$\Phi_{i,j} \triangleq \begin{cases} \mathbf{I}_{n+p+q}, & \text{if } i = j, \\ \sqrt{1+\eta_j}\Phi_{i,j+1}\bar{A}, & \text{if } i \neq j. \end{cases}$$

Meanwhile, there also is

$$\bar{\gamma} = \max\left\{\lambda_{\max}(Q), \frac{1+\underline{\eta}}{\underline{\eta}}\lambda_{\max}(Q_\delta)\right\} \sum_{i=0}^{N} (1+\overline{\eta})^i \|A\|^{2i},$$

for any $k \geq N$, such that $\sum_{i=k-N}^{k} \Phi_{k,i}\bar{Q}_i\Phi_{k,i}^T \leq \bar{\gamma}\mathbf{I}$. Then according to the Kalman filter theory (Jazwinski, 1970), for any $k \geq N$, there is

$$P_k \leq \frac{1 + (n+p+q)^2\beta\bar{\gamma}}{\bar{\alpha}}\mathbf{I}_{n+p+q},$$

i.e., P_k is upper bounded. $\square$

The following theorem displays the bias estimation of Algorithm 1 is convergent in the mean square sense.

Theorem 3. Considering the system (1) with Assumptions 1-2 and (13), and the Algorithm 1 with positive constant η, there is

$$\lim_{k\to\infty} \mathbb{E}\left\{\left(\hat{b}_k - b\right)\left(\hat{b}_k - b\right)^T\right\} = \mathbf{0}. \tag{17}$$

Proof. Since $P_k \preceq \bar{P}_k$, by the consistency of Algorithm 1, we just need to prove $\lim_{k\to\infty} H\bar{P}_kH^T = \mathbf{0}$, where $H \triangleq \left[\mathbf{0}_{q\times(n+p)} \ \mathbf{I}_q\right]$. Combination of (8a) and (8b) yields

$$\begin{aligned}\bar{P}_{k+1} = &(1+\eta)\bar{A}\bar{P}_k\bar{A}^T - (1+\eta)\bar{A}\bar{P}_k\bar{C}^T \\ &\left(\bar{C}\bar{P}_k\bar{C}^T + R\right)^{-1}\bar{C}\bar{P}_k\bar{A}^T + \bar{Q}.\end{aligned} \tag{18}$$

To analyze the asymptotic properties of $\bar{P}_k$, for any positive scalar ϵ, consider the following Riccati equation

$$\begin{aligned}\bar{P}_{k+1}^{(\epsilon)} = &(1+\eta)\bar{A}\bar{P}_k^{(\epsilon)}\bar{A}^T - (1+\eta)\bar{A}\bar{P}_k^{(\epsilon)}\bar{C}^T \\ &\left(\bar{C}\bar{P}_k^{(\epsilon)}\bar{C}^T + R\right)^{-1}\bar{C}\bar{P}_k^{(\epsilon)}\bar{A}^T + \bar{Q} + \epsilon\mathbf{I}\end{aligned} \tag{19}$$

with $\bar{P}_0^{(\epsilon)} \succeq \bar{P}_0$. With the assistance of the matrix inverse formula, according to (18) and (19), it is easy to verify that for any positive scalar ϵ, $\bar{P}_k \prec \bar{P}_k^{(\epsilon)}$. Therefore, to prove (17), we just need to validate $\lim_{\epsilon\to 0}\lim_{k\to\infty} H\bar{P}_k^{(\epsilon)}H^T = \mathbf{0}$. Since $\left((1+\eta)\bar{A}, \bar{C}\right)$ is observable and $\bar{Q} + \epsilon\mathbf{I}$ is positive definite, $\bar{P}_k^{(\epsilon)}$ has a limitation which is the unique positive definite solution of the following Riccati equation

$$\begin{aligned}\bar{P}^{(\epsilon)} = &(1+\eta)\bar{A}\bar{P}^{(\epsilon)}\bar{A}^T - (1+\eta)\bar{A}\bar{P}^{(\epsilon)}\bar{C}^T \\ &\left(\bar{C}\bar{P}^{(\epsilon)}\bar{C}^T + R\right)^{-1}\bar{C}\bar{P}^{(\epsilon)}\bar{A}^T + \bar{Q} + \epsilon\mathbf{I}.\end{aligned} \tag{20}$$

According to the expression of $\bar{A}$, $\bar{C}$ and $\bar{Q}$, the Riccati equation (20) implies that

$$\begin{aligned}\bar{P}_{1,2}^{(\epsilon)} = &\tilde{A}\bar{P}_{1,2}^{(\epsilon)} - \tilde{A}\left(\bar{P}_{1,1}^{(\epsilon)}\tilde{C}^T + \bar{P}_{1,2}^{(\epsilon)}D^T\right) \\ &\left(\bar{C}\bar{P}^{(\epsilon)}\bar{C}^T + R\right)^{-1}\left(\tilde{C}\bar{P}_{1,2}^{(\epsilon)} + D\bar{P}_{2,2}^{(\epsilon)}\right), \\ \bar{P}_{2,2}^{(\epsilon)} = &\bar{P}_{2,2}^{(\epsilon)} - \left(\bar{P}_{2,1}^{(\epsilon)}\tilde{C}^T + \bar{P}_{2,2}^{(\epsilon)}D^T\right)\left(\bar{C}\bar{P}^{(\epsilon)}\bar{C}^T\right. \\ &\left.+R\right)^{-1}\left(\tilde{C}\bar{P}_{1,2}^{(\epsilon)} + D\bar{P}_{2,2}^{(\epsilon)}\right) + \epsilon\mathbf{I}_q,\end{aligned} \tag{21}$$

where

$$\begin{aligned}&\bar{P}_{1,1}^{(\epsilon)} = G\bar{P}^{(\epsilon)}G^T, \quad \bar{P}_{1,2}^{(\epsilon)} = G\bar{P}^{(\epsilon)}H^T, \\ &\bar{P}_{2,1}^{(\epsilon)} = H\bar{P}^{(\epsilon)}G^T, \quad \bar{P}_{2,2}^{(\epsilon)} = H\bar{P}^{(\epsilon)}H^T, \\ &\tilde{A} \triangleq \sqrt{1+\eta}\begin{bmatrix} A & B \\ \mathbf{0}_{p\times n} & \mathbf{I}_p \end{bmatrix}, \quad \begin{aligned}&\tilde{C} \triangleq \left[C \ \mathbf{0}_{m\times p}\right], \\ &G \triangleq \left[\mathbf{I}_{(n+p)} \ \mathbf{0}_{(n+p)\times q}\right].\end{aligned}\end{aligned}$$

Let $\epsilon \to 0$, by the positive of R, equation (21) yields

$$\begin{bmatrix} \tilde{A} - \mathbf{I}_{n+p} & \mathbf{0}_{(n+p)\times q} \\ \tilde{C} & D \end{bmatrix}\begin{bmatrix} \lim\limits_{\epsilon\to 0}\bar{P}_{1,2}^{(\epsilon)} \\ \lim\limits_{\epsilon\to 0}\bar{P}_{2,2}^{(\epsilon)} \end{bmatrix} = \mathbf{0}_{(n+p+q)\times q}. \tag{22}$$

On the other hand, according to condition (13),

$$\begin{bmatrix} \tilde{A} - \mathbf{I}_{n+p} & \mathbf{0}_{(n+p)\times q} \\ \tilde{C} & D \end{bmatrix}$$

is column full rank. Therefore $\lim_{\epsilon\to 0}\bar{P}_{2,2}^{(\epsilon)} = \mathbf{0}$. $\square$

5. NUMERICAL SIMULATION

To illustrate the effectiveness of ESKF, the simulation for the following stochastic system with uncertain dynamics and measurement bias is considered:

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_k + \omega_{k+1}, \\ y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} b + \nu_k. \end{cases} \tag{23}$$

The covariance matrices of process noise ω_k and measurement noise ν_k are set to be $Q = \mathbf{I}_2$ and $R = \mathbf{I}_2$, respectively. It is assumed that the initial value of the system state satisfies $\mathbb{E}\{x_0\} = [0\ 0]^T$ and $\mathrm{var}(x_0) = 100\mathbf{I}_2$. The measurement bias is assumed to be $b = 10$. As for the uncertain dynamics, two different cases are carried out:

$$\text{Case 1:} \quad \xi_k = 5\sin(0.02k) + \frac{x_k(1) + x_k(2)}{1 + x_k^T x_k} - 10, \tag{24a}$$

$$\text{Case 2:} \quad \xi_k = -0.1k + \frac{x_k(1) + x_k(2)}{1 + x_k^T x_k} - 10, \tag{24b}$$

where $x_k(1)$ and $x_k(2)$ are the first and second elements of x_k, respectively. And in ESKF the parameters are selected as $X_0 = \mathbf{0}_{4\times 1}$, $P_0 = 100\mathbf{I}_4$, $Q_\delta = 0.01$ and $\eta_k = \sqrt{\frac{\mathrm{tr}(Q_\delta)}{\mathrm{tr}(\bar{A}P_{k-1}\bar{A}^T)}}$, respectively.

Fig. 1 and Fig. 2 display the estimation results of ESKF in Case 1 and Case 2, respectively. In the left half of these two figures, the blue solid lines represent one sample of the system state data generated by system (23) and the red dash lines are the corresponding estimation result of ESKF. As for the right half parts, the red dash lines stand for the mean square errors (MSE) of ESKF obtained from 500 statistical experiments, and the blue solid lines represent the diagonal elements of P_k provided by ESKF. As can be seen from these two figure, the estimation result of ESKF can track the true state well. Meanwhile, the estimation error covariances of ESKF keep stable in the given period and the consistency remains, which validates Theorem 1 and Theorem 2. Fig. 3 compares the estimation errors of Kalman filter, augmented state Kalman filter (ASKF) which treats bias as an augmented state and ESKF in Case 1. And Fig. 4 shows the results in Case 2. In these two figures, the black solid lines, green solid lines and blue solid lines represent the maximum and minimum estimation errors obtained from 500 statistical experiments of KF, ASKF and ESKF, respectively. And the dark purple dash lines, pink dash lines and red dash lines are one sample from each 500 experiments. It can be seen in these two figures, the estimation results of KF and ASKF both have biases affected by uncertain dynamics and measurement bias, but with the assistance of timely estimating uncertain dynamics and measurement bias, ESKF can eliminate such estimation bias. In addition, in Fig. 4, the estimation error of ASKF is even greater than KF, which means that for the system with large uncertain dynamics, the measurement bias cannot be estimated independently regardless of the uncertain dynamics.

6. CONCLUSION

This paper studied the state estimation problem for a class of stochastic systems with uncertain dynamics and unknown measurement bias. With the idea of timely estimating the uncertain dynamics and unknown bias, an extended state based filter structure was developed. The consist filter was proposed. In addition, a necessary and sufficient condition for the observability of augmented system was presented, such that the stability of the proposed filter was analyzed. It is shown that the designed filter can realize the estimation of measurement bias. A numerical simulation was also carried out to illustrate the

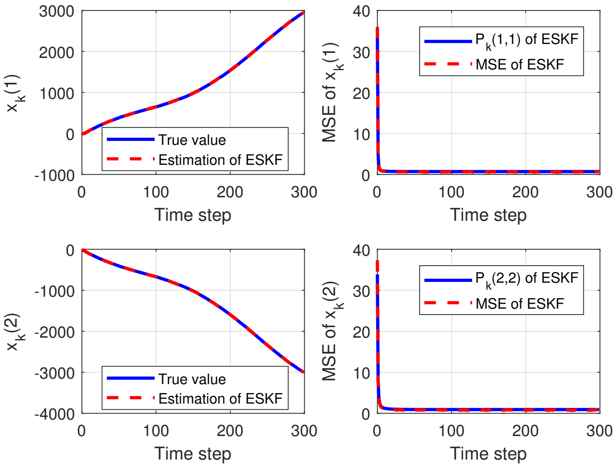

Fig. 1. *Left part:* The system state data generated by system (23) and the estimation of ESKF in Case 1; *Right part:* The mean square errors of ESKF and the diagonal elements of P_k provided by ESKF in Case 1.

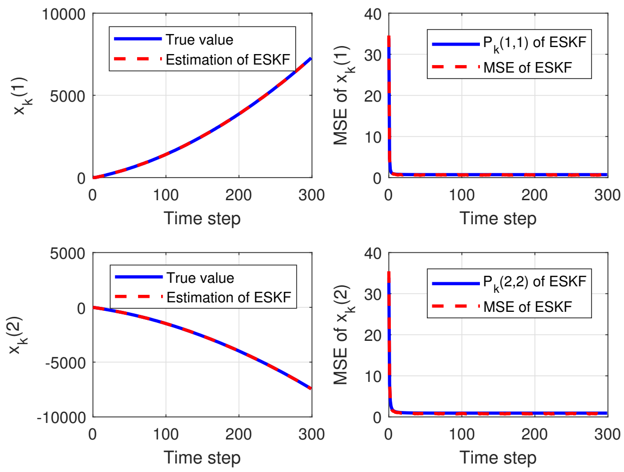

Fig. 2. *Left part:* The system state data generated by system (23) and the estimation of ESKF in Case 2; *Right part:* The mean square errors of ESKF and the diagonal elements of P_k provided by ESKF in Case 2.

effectiveness of proposed method. In this paper, the bias is assumed to be a unknown constant, and it can be extended to the situation that the bias satisfies certain model without much work. While the bias model information is incomplete, it may be another interesting problem.

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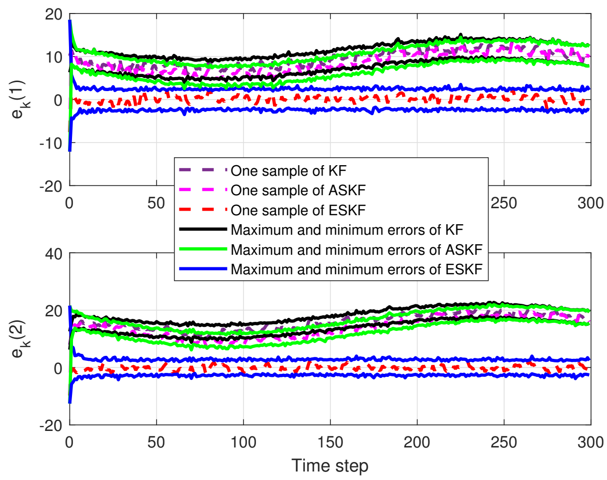

Fig. 3. The estimation errors of KF, ASKF and ESKF in Case 1.

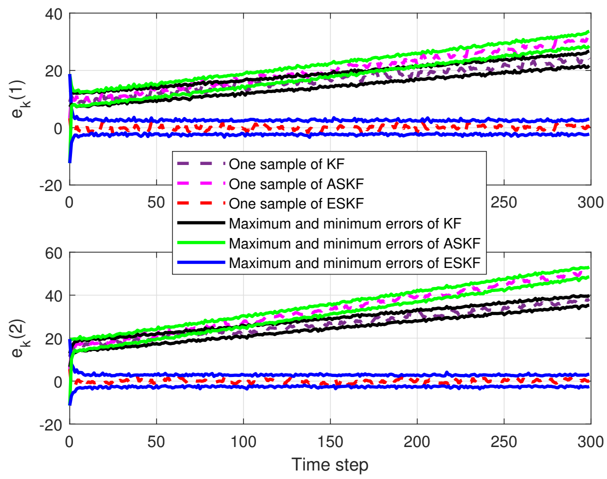

Fig. 4. The estimation errors of KF, ASKF and ESKF in Case 2.

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