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Consistent Parameter Estimators for Second-order Modulus Systems with Non-additive Disturbances ⋆

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Abstract: This work deals with a class of nonlinear regression models called second-order modulus models. It is shown that the possibility of obtaining consistent parameter estimators for these models depends on how process disturbances enter the system. Two scenarios where consistency can be achieved for instrumental variable estimators despite non-additive system disturbances are demonstrated, both in theory and by simulation examples.

Keywords: System identification, nonlinear models, physical models, marine systems, disturbance rejection.

1. INTRODUCTION

As marine vessels are becoming increasingly autonomous, having accurate simulation models available is turning into an absolute necessity. This holds both for facilitation of development and for achieving satisfactory model-based control. Linear theory is useful for analyzing ship motion performed within close proximity to an equilibrium point. It is however not useful for accurately predicting the characteristics of tight maneuvers, that are for example used during docking at ports.

Ship dynamics depend on the forces and moments acting on the ship according to Newton's laws of motion. Except for actuators, like thrusters and rudders, also environmental forces affect the steering dynamics in this way. Dealing with these, typically quite impactful process disturbances, in a correct way is quite challenging already in the linear case and becomes even more difficult when models are nonlinear.

The challenges of parameter estimation for nonlinear model classes are widely known, see for example Ljung (2010). As a consequence there is a substantial research effort focused on the problem. One possible way of approaching it is to consider cases where the Maximum Likelihood (ML) problem can be formulated and solved. In Schön et al. (2011) this was done using the Expectation Maximization algorithm and particle smoothing. In Abdalmoaty (2019) a prediction-error perspective with suboptimal predictors was explored. The results showed that linear predictors can give consistent estimators in a prediction-error framework, for a quite large class of nonlinear models. Larsson (2019) investigated the possibility of having a parameterized linear observer capturing unmodelled disturbance characteristics. This linear observer was an easily accessible way of compensating for miss-specified predictors.

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Different parametric model structures for ship dynamics have been proposed in the past, see for example Fossen (1994). The aim of this work is not to develop new theory in that regard. Instead, focus is on a fairly general class of nonlinear regression models for marine vessels called second-order modulus models. For this model class consistent parameter estimators are suggested, that are robust to data being influenced by environmental disturbances in the form of wind, waves, and ocean currents.

In general, formulating the ML problem for parameter estimation requires prior knowledge about the disturbances' probability distributions. Since the environmental disturbances considered here usually are not well-described as white Gaussian processes, the required prior knowledge is not even necessarily restricted to first and second-order moments. This makes the ML method unsuitable in the studied scenario. Another conceivable approach is the least squares (LS) method. Since the work deals with regression models the LS estimate can readily be formulated. However, such an approach usually provides biased estimators under errors-in-variables conditions.

In Ljungberg and Enqvist (2019) the issue of obtaining consistent parameter estimators for this class of model structures under additive measurement noise, was explored. It was shown that the accuracy of an instrumental variable (IV) estimator could be improved by conducting experiments where the input signal had a static offset of sufficient amplitude and the instruments were forced to have zero mean. This work complements that study by acknowledging the possibility of non-additive process disturbances with unknown probability distributions.

2. PROBLEM FORMULATION

Most model structures used for ships stem from one of two basic ideas. The first is to base the model on a Taylor expansion and was first suggested by Abkowitz (1964). If Taylor expansions are considered, the even-order terms are often neglected. This is done in order to enforce that the model behaves in the same way for positive and negative relative velocities, something that is necessary due to ship symmetry. The other type of model structure was first proposed by Fedyaevsky and Sobolev (1964) and is the one studied in this work. This type of model structure is based on a combination of physical effects such as circulation and cross-flow drag principles, properties that are usually well-described by quadratic functions. The constraint of having a symmetric model is therefore instead resolved by use of the modulus function. Models of this type typically do not include any terms of higher order than two and are therefore referred to as second-order modulus models. For describing a general second-order modulus model, it is convenient to first define a second-order modulus function.

Definition 2.1. A second-order modulus function is a function, $f\ :\ \mathbb{R}^{n+p} \to \mathbb{R}^m$ that can be written as

$$f(\mathbf{x}, \theta) = \Phi^T(\mathbf{x})\theta,$$

where each element of the $p \times m$ matrix $\Phi(\mathbf{x})$ is on one of the forms $x_i, |x_i|, x_i x_j, x_i|x_j|$ for $i, j \leq n$ or zero and $\theta \in \mathbb{R}^p$ is a vector of coefficients.

Since the second-order modulus models are based on physical principles, they are usually first formulated in continuous time. A continuous-time second-order modulus model can however be cast as a discrete-time model with the same type of terms, using for example Euler's explicit method. If the sampling frequency is sufficiently much faster than the frequency of the signal variations, the accuracy of this approximation will be good. The remainder of the paper will deal exclusively with discrete-time models.

Consider a nonlinear discrete-time state-space system with n states, m inputs and n outputs

$$\mathbf{x}(k+1) = f(\begin{bmatrix}\mathbf{x}(k) + \mathbf{v}(k) \\ \mathbf{u}(k)\end{bmatrix}, \theta_0) + \mathbf{w}(k), \tag{1a}$$

$$\mathbf{y}(k) = \mathbf{x}(k) + \mathbf{e}(k), \tag{1b}$$

where $\mathbf{u}(k)$ is the known input signal, $\mathbf{x}(k)$ is a vector consisting of the latent system states, all of which are measured directly (with noise) and collected in the output vector $\mathbf{y}(k)$. Moreover, $\mathbf{v}(k)$ and $\mathbf{w}(k)$ are external signals that are assumed to be unknown (process disturbances) and $\mathbf{e}(k)$ constitutes an additive measurement error, which is also assumed to be unknown. The system is described by the parameter vector θ_0, which does not vary over time. Further, the following premises regarding the system are assumed to be imposed.

A1. f is a second-order modulus function in agreement with Definition 2.1 and its structure is known.

A2. The measurement noise $\mathbf{e}(k)$ is a stationary signal with zero mean and well-defined moments of any order. Also, its amplitude is limited, $-\eta_e < \mathbf{e}(k) < \eta_e$.

A3. The process disturbance $\mathbf{w}(k)$ is a stationary signal with well-defined moments of any order. Moreover, $\mathbf{w}(k)$ is independent of $\mathbf{e}(k)$.

A4. The system is operating in open loop, *i.e.* the input, $\mathbf{u}$, does not depend on the measured states, $\mathbf{y}$, and is consequently assumed to be independent of the disturbances.

Following Definition 2.1, this system can be expressed as

$$\mathbf{x}(k+1) = \Phi^T(\begin{bmatrix}\mathbf{x}(k) + \mathbf{v}(k) \\ \mathbf{u}(k)\end{bmatrix})\theta_0 + \mathbf{w}(k), \tag{2a}$$

$$\mathbf{y}(k) = \mathbf{x}(k) + \mathbf{e}(k). \tag{2b}$$

Since the structure of the true system is known by Assumption A1, it is reasonable to consider the one-step ahead predictor model

$$\hat{\mathbf{y}}(k|\theta) = \Phi^T(\begin{bmatrix}\mathbf{y}(k-1) \\ \mathbf{u}(k-1)\end{bmatrix})\theta, \tag{3}$$

for which an additional assumption is made.

A5. The model structure is globally identifiable according to the definition in Ljung (1999).

The problem considered in this work is to develop consistent estimators of the unknown parameter vector θ. This is done based on two different assumptions regarding the process disturbance $\mathbf{v}(k)$. First it will be assumed that $\mathbf{v}(k)$ has zero mean. Then a situation where $\mathbf{v}(k)$ is of more general character but is measured will be studied.

The suggested estimators will be based on the IV method. The IV estimate is

$$\hat{\theta}_N^{IV} = \text{sol}\left\{\frac{1}{N}\sum_{k=1}^{N}\mathbf{Z}(k)[\mathbf{y}(k) - \Phi^T(k)\theta] = 0\right\}, \tag{4}$$

where $\mathbf{Z}(k)$ is called the instrument matrix and the notation sol$\{f(\mathbf{x}) = 0\}$ is used for the solution to the system of equations $f_i(\mathbf{x}) = 0,\ i = 1, \ldots,\ n$. From (4) it can be noted that the IV estimator will be consistent if

$$\bar{E}\{\mathbf{Z}(k)\Phi^T(k)\} \text{ is full rank}, \tag{5a}$$

$$\bar{E}\{\mathbf{Z}(k)[\mathbf{y}(k) - \Phi^T(k)\theta_0]\} = 0, \tag{5b}$$

i.e. if the instruments are correlated with the regressors but uncorrelated with the optimal model residual. See Ljung (1999) for more details, where also the notation $\bar{E}\{.\} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}E\{.\}$ was adopted from.

Assume that N_E experiments are performed, where in each N_D data points are collected and that for each experiment, E, there is an $p \times n$ instrument matrix

$$\mathbf{Z}_E(k) = \begin{bmatrix}\zeta_{E,1}(k) & \ldots & \zeta_{E,n}(k)\end{bmatrix},$$

that fulfills the following assumptions.

A6. $\mathbf{Z}_E$ is independent of the noise signals $\mathbf{e}$, $\mathbf{v}$, and $\mathbf{w}$.

A7. $\bar{E}\{\mathbf{Z}_E(k)\} = 0$ and all the moments of higher order are well-defined.

Since an exact solution to (4) might not exist, the IV estimate is obtained as the least-squares solution to the system of pN_E equations

$$\begin{cases}\dfrac{1}{N_D}\displaystyle\sum_{k=1}^{N_D}\mathbf{Z}_1(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix}\mathbf{y}(k-1) \\ \mathbf{u}(k-1)\end{bmatrix})\theta] = 0, \\ \qquad\vdots \\ \dfrac{1}{N_D}\displaystyle\sum_{k=(N_E-1)N_D+1}^{N_E N_D}\mathbf{Z}_{N_E}(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix}\mathbf{y}(k-1) \\ \mathbf{u}(k-1)\end{bmatrix})\theta] = 0.\end{cases} \tag{6}$$

Finally it is assumed that when $N_D \to \infty$ the parameters can be determined uniquely, *i.e.* that the data from all the experiments combined are sufficiently informative.

A8. $\bar{E}\{\begin{bmatrix}\mathbf{Z}_1(k)\\ \vdots \\ \mathbf{Z}_{N_E}(k)\end{bmatrix}\Phi^T(\begin{bmatrix}\mathbf{y}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\}$ is full rank.

3. ZERO-MEAN DISTURBANCE

Now the scenario where the first-order moment of $\mathbf{v}(k)$ is zero will be addressed.

A9. The process disturbance $\mathbf{v}(k)$ is a stationary white-noise signal with zero mean and well-defined moments of any higher order. Also, the amplitude of $\mathbf{v}(k)$ is limited, $-\eta_v < \mathbf{v}(k) < \eta_v$ and it is independent of $\mathbf{e}(k)$ as well as of earlier values of $\mathbf{w}(k)$, *i.e.* $\bar{E}\{\mathbf{v}(k)\mathbf{w}(l)\} = \bar{E}\{\mathbf{v}(k)\}\bar{E}\{\mathbf{w}(l)\}$, $\forall\ k > l$.

The proposed estimator will be based on an assumption regarding experiment design.

A10. The input in each experiment is such that it excites the system to the extent that each of its states, $x_1(k), \ \dots \ x_n(k)$, continuously has an amplitude that is sufficiently well-separated from the origin

$$\big|x_i(k)\big| > \max(\eta_{e,i}, \eta_{v,i}) \triangleq \eta_i,$$

for $k = 1, \ \dots \ N_D$ and $i = 1, \ \dots \ n$.

Performing experiments in accordance with A10 was a key step proposed in Ljungberg and Enqvist (2019) that is central, also for the ideas presented here. The assumption makes it possible to temporarily treat second-order modulus functions as normal second-order functions, during the analysis of the parameter estimation.

Lemma 3.1. Provided that Assumptions A1-A10 are fulfilled, the IV method defined by (6) is a consistent estimator of θ.

Proof. Under the assumptions, the consistency of the IV method can be investigated by analyzing the unbiasedness of the asymptotic IV estimator, *i.e.* the IV method defined by (6) when $N_D \to \infty$. The requirement (5a) is already fulfilled by A8 which means that a sufficient condition for consistency is that also (5b) holds, *i.e.* that

$$\bar{E}\{\mathbf{Z}_E(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix}\mathbf{y}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0]\} = 0, \qquad (7)$$

for all the experiments $E = 1, \ \dots \ N_E$. Denoting the columns of the regression matrix as $\Phi(.) = \big[\varphi_1(.) \ \dots \ \varphi_n(.)\big]$ it can be seen that (7) is fulfilled if

$$\bar{E}\{\zeta_{E,i}(k)[y_i(k) - \varphi_i^T(\begin{bmatrix}\mathbf{y}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0]\} = 0,$$

for $i = 1, \ \dots \ n$ and $E = 1, \ \dots \ N_E$. Here

$$y_i(k) - \varphi_i^T(\begin{bmatrix}\mathbf{y}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0 = \varphi_i^T(\begin{bmatrix}\mathbf{x}(k-1) + \mathbf{v}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0 + w_i(k-1) + e_i(k) - \varphi_i^T(\begin{bmatrix}\mathbf{y}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0.$$

Since $\bar{E}\{\zeta_{E,i}(k)w_i(k-1)\} = \bar{E}\{\zeta_{E,i}(k)e_i(k)\} = 0$, by A2, A3, A6, and A7 it remains to show that

$$\bar{E}\{\zeta_{E,i}(k)[\varphi_i^T(\begin{bmatrix}\mathbf{x}(k-1) + \mathbf{v}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix}) - \varphi_i^T(\begin{bmatrix}\mathbf{x}(k-1) + \mathbf{e}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})]\} = 0, \qquad (8)$$

holds for all $i = 1, \ \dots, \ n.$ and $E = 1, \ \dots, \ N_E$. This residual vector will consist of a combination of different kinds of elements. Elements on the form u_j, $|u_j|$, $u_j u_l$, or $u_j|u_l|$ are trivially zero since the input is assumed to be perfectly known. Elements on the form $|x_j|$ give

$$\bar{E}\{\zeta_{E,i}(k)[\,\big|x_j(k-1) + v_j(k-1)\big| - \big|x_j(k-1) + e_j(k-1)\big|]\} = \bar{E}\{\zeta_{E,i}(k)[x_j(k-1) + v_j(k-1) - (x_j(k-1) + e_j(k-1))]\} = 0,$$

if $x_j > \eta_j$. This follows by A2, A6, A9, and A10. For the case when $x_j < -\eta_j$ only the sign of the expression changes. Cross-elements on the form $x_j|u_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[(x_j(k-1) + v_j(k-1))\big|u_l(k-1)\big| - (x_j(k-1) + e_j(k-1))\big|u_l(k-1)\big|]\} = \bar{E}\{\zeta_{E,i}(k)\big|u_l(k-1)\big| \cdot (v_j(k-1) - e_j(k-1))\} = 0,$$

which follows from A2, A4, A6, and A9. Cross-elements on the form $u_j|x_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[u_j(k-1)\big|x_l(k-1) + v_l(k-1)\big| - u_j(k-1) \cdot \big|x_l(k-1) + e_l(k-1)\big|]\} = \bar{E}\{\zeta_{E,i}(k)u_j(k-1) \cdot (x_l(k-1) + v_l(k-1) - (x_l(k-1) + e_l(k-1)))\} = \bar{E}\{\zeta_{E,i}(k)u_j(k-1)(v_l(k-1) - e_l(k-1))\} = 0,$$

if $x_l > \eta_l$. This follows by A2, A4, A6, A9, and A10. For the case when $x_l < -\eta_l$ only the sign of the expression changes. Finally elements on the form $x_j|x_l|$ give

$$\begin{aligned}&\bar{E}\{\zeta_{E,i}(k)[(x_j(k-1) + v_j(k-1))\big|x_l(k-1) + v_l(k-1)\big| - (x_j(k-1) + e_j(k-1))\big|x_l(k-1) + e_l(k-1)\big|]\}\\ &= \bar{E}\{\zeta_{E,i}(k)[x_j(k-1)x_l(k-1) + x_j(k-1)v_l(k-1) + v_j(k-1)x_l(k-1) + v_j(k-1)v_l(k-1) - x_j(k-1) \cdot x_l(k-1) - x_j(k-1)e_l(k-1) - e_j(k-1)x_l(k-1) - e_j(k-1)e_l(k-1)]\}\\ &= \bar{E}\{\zeta_{E,i}(k)(x_j(k-1)[v_l(k-1) - e_l(k-1)] + x_l(k-1)[v_j(k-1) - e_j(k-1)] + v_j(k-1)v_l(k-1) - e_j(k-1)e_l(k-1))\} = 0,\end{aligned}$$

if $x_l > \eta_l$. This follows by A2, A3, A4, A6, A7, A9, and A10. For the case when $x_l < -\eta_l$ only the sign of the expression changes.

First and second-order elements without the modulus operator can be seen to equal zero following to the same type of reasoning. Hence, all elements in (8) will be zero, regardless of i, j, l, and E. Conclusively (7) is fulfilled so the estimator for θ is consistent. This concludes the proof.

Remark 3.1. Only the biggest of $\eta_{i,e}$ and $\eta_{i,v}$ is necessary to consider when the experiment is designed.

4. GENERAL DISTURBANCE

Now the scenario where the process disturbance $\mathbf{v}(k)$ is of more general character will be addressed. It will however be assumed that an independent measurement of $\mathbf{v}(k)$ is available. Therefore let $\mathbf{y}_1(k) = \mathbf{x}(k) + \mathbf{e}_1(k)$, $-\eta_{e_1} < \mathbf{e}_1(k) < \eta_{e_1}$, denote the original state measurement.

A11. The process disturbance $\mathbf{v}(k)$ is a signal with well-defined moments of any order and its amplitude is limited, $-\eta_v < \mathbf{v}(k) < \eta_v$.

A12. An unbiased measurement $\mathbf{y}_2(k) = \mathbf{v}(k) + \mathbf{e}_2(k)$ is available. The measurement noise $\mathbf{e}_2(k)$ is a stationary signal with zero mean and well-defined moments of any order. Also, its amplitude is limited, $-\eta_{e_2} < \mathbf{e}_2(k) < \eta_{e_2}$.

A13. $\mathbf{Z}_E$ is independent of $\mathbf{e}_2$.

A14. The measurement disturbances, $\mathbf{e}_1(k)$ and $\mathbf{e}_2(k)$, are independent of the process disturbances, $\mathbf{v}(k)$ and $\mathbf{w}(k)$.

A15. The input in each experiment is such that it excites the system to the extent that each of its states, $x_1(k), \ldots x_n(k)$, continuously has an amplitude that is sufficiently well-separated from the origin

$$\left|x_i(k)\right| > \eta_{e_1,i} + \eta_{e_2,i} + \eta_{v,i} \triangleq \eta_{i,\check{e}} + \eta_{v,i}$$

for $k = 1, \ldots N_D$ and $i = 1, \ldots n$.

Let $\check{\mathbf{y}}(k) = \mathbf{y}_1(k) + \mathbf{y}_2(k)$ and consider the predictor

$$\hat{\mathbf{y}}(k|\theta) = \Phi^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta. \tag{9}$$

Define the IV estimator as the least-squares solution to the system of pN_E equations

$$\begin{cases} \dfrac{1}{N_D}\displaystyle\sum_{k=1}^{N_D} \mathbf{Z}_1(k)[\mathbf{y}_1(k) - \Phi^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta] = 0, \\ \vdots \\ \dfrac{1}{N_D}\displaystyle\sum_{k=(N_E-1)N_D+1}^{N_E N_D} \mathbf{Z}_{N_E}(k)[\mathbf{y}_1(k) - \Phi^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta] = 0. \end{cases} \tag{10}$$

Lemma 4.1. Provided that Assumptions A1-A8 and A11-A15 are fulfilled, the IV method defined by (10) is a consistent estimator of θ.

Proof. Under the assumptions, the consistency of the IV method can be investigated by analyzing the unbiasedness of the asymptotic IV estimator, *i.e.* the IV method defined by (10) when $N_D \to \infty$. The requirement (5a) is already fulfilled by A8 which means that a sufficient condition for consistency is that also (5b) holds, *i.e.* that

$$\bar{E}\{\mathbf{Z}_E(k)[\mathbf{y}_1(k) - \Phi^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0]\} = 0, \tag{11}$$

for all the experiments $E = 1, \ldots N_E$. Again denoting the columns of the regression matrix as $\Phi(.) = [\varphi_1(.) \ldots \varphi_n(.)]$ it can be seen that (11) is fulfilled if

$$\bar{E}\{\zeta_{E,i}(k)[y_{1,i}(k) - \varphi_i^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0]\} = 0,$$

for $i = 1, \ldots n$ and $E = 1, \ldots N_E$. Here

$$y_{1,i}(k) - \varphi_i^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0 = \varphi_i^T(\begin{bmatrix}\mathbf{x}(k-1)+\mathbf{v}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0 + w_i(k-1) + e_{1,i}(k) - \varphi_i^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})\theta_0.$$

Since $\bar{E}\{\zeta_{E,i}(k)w_i(k-1)\} = \bar{E}\{\zeta_{E,i}(k)e_{1,i}(k)\} = 0$, by A2, A3, A6, and A7 it remains to show that

$$\bar{E}\{\zeta_{E,i}(k)[\varphi_i^T(\begin{bmatrix}\mathbf{x}(k-1)+\mathbf{v}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix}) - \varphi_i^T(\begin{bmatrix}\check{\mathbf{y}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})]\} = 0, \tag{12}$$

holds for all $i = 1, \ldots, n.$ and $E = 1, \ldots, N_E$. Let

$$\mathbf{r}(k) \triangleq \mathbf{x}(k) + \mathbf{v}(k), \tag{13}$$

$$\check{\mathbf{e}}(k) \triangleq \mathbf{e}_1(k) + \mathbf{e}_2(k). \tag{14}$$

Then it holds that $\check{\mathbf{y}}(k) = \mathbf{r}(k) + \check{\mathbf{e}}(k)$. Further note that A15 and (13) implies that

$$\left|x_i(k)\right| > \eta_{\check{e},i} + \left|v_i(k)\right| \implies \left|r_i(k)\right| > \eta_{\check{e},i}.$$

Verifying (12) is equivalent to verifying

$$\bar{E}\{\zeta_{E,i}(k)[\varphi_i^T(\begin{bmatrix}\mathbf{r}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix}) - \varphi_i^T(\begin{bmatrix}\mathbf{r}(k-1)+\check{\mathbf{e}}(k-1)\\ \mathbf{u}(k-1)\end{bmatrix})]\} = 0.$$

The remainder of the proof is similar to that of Lemma 3.1 but is still included for completeness. The residual vector above will consist of a combination of different kinds of elements. Elements on the form $u_j, \left|u_j\right|, u_j u_l$ or $u_j|u_l|$ are trivially zero since the input is assumed to be perfectly known. Elements on the form $\left|r_j\right|$ give

$$\bar{E}\{\zeta_{E,i}(k)[\left|r_j(k-1)\right| - \left|r_j(k-1) + \check{e}_j(k-1)\right|]\} = \bar{E}\{\zeta_{E,i}(k)[r_j(k-1) - (r_j(k-1) + \check{e}_j(k-1))]\} = 0,$$

if $r_j > \eta_{\check{e},j}$. This follows by A2, A6, A12, A13, and A15. For the case when $r_j < -\eta_{\check{e},j}$ only the sign of the expression changes. Cross-elements on the form $r_j|u_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[r_j(k-1)\left|u_l(k-1)\right| - (r_j(k-1) + \check{e}_j(k-1)) \cdot\left|u_l(k-1)\right|]\} = -\bar{E}\{\zeta_{E,i}(k)\left|u_l(k-1)\right|\check{e}_j(k-1)\} = 0,$$

which follows from A2, A4, A6, A12 and A13. Cross-elements on the form $u_j|r_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[u_j(k-1)\left|r_l(k-1)\right| - u_j(k-1)\cdot\left|r_l(k-1) + \check{e}_l(k-1)\right|]\} = \bar{E}\{\zeta_{E,i}(k)u_j(k-1)(r_l(k-1) - (r_l(k-1) + \check{e}_l(k-1)))\} = -\bar{E}\{\zeta_{E,i}(k)u_j(k-1)\check{e}_l(k-1)\} = 0,$$

if $r_l > \eta_{\check{e},l}$. This follows by A2, A4, A6, A12, A13, and A15. For the case when $x_l < -\eta_{\check{e},l}$ only the sign of the expression changes. Finally elements on the form $r_j|r_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[r_j(k-1)\left|r_l(k-1)\right| - (r_j(k-1) + \check{e}_j(k-1)) \cdot\left|r_l(k-1) + \check{e}_l(k-1)\right|]\} = \bar{E}\{\zeta_{E,i}(k)[r_j(k-1)r_l(k-1) - (r_j(k-1) + \check{e}_j(k-1))(r_l(k-1) + \check{e}_l(k-1))]\} = -\bar{E}\{\zeta_{E,i}(k)[r_j(k-1)\check{e}_l(k-1) + \check{e}_j(k-1)r_l(k-1) + \check{e}_j(k-1)\check{e}_l(k-1)]\} = 0,$$

if $r_l > \eta_{\check{e},l}$. This follows by A2, A4, A6, A7, A11, A12, A13, A14 and A15. For the case when $x_l < -\eta_{\check{e},l}$ only the sign of the expression changes.

First and second-order elements without the modulus operator can be seen to equal zero following to the same type of reasoning. Hence, all elements in (12) will be zero, regardless of i, j, l, and E. Conclusively (11) is fulfilled so the estimator for θ is consistent. This concludes the proof.

Remark 4.1. In this scenario, it was never assumed that the disturbances, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{v}$, and $\mathbf{w}$, are white. In fact, $\mathbf{v}$ can even include a deterministic time-dependent component.

Remark 4.2. A15 is a more restricting condition for the experiment design than A10.

Remark 4.3. The main idea of the proof is to consider the aggregation $\mathbf{r}(k) = \mathbf{x}(k) + \mathbf{v}(k)$ as state, temporarily during estimation. Instead of adding a second measurement of the disturbance $\mathbf{v}(k)$ it is sometimes possible to add a measurement of the aggregated state $\mathbf{r}(k)$.

Remark 4.4. Whether it is feasible to measure a system disturbance or not is highly application dependent. For example wind speed and direction can be measured using an anemometer together with a weathervane. It is not uncommon that ships are equipped with these sensors.

5. ILLUSTRATIVE EXAMPLE

It is clearly the case that $\mathbf{v}(k)$, which enters the system through the nonlinearity, is the challenging disturbance to deal with. This will be illustrated by an example. Assume that data is generated based on the single-input single-output (SISO) system

$$x(k+1) = n_0(x(k)+v(k))\big|x(k)+v(k)\big| + f_0u(k) + w(k),$$
$$y_1(k) = x(k) + e_1(k),$$

where $v(k) = \bar{v} + \tilde{v}(k)$ and $\tilde{v}(k)$ is white, uniformly distributed and has zero mean, $-\eta_{\tilde{v}} < \tilde{v}(k) < \eta_{\tilde{v}}$, $w(k)$ is a stationary signal with well-defined moments of any order and $e_1(k)$ is uniformly distributed with zero mean, $-\eta_{e_1} < e_1(k) < \eta_{e_1}$. Consider the predictor

$$\hat{y}(k|\theta) = \begin{bmatrix} y_1(k-1)\big|y_1(k-1)\big| & u(k-1)\end{bmatrix}\begin{bmatrix} n \\ f\end{bmatrix} \triangleq \varphi^T(k)\theta,$$

and the input $u(k) = \bar{u}+\tilde{u}(k)$, where $\tilde{u}(k)$ is uniformly distributed with zero mean, $-\eta_{\tilde{u}} < \tilde{u}(k) < \eta_{\tilde{u}}$. If the system is stable this will yield an output that, with the notation $\bar{E}\{x(k)\} = \bar{x}$, can be written as

$$x(k) = \bar{x} + \tilde{x}(k),\ \ \bar{E}\{\tilde{x}(k)\} = 0.$$

For simplicity assume that $\bar{u} > 0$, $\bar{x} > 0$ and

$$x(k) = \bar{x} + \tilde{x}(k) > \max(|\bar{v}| + \eta_{\tilde{v}}, \eta_{e_1}) > 0.$$

Under the stated circumstances it can be concluded that $x(k)+v(k) > 0$ and $x(k)+e_1(k) > 0$ and as a consequence, all occurrences of the modulus operator, both in the system and in the predictor can be ignored. Provided that the input is informative such that (5a) holds, the consistency of an IV estimator can be studied by the fulfillment of (5b), which left-hand-side in this situation is

$$\begin{aligned}
&\bar{E}\{\zeta(k)(y_1(k)-\varphi^T(k)\theta_0)\} = \bar{E}\{\zeta(k)(n_0(x(k-1)+v(k-1))^2\\
&+ f_0u(k-1) + w(k-1) + e_1(k) - n_0(x(k-1)+e_1(k-1))^2\\
&- f_0u(k-1))\} = 2n_0\bar{E}\{\zeta(k)x(k-1)(v(k-1)-e_1(k-1))\}\\
&+ n_0\bar{E}\{\zeta(k)(v(k-1)^2 - e_1(k-1)^2)\} + \bar{E}\{\zeta(k)w(k-1)\}\\
&+ \bar{E}\{\zeta(k)e_1(k)\} = 2n_0\bar{v}\bar{E}\{\zeta(k)x(k-1)\},
\end{aligned}$$

where the last equality follows if $u(k)$ and $\zeta(k)$ are independent of the disturbances, $\bar{E}\{\zeta(k)\} = 0$, and the disturbances are mutually independent. If (5a) holds it is the case that $\bar{E}\{\zeta(k)x(k-1)\} \neq 0$. This means that the estimator will be consistent if $\bar{v} = 0$ but not otherwise.

As described in Section 4, one way to get consistency in the general case when $\bar{v} \neq 0$, is to add a second sensor for measuring the disturbance, $y_2(k) = v(k) + e_2(k)$. Here it will be assumed that also $e_2(k)$ is uniformly distributed with zero mean, $-\eta_{e_2} < e_2(k) < \eta_{e_2}$. Now consider the aggregated-state predictor

$$\hat{y}_r(k|\theta) = \begin{bmatrix} \check{y}(k-1)\big|\check{y}(k-1)\big| & u(k-1)\end{bmatrix}\begin{bmatrix} n \\ f\end{bmatrix} \triangleq \varphi_r^T(k)\theta,$$

where $\check{y}(k) = x(k) + v(k) + \check{e}(k)$ and $\check{e}(k) = e_1(k) + e_2(k)$. In order to be able to ignore the modulus functions it is assumed that

$$x(k) > |\bar{v}| + \eta_{\tilde{v}} + \eta_{e_1} + \eta_{e_2}.$$

This gives another left-hand-side of (5b)

$$\begin{aligned}
&\bar{E}\{\zeta(k)(y_1(k) - \varphi_r^T(k)\theta_0)\} = \bar{E}\{\zeta(k)(n_0(x(k-1)\\
&+ v(k-1))^2 + f_0u(k-1) + w(k-1) + e_1(k)\\
&- n_0(x(k-1) + v(k-1) + \check{e}(k-1))^2 - f_0u(k-1))\}\\
&= -2n_0\bar{E}\{\zeta(k)(x(k-1) + v(k-1))\check{e}(k-1))\}\\
&- n_0\bar{E}\{\zeta(k)\check{e}(k-1)^2\} + \bar{E}\{\zeta(k)w(k-1)\} + \bar{E}\{\zeta(k)e_1(k)\}\\
&= -2n_0\bar{E}\{\check{e}(k-1)\}\bar{E}\{\zeta(k)(v(k-1) + x(k-1))\} = 0,
\end{aligned}$$

where the last equality holds provided that the same assumptions regarding instrument vector, input, and disturbances are imposed. This means that the estimator is consistent.

6. SIMULATIONS

In order to further illustrate the results, simulations were performed using a small-scale second-order modulus system with a single state. According to Fossen (1994), the model

$$\begin{aligned}
\frac{w_a(k+T_s) - w_a(k)}{T_s} &= \tilde{a}_0(w_a(k) - w_c(k)) + n_0(w_a(k)\\
&- w_c(k))\big|w_a(k) - w_c(k)\big| + f_0\tau(k) + \tilde{\xi}(k), \quad (15)
\end{aligned}$$

is describing the hydrodynamic damping of an underwater vehicle's heave motion. Here w is the heave velocity and should not be confused with $\tilde{\xi}$ which is an unmodelled discrepancy between gravity and buoyancy. The subscript a signifies that w_a is the vehicle's absolute velocity which is assumed to be measured, whereas w_c is the speed of the surrounding water. Further, τ is a controllable input thrust.

Using the notation $w_r(k) = w_a(k) - w_c(k)$, assuming unit sampling time, and restructuring of (15) gives the system

$$w_a(k+1) = a_0w_r(k) + n_0w_r(k)\big|w_r(k)\big| + f_0\tau(k) + \xi(k),$$
$$y_1(k) = w_a(k) + e_1(k),$$

which data was generated based on. Here $a_0 = 1 + \tilde{a}_0$, $\xi(k) = w_c(k) + \tilde{\xi}(k)$, and an additive uncertainty, e_1, was associated with the measurement of the absolute velocity. This system is sufficiently simple and transparent for analysis but yet the estimation of its parameters, $\theta_0 = \begin{bmatrix} a_0 & n_0 & f_0\end{bmatrix}^T$, is a non-trivial task which includes all the challenges discussed in the work. Initially, the one-step ahead predictor

$$\hat{y}(k|\theta) = ay_1(k-1) + ny_1(k-1)\big|y_1(k-1)\big| + f\tau(k-1),$$

was considered.

The input, $\tau(k)$, was a set of pulses with amplitudes between 0.1 and 0.3. The pulses were of varying width and excited the system well. The true system parameters were $a_0 = 0.99$, $n_0 = -0.1$ and $f_0 = 1$. The three noise sources were sampled from Gaussian distributions $w_c(k) \sim \mathcal{N}(\bar{w}_c, 0.01)$, $\xi(k) \sim \mathcal{N}(\bar{w}_c - 0.05, 0.01)$, and $e_1(k) \sim \mathcal{N}(0, 0.01)$. This means that neither the distribution of the measurement noise nor of the underwater current did have finite support, a choice made in order to test the robustness of the method. Each of the simulation sets used $N = 10^4$ data points for each parameter estimation step, and was repeated 1000 times, using new noise sequences, in a Monte Carlo manner.

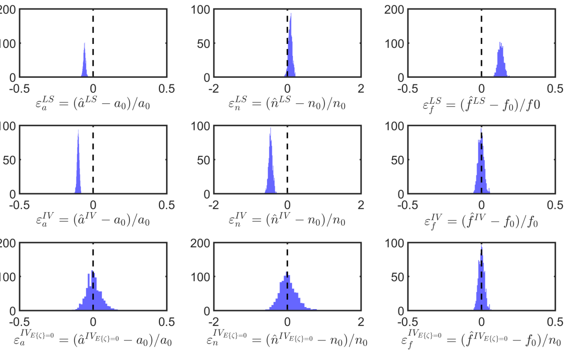

Fig. 1. Histograms of normalized estimation errors for the set of Monte Carlo runs with $\bar{w}_c = 0$.

A common way of obtaining instruments in practice is by simulation of an auxiliary model, as in Thil et al. (2008). In this work the model obtained by taking the LS estimate

$$\hat{\theta}_N^{LS} = \underset{\theta}{\arg\min} \sum_{k=1}^{N} [y_1(k) - \hat{y}(k|\theta)]^2, \tag{16}$$

for the parameters was used for this so that

$$\zeta(k) = \left[\hat{w}_a^{LS}(k-1)\ \hat{w}_a^{LS}(k-1)\left|\hat{w}_a^{LS}(k-1)\right|\ \tau(k-1)\right]^T,$$

where

$$\hat{w}_a^{LS}(k) = \hat{a}^{LS}\hat{w}_a^{LS}(k-1) + \hat{n}^{LS}\hat{w}_a^{LS}(k-1)\left|\hat{w}_a^{LS}(k-1)\right| + \hat{f}^{LS}\tau(k-1),\ k = 1,\ \ldots,\ N.$$

The parameters were then refined by iteratively letting the instruments instead be simulated from the model parameterized by the latest version of $\hat{\theta}_N^{IV}$ until convergence, as described in Young (2008).

Three estimators were compared in the simulations, one LS estimator, and two IV estimators. The IV estimators differed by having zero-mean instruments or not. In order to obtain zero-mean instruments, the average value of each component of $\zeta(k)$ was simply subtracted

$$\tilde{\zeta}_i(k) = \zeta_i(k) - \frac{1}{N}\sum_{k=1}^{N}\zeta_i(k),\ i = 1,\ 2,\ 3.$$

The first set of simulations were performed with, $\bar{w}_c = 0$ and histograms showing the parameter errors for the three estimators are provided in Figure 1 and Table 1. Only the IV estimator with zero-mean instruments can be seen to be consistent. The consistency does however come with a price, since the corresponding estimation errors also have higher variance.

After this a set of simulations were performed with $\bar{w}_c = 0.1$. In this setup it was assumed that an additional measurement $y_2(k) = w_c(k) + e_2(k)$ was available, where $e_2(k) \sim \mathcal{N}(0, 0.01)$. The relative measurement $\check{y}(k) = y_1(k) - y_2(k)$ was formed and the predictor

$$\hat{y}(k|\theta) = a\check{y}(k-1) + n\check{y}(k-1)\left|\check{y}(k-1)\right| + f\tau(k-1),$$

was considered. Estimating the parameters under these premises gave results similar to those obtained for the first set of simulations. The estimation errors are presented in Table 2.

Table 1. The mean plus/minus one standard deviation of normalized estimation errors for the set of Monte Carlo runs with $\bar{w}_c = 0$

	$(\hat{a} - a_0)/a_0$	$(\hat{n} - n_0)/n_0$	$(\hat{f} - f_0)/f_0$
LS	-0.0586 ± 0.0082	0.0794 ± 0.0526	0.1309 ± 0.0191
IV	-0.1013 ± 0.0079	-0.4553 ± 0.0536	-0.0034 ± 0.0197
$IV_{E\{\zeta\}=0}$	0.0041 ± 0.0504	0.0192 ± 0.2309	-0.0002 ± 0.0202

Table 2. The mean plus/minus one standard deviation of normalized estimation errors for the set of Monte Carlo runs with w_c measured and $\bar{w}_c = 0.1$.

	$(\hat{a} - a_0)/a_0$	$(\hat{n} - n_0)/n_0$	$(\hat{f} - f_0)/f_0$
LS	-0.0460 ± 0.0104	0.3413 ± 0.0627	0.2502 ± 0.0244
IV	-0.0994 ± 0.0088	-0.4470 ± 0.0591	-0.0032 ± 0.0252
$IV_{E\{\zeta\}=0}$	0.0023 ± 0.0576	0.0115 ± 0.2642	0.0001 ± 0.0259

7. CONCLUSIONS

Two scenarios where consistent parameter estimators for second-order modulus models can be obtained have been shown. These results are based on the assumption that all system states are measured directly. In practice it is more common that only some components of the state vector are available immediately, whereas other components must be obtained by filtering techniques. This is a scenario that it would be of interest to explore further. Another possible area of future work is to address the problem of parameter estimation in the class of model structures suggested by Abkowitz (1964), where there are cubic regressors present.

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