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Lie-Algebraic Criterion for Stability of Switched Differential-Algebraic Equations

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Abstract: In this paper, we prove a Lie algebraic result for stability of switched DAEs with a common descriptor matrix (common E matrix). We first show that if a switched DAE with a common descriptor matrix is *asymptotically stable*, then it is also *globally uniformly exponentially stable*. We then show that switched DAEs with common descriptor matrix and consistent block upper triangular structure is *globally uniformly exponentially stable* if and only if the switched DAEs corresponding to the diagonal blocks are *globally uniformly exponentially stable*. Finally, we show that a switched DAE with common descriptor matrix, stable and impulse free DAE subsystems, is *globally uniformly exponentially stable (GUES)* if there exists an invertible matrix N such that the Lie algebra $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable.

Keywords: Switched differential-algebraic equations, hybrid systems, exponential stability.

1. INTRODUCTION

In this paper, we consider stability of a switched DAE with common descriptor matrix as given below

$$E\dot{x} = A_{\sigma(t)}x, \tag{1}$$

where $E, A_\sigma \in \mathbb{C}^{n\times n}$, E may be singular, $\sigma : \mathbb{R} \to \mathcal{P}$, $\mathcal{P}$ is an index set, σ is a piece-wise constant and right continuous function having finitely many switchings for any finite duration. The number of switchings may tend to infinity as t tends to infinity.

Switched DAEs form a more general class of switched systems and they arise in situations where the underlying subsystems are Differential-Algebraic Equations (DAE). Stability of switched systems has been studied extensively (see Narendra and Balakrishnan (1994); Liberzon (2003); Sun (2005); Feng et al. (2019)). It has been shown that linear switched systems, where-in the underlying subsystems are stable and the system matrices commute pairwise, share a common quadratic Lyapunov function (CQLF) (Narendra and Balakrishnan (1994)). A necessary and sufficient condition for the existence of CQLF for second order switched systems with two underlying subsystems has been reported (Shorten and Narendra (1997)). It has been shown that switched systems with stable underlying subsystems and simultaneously upper triangularizable subsystem matrices share a CQLF (Liberzon et al. (1999)). Stability of switched DAEs has been considered in (Liberzon and Trenn (2009); Zhai et al. (2009); Liberzon et al. (2011); Trenn (2009)). It has been shown that existence of a suitable Lyapunov function guarantees asymptotic stability of switched DAEs (Liberzon and Trenn (2009)). A dwell time based result for stability of switched DAEs has also been reported (Liberzon and Trenn (2009)). It has been shown that switched DAE with common descriptor matrix, stable underlying subsystems and pairwise commuting subsystem matrices (including E matrix), is asymptotically stable (Zhai et al. (2009)). It has been shown that if the differential flows associated with each DAE subsystem commute pairwise, then the switched DAE is asymptotically stable (Liberzon et al. (2011)). For the case of conventional switched systems it is known that attractivity implies global uniform exponential stability. For the case of switched DAEs it has been shown that attractivity implies asymptotic stability (Trenn (2009)).

Most results for stability of switched DAEs, to the best of our knowledge, are based on commutativity (Mironchenko et al. (2015)). In this paper we prove a more general stability criterion based on Lie algebra for switched DAEs with the same descriptor matrix. Note that the commutativity based result reported in Zhai et al. (2009), is a special case of the Lie algebraic result presented in this paper. Furthermore, the result presented in this paper is also applicable to switched DAEs where the differential flows do not necessarily commute (Liberzon et al. (2011)).

The structure of the paper is as follows. In Sections 2.1, 2.2 and 2.3 necessary prerequisites for switched systems, differential-algebraic equations and switched DAEs are presented. In Section 3.1 we show that, for switched DAEs with common descriptor matrix, asymptotic stability is equivalent to global uniform exponential stability. In Section 3.1 we also show that switched DAEs with common descriptor matrix and consistent block upper triangular structure is GUES if and only if the switched DAEs corresponding to the diagonal blocks are GUES. In Section 3.2 we state and prove the main result that switched DAEs with common descriptor and stable underlying subsystem DAEs is GUES if there exists an invertible matrix N such that the Lie algebra $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable. Results are discussed in Section 4 with some examples.

Following notations are used throughout the paper. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote sets of natural numbers, real numbers and complex numbers, respectively. If $A \in \mathbb{C}^{n\times n}$, then A^T and A^* denote transpose and *Hermitian* transpose of matrix A, respectively. Kernel or nullspace of a matrix A is denoted as $ker\ A$ and image or column space of a matrix A is denoted as $im\ A$. Let $p(s)$ be a polynomial, then $\deg(p(s))$ denotes the largest exponent of the polynomial $p(s)$. For a piece-wise smooth function f, the left sided evaluation and right sided evaluation at $t \in \mathbb{R}$ are denoted as $f(t^-)$ and $f(t^+)$, respectively.

2. PRELIMINARIES

In this section we go through the necessary prerequisites required to state and prove the results. We go through necessary prerequisites for switched systems in Section 2.1, DAEs in Section 2.2 and switched DAEs in Section 2.3.

2.1 Switched systems

In this section we revisit a few well-established results for conventional switched systems, which are useful in proving the main result.

Definition 1. (Attractivity). The equilibrium x_e of a switched system (also a switched DAE), where $x_e \in \mathbb{R}^n$, is said to be attractive if the solution $x(t)$, for any initial condition and any arbitrary switching sequence, is such that $||x(t) - x_e||$ tends to zero as t tends to infinity.

Lemma 2. (Sun (2005)). Consider a switched system $\dot{x} = A_\sigma x$, where $\sigma : \mathbb{R} \to \mathcal{P}$, $\mathcal{P}$ is an index set. The following are equivalent:

(1) Origin is attractive.
(2) The switched system is globally uniformly asymptotically stable (GUAS).
(3) The switched system is globally uniformly exponentially stable (GUES).

Lemma 3. (Matni and Oishi (2011)). Consider a switched system $\dot{x} = A_i x$, where $i \in \mathcal{P}$, each $A_i = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ 0 & A_{22}^{(i)} \end{bmatrix}$, $A_{11}^{(i)} \in \mathbb{R}^{m\times m}$, $A_{22}^{(i)} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $A_{12}^{(i)} \in \mathbb{R}^{m\times(n-m)}$ such that $\{A_{12}^{(i)} : i \in \mathcal{P}\}$ is bounded. The switched system is GUES if and only if the switched systems corresponding to the diagonal blocks $\dot{x_1} = A_{11}^{(i)} x_1$ and $\dot{x_2} = A_{22}^{(i)} x_2$ are GUES.

Lemma 4. Consider a switched system $\dot{x} = A_\sigma x$, where $\sigma : \mathbb{R} \to \mathcal{P}$, $\mathcal{P}$ is an index set. Let the solution of switched system be $x(t) = \Phi(t,\sigma)x(0)$. If the switched system is GUES, then there exist $c, \lambda \in \mathbb{R}_+$ such that $||\Phi(t,\sigma)|| \le ce^{-\lambda t}$ for all $t \in \mathbb{R}_+$ and any arbitrary switching sequence.

Proof. Since the switched system is GUES, there exist $c, \lambda \in \mathbb{R}_+$ such that $||\Phi(t,\sigma)x(0)|| < ce^{-\lambda t}||x(0)||$ for any arbitrary switching sequence and any $t \in \mathbb{R}_+$. Re-writing the inequality, we have

$$\frac{||\Phi(t,\sigma)x(0)||}{||x(0)||} < ce^{-\lambda t} \text{ and } \sup_{||x(0)||\neq 0} \frac{||\Phi(t,\sigma)x(0)||}{||x(0)||} \le ce^{-\lambda t}.$$

Consequently $||\Phi(t,\sigma)|| \le ce^{-\lambda t}$ for any switching sequence and any $t \in \mathbb{R}_+$. Furthermore, it is easy to see that $||\Phi^T(t,\sigma)||$ is also uniformly exponentially bounded. $\square$

Lemma 5. The switched system $\dot{x} = A_\sigma x$, where $\sigma : \mathbb{R} \to \mathcal{P}$, $\mathcal{P}$ is an index set, is GUES if and only if the dual switched system $\dot{z} = A_\sigma^T z$ is GUES.

Proof. Let $\tilde{\Phi}(t,\sigma)z(0)$ be the solution of the dual switched system due to a switching sequence σ and an initial condition $z(0)$ and on the same lines let $\Phi(t,\sigma)x(0)$ be the solution of switched system $\dot{x} = A_\sigma x$. It is easy to see that for any switching sequence σ and any time instant $t \in \mathbb{R}_+$, there always exists a switching sequence $\sigma_{d(t)}$ such that $\tilde{\Phi}(t,\sigma) = \Phi^T(t,\sigma_{d(t)})$. As the switched system $\dot{x} = A_\sigma x$ is GUES, using Lemma 4, we have that there exist $c, \lambda \in \mathbb{R}_+$ such that $||\Phi^T(t,\sigma_{d(t)})|| \le ce^{-\lambda t}$ for all $t \in \mathbb{R}_+$. Thus $||\tilde{\Phi}(t,\sigma)|| \le ce^{-\lambda t}$ for any switching sequence and consequently the dual switched system is GUES. $\square$

2.2 Differential-Algebraic Equations

In this section we go through necessary prerequisites for DAEs. We go through the well known notions of regularity, impulse freeness and stability of linear DAEs. Linear DAEs are as given below

$$E\dot{x} = Ax, \tag{2}$$

where $E, A \in \mathbb{C}^{n\times n}$, and E may be singular.

Definition 6. (Regularity). DAE is said to be regular if and only if the polynomial $\det(sE - A)$ is nonzero (see Kunkel and Mehrmann (2006) for more on regularity).

Definition 7. (Piece-wise smooth functions). A function $f : \mathbb{R} \to \mathbb{C}$ is said to be piece-wise smooth if it is smooth for all but finitely many $t \in \mathbb{R}$. The set of all piece-wise smooth functions is denoted as $\mathscr{C}_{pw}^\infty$.

Definition 8. (Piece-wise smooth distributions). The space of piece-wise smooth distributions is as defined below

$$\mathbb{D}_{pw\mathscr{C}^\infty} := \left\{ f + \sum_{t\in T} D_t \,\middle|\, f \in \mathscr{C}_{pw}^\infty,\ T \subset \mathbb{R} \text{ locally finite}, \right.$$
$$\left. D_t \in span\{\delta_t, \delta_t^{(1)}, \delta_t^{(2)}, \dots\} \right\},$$

where δ_t is Dirac delta impulse function supported at time instant t and $\delta_t^{(i)}$ is the i^{th} distributional derivative of δ_t (Trenn (2012)).

Theorem 9. (Weierstrass (1868)). Consider a DAE $E\dot{x} = Ax$, where $E, A \in \mathbb{C}^{n\times n}$. The following are equivalent:

(1) (E, A) pair is regular.
(2) There exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$, such that $(SET, SAT) = \left(\begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-r} \end{bmatrix}\right)$, where N is nil-potent. (The pair (SET, SAT) is said to be in *quasi-Weierstrass* form.)
(3) The solution $x(t)$ exists in $(\mathbb{D}_{pw\mathscr{C}^\infty})^n$ and is uniquely determined by the initial condition $x(0^-)$.

Let (E, A) pair be regular. The invertible matrices $S, T \in \mathbb{C}^{n\times n}$, such that (SET, SAT) is in quasi-Weierstrass form, can be obtained using the following iterative process

$$\mathscr{V}^{(k+1)} = A^{-1}\left(E\mathscr{V}^{(k)}\right) \text{ and } \mathscr{W}^{(k+1)} = E^{-1}\left(A\mathscr{W}^{(k)}\right),$$

where $A^{-1}(\mathscr{S}_1)$ and $E^{-1}(\mathscr{S}_2)$ denote pre-images of $\mathscr{S}_1$ under A, and $\mathscr{S}_2$ under E, respectively, $\mathscr{V}^{(0)} = \mathbb{C}^n$ and $\mathscr{W}^{(0)} = \{0\}$. The iterative process simultaneously converges to subspaces $\mathscr{V}^{(*)}$ and $\mathscr{W}^{(*)}$ such that $\mathscr{V}^{(*)} \oplus \mathscr{W}^{(*)} = \mathbb{C}^n$. The subspaces $\mathscr{V}^{(*)}$ and $\mathscr{W}^{(*)}$ are called *consistency space* and *inconsistency space*, respectively. Let V and W be matrices consisting basis vectors of $\mathscr{V}^{(*)}$ and $\mathscr{W}^{(*)}$ as columns, respectively, $T := [V\ W]$ and $S := [EV\ AW]^{-1}$. Then (SET, SAT) is in *quasi-Weierstrass* form (Armento (1986)).

Remark 10. If the initial condition is in the *consistency space*, then the unique solution is indeed smooth. On the other hand if the initial condition has a nonzero component in the *inconsistency space*, then there will be discontinuities and/or impulses at $t = 0$.

Definition 11. (Impulse free). DAE is said to be impulse free if and only if solution due to every initial condition

$x(0^-)$ is impulse free, i.e., solution $x(t)$ due to each initial condition is in the space of piece-wise smooth functions.

Theorem 12. (Kunkel and Mehrmann (2006)). DAE is impulse free if and only if rank $E = \deg(\det(sE - A))$.

Remark 13. If (E, A) pair is regular and impulse free, then there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that $(SET, SAT) = \left(\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \Lambda & 0 \\ 0 & I_{n-r} \end{smallmatrix}\right]\right)$ (Kunkel and Mehrmann (2006)).

Definition 14. (Stability of DAEs). DAE is said to be asymptotically stable if and only if, for all initial conditions $x(0) \in \mathscr{V}^{(*)}$ the corresponding unique solution $x(t)$ tends to zero as t tends to infinity.

Theorem 15. (Owens and Debeljkovic (1985)). Consider a DAE $E\dot{x} = Ax$, where (E, A) pair is regular. The DAE is asymptotically stable if and only if there exists a positive definite matrix $P = P^* \in \mathbb{C}^{n\times n}$ such that $A^*PE + E^*PA = -Q$, where Q is positive definite on $\mathscr{V}^{(*)}$ (*consistency space*).

2.3 Switched DAE

In this section we go through necessary prerequisites for switched DAEs. First, we define asymptotic stability and global uniform exponential stability for switched DAEs, then we restate from the literature two commutativity based sufficient conditions for asymptotic stability of switched DAEs.

Definition 16. (Asymptotic stability, Trenn (2012)). The switched DAE $E_\sigma \dot{x} = A_\sigma x$ with regular DAE subsystems, is said to be asymptotically stable if and only if for any arbitrary switching sequence, following conditions hold:

$$\forall\epsilon > 0\ \exists\delta > 0\ :\ ||x(0^-)|| < \delta \implies ||x(t^\pm)|| < \epsilon, \forall t > 0,$$
$$||x(t^\pm)|| \to 0 \text{ as } t \to \infty \text{ and } x(t) \in \mathscr{C}^\infty_{pw}.$$

Definition 17. (Global uniform exponential stability). The switched DAE is globally uniformly exponentially stable (GUES) if the following condition holds for all switching sequence and any initial condition:

$$||x(t)|| \le ce^{-\lambda t}||x(0^-)||\ \forall t \ge 0, \text{where } c, \lambda > 0.$$

Remark 18. From the definition of global uniform exponential stability, it is obvious the $x(t)$ should be impulse free for all $t \ge 0$. The solution due to any initial condition should be impulse free in particular at time instant $t = 0$. Thus the underlying subsystem DAEs should be necessarily impulse free.

Definition 19. (Liberzon et al. (2011)). Consider a regular (E, A) pair and let $S,\ T \in \mathbb{C}^{n\times n}$ be invertible matrices such that (SET, SAT) pair is in quasi-Weierstrass form. The consistency projector, differential projector, impulse projector and differential flow for (E, A) pair are defined, respectively, as $\Pi := T\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]T^{-1}$, $\Pi^{\text{diff}} := T\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]S$, $\Pi^{\text{imp}} := T\left[\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-r} \end{smallmatrix}\right]S$ and $A^{\text{diff}} := \Pi^{\text{diff}}A$.

It should be noted that the differential projector and the impulse projector are not necessarily projectors but are called so just for convenience. For a given (E, A_i) pair we denote consistency projector, differential projector, impulse projector and differential flow as Π_i, Π_i^{diff}, Π_i^{imp} and A_i^{diff}, respectively.

Remark 20. It has been shown that a switched DAE with common descriptor matrix, and having regular, impulse free and stable subsystems, is asymptotically stable if the system matrices commute pairwise (Zhai et al. (2009)). It has also been shown that switched DAE $E_\sigma \dot{x} = A_\sigma x$, with finitely many regular and stable subsystems, is asymptotically stable if the differential flows (see Definition 19) associated with each subsystem commute pairwise under some impulse freeness conditions (Liberzon et al. (2011)).

3. RESULTS

In this section we state and prove the main Lie algebraic result. But before doing so we state and prove necessary preliminary results in the following section.

3.1 Preliminary results

In this section we prove preliminary results useful in proving the main result. First we show that switched DAEs with same descriptor matrix, and having regular and impulse free DAE subsystems, can be decomposed into conventional switched systems and switched algebraic equations. Then we show that a switched DAE with common descriptor matrix is GUES if it is asymptotically stable. We also show that if a switched DAE with common descriptor matrix is GUES then its dual is also GUES. Finally using these results we show that a switched DAE with consistent block upper triangular structure is GUES if and only if the switched DAEs corresponding to the diagonal blocks are GUES under some assumptions on the off-diagonal blocks of the common descriptor matrix.

Lemma 21. Consider a switched DAE $E\dot{x} = A_i x$, where $i \in \mathcal{P}$, $\mathcal{P}$ is an index set and each (E, A_i) pair is regular and impulse free. There exist invertible $S, T \in \mathbb{C}^{n\times n}$ such that $(SET, SA_iT) = \left(\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{smallmatrix}\right]\right)$ and $\Lambda_{22}^{(i)}$ is invertible for each $i \in \mathcal{P}$.

Proof. Since the subsystems of the switched DAE are individually regular and impulse free, there exist $S,\ T \in \mathbb{C}^{n\times n}$ such that $(SET, SA_iT) = \left(\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{smallmatrix}\right]\right)$. Furthermore since each (SET, SA_iT) pair is regular and impulse free, there exist $S_i, T_i \in \mathbb{C}^{n\times n}$ such that

$$S_i \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and} \tag{3}$$

$$S_i \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{bmatrix} T_i = \begin{bmatrix} \tilde{\Lambda}_1^{(i)} & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{4}$$

Re-writing S_i and T_i in block structure as given below

$$S_i = \begin{bmatrix} S_{11}^{(i)} & S_{12}^{(i)} \\ S_{21}^{(i)} & S_{22}^{(i)} \end{bmatrix} \text{ and } T_i = \begin{bmatrix} T_{11}^{(i)} & T_{12}^{(i)} \\ T_{21}^{(i)} & T_{22}^{(i)} \end{bmatrix}, \tag{5}$$

where $S_{11}^{(i)}, T_{11}^{(i)} \in \mathbb{C}^{r\times r}$ and $S_{22}^{(i)}, T_{22}^{(i)} \in \mathbb{C}^{(n-r)\times(n-r)}$, and using (3) we have that $\begin{bmatrix} S_{11}^{(i)}T_{11}^{(i)} & S_{11}^{(i)}T_{12}^{(i)} \\ S_{21}^{(i)}T_{11}^{(i)} & S_{21}^{(i)}T_{12}^{(i)} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Thus $S_{11}^{(i)}$ and $T_{11}^{(i)}$ are invertible, $S_{21}^{(i)} = 0$ and $T_{12}^{(i)} = 0$ for each $i \in \mathcal{P}$. Furthermore, $S_{22}^{(i)}$ and $T_{22}^{(i)}$ are also invertible for each $i \in \mathcal{P}$. Using these facts, equation (4) can be written as given below

$$S_i \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{bmatrix} T_i = \begin{bmatrix} * & * \\ * & S_{22}^{(i)}\Lambda_{22}^{(i)}T_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_1^{(i)} & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

Thus $S_{22}^{(i)}\Lambda_{22}^{(i)}T_{22}^{(i)}$ is equal to I_{n-r} and hence $\Lambda_{22}^{(i)}$ is invertible for each $i \in \mathcal{P}$. $\square$

Lemma 21 shows that switched DAEs with common descriptor matrix that are regular and impulse free can be decomposed into conventional switched system and switched algebraic equations. Using this result we show that switched DAEs with common descriptor matrix is GUES if it is asymptotically stable.

Theorem 22. If the switched DAE $E\dot{x} = A_i x$ is asymptotically stable, then it is also GUES.

Proof. Since the switched DAE is asymptotically stable it must be that the individual DAEs are regular, impulse free and stable (see Definition 16). Hence there exist invertible $S, T \in \mathbb{C}^{n\times n}$ such that $(SET, SA_iT) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{bmatrix}\right)$. Furthermore using Lemma 21, we have that $\Lambda_{22}^{(i)}$ is invertible for each $i \in \mathcal{P}$. The equivalent switched DAE can then be written as

$$\dot{z}_1 = \Lambda_{11}^{(i)} z_1 + \Lambda_{12}^{(i)} z_2 \tag{6}$$

$$z_2 = -{\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)} z_1, \tag{7}$$

where $z(t) = \begin{bmatrix} z_1(t)^T & z_2(t)^T \end{bmatrix}^T = T^{-1}x(t)$. Using (7) in (6), we have

$$\dot{z}_1 = (\Lambda_{11}^{(i)} - \Lambda_{12}^{(i)} {\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)}) z_1. \tag{8}$$

Since the switched DAE is asymptotically stable, it must be that its equivalent is also asymptotically stable. Thus the switched system corresponding to equation (8) is asymptotically stable and using Lemma 2, it is in fact GUES. Since $z_1(t)$ decays uniformly exponentially, $z_2(t)$ also decays uniformly exponentially. Thus the switched DAE $E\dot{x} = A_i x$ is GUES. □

As an implication, we have the following corollary.

Corollary 23. Consider a switched DAE with common descriptor matrix, regular and impulse free DAE subsystems. The following are equivalent:

(1) Origin is attractive.
(2) The switched DAE is asymptotically stable.
(3) The switched DAE is GUES.

Now, we show that if a switched DAE with common descriptor matrix is GUES then its dual $E^T \dot{z} = A_i^T z$ is also GUES.

Theorem 24. The switched DAE $E\dot{x} = A_i x$ is GUES if and only if the dual switched DAE $E^T \dot{y} = A_i^T y$ is GUES.

Proof. Using the proof of Theorem 22, an equivalent of switched DAE $E\dot{x} = A_i x$ is as given below

$$\dot{z}_1 = (\Lambda_{11}^{(i)} - \Lambda_{12}^{(i)} {\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)}) z_1 \tag{9}$$

$$z_2 = -{\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)} z_1. \tag{10}$$

Furthermore, an equivalent of the dual switched DAE then comes out to be $(T^T E^T S^T)\dot{\tilde{y}} = (T^T A_i^T S^T)\tilde{y}$, where $\tilde{y}(t) = S^{-T} y(t)$. The equivalent of $E^T \dot{y} = A_i^T y$, can be further re-written as given below

$$\dot{\tilde{y}}_1 = (\Lambda_{11}^{(i)} - \Lambda_{12}^{(i)} {\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)})^T \tilde{y}_1 \tag{11}$$

$$\tilde{y}_2 = -{\Lambda_{22}^{(i)}}^{-T} {\Lambda_{12}^{(i)}}^T \tilde{y}_1, \tag{12}$$

where $\tilde{y} = \begin{bmatrix} \tilde{y}_1^T & \tilde{y}_2^T \end{bmatrix}^T$. The switched system corresponding to equation (11) is dual of switched system corresponding to (9). Using Lemma 5, the switched system corresponding to equation (11) is GUES and hence the dual of the switched DAE is GUES. □

Using Theorems 22 and 24, we show that a switched DAE with consistent block upper triangular structure is GUES if and only if the switched DAEs corresponding to the diagonal blocks are GUES, under some assumptions on the off-diagonal blocks of the common descriptor matrix (see Remark 27).

Theorem 25. Consider the switched DAE $E\dot{x} = A_i x$, where each (E, A_i) pair is regular and impulse free, $E = \begin{bmatrix} E_1 & E_{12} \\ 0 & E_2 \end{bmatrix}$, $A_i = \begin{bmatrix} A_1^{(i)} & A_{12}^{(i)} \\ 0 & A_2^{(i)} \end{bmatrix}$, E_1 and each $A_1^{(i)} \in \mathbb{R}^{n_1\times n_1}$, E_2 and each $A_2^{(i)} \in \mathbb{R}^{n_2\times n_2}$, $E_{12} \in \mathbb{R}^{n_1\times n_2}$ such that $ker\ E_{12} \supseteq ker\ E_2$ or $im\ E_{12} \subseteq im\ E_1$, and $A_{12}^{(i)} \in \mathbb{R}^{n_1\times n_2}$ such that $\{A_{12}^{(i)} : i \in \mathcal{P}\}$ is bounded. The switched DAE is GUES if and only if the switched DAEs corresponding to the diagonal blocks $E_1\dot{x}_1 = A_1^{(i)} x_1$ and $E_2\dot{x}_2 = A_2^{(i)} x_2$ are GUES.

Proof. *(If)* First we prove the result assuming $ker\ E_{12} \supseteq ker\ E_1$ and then use duality property to prove the same when $im\ E_{12} \subseteq im\ E_1$. As each (E, A_i) pair is regular and impulse free, using Lemma 21, there exist invertible matrices $S_1, T_1 \in \mathbb{R}^{n_1\times n_1}$ such that $(S_1E_1T_1, S_1A_1^{(i)}T_1) = \left(\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} \end{bmatrix}\right)$, where $\Lambda_{22}^{(i)}$ is invertible. Define $S := \begin{bmatrix} S_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$ and $T := \begin{bmatrix} T_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$. The equivalent of the switched DAE (SET, SA_iT) is as given below

$$\begin{bmatrix} I_{r_1} & 0 & \tilde{V}_1 \\ 0 & 0 & \tilde{V}_2 \\ 0 & 0 & E_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} & \tilde{A}_1 \\ \Lambda_{21}^{(i)} & \Lambda_{22}^{(i)} & \tilde{A}_2 \\ 0 & 0 & A_2^{(i)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ x_2 \end{bmatrix}, \tag{13}$$

where $\Lambda_{22}^{(i)}$ is invertible, $\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = S_1 E_{12}$ and $\begin{bmatrix} \tilde{A}_1^{(i)} \\ \tilde{A}_2^{(i)} \end{bmatrix} = S_1 A_{12}^{(i)}$. Re-writing equation (13), we have

$$\dot{z}_1 = \left(\Lambda_{11}^{(i)} - \Lambda_{12}^{(i)} {\Lambda_{22}^{(i)}}^{-1} \Lambda_{21}^{(i)}\right) z_1 + \left(\tilde{A}_1 - {\Lambda_{22}^{(i)}}^{-1} \tilde{A}_2\right) x_2 - \left(\tilde{V}_1 - {\Lambda_{22}^{(i)}}^{-1} \tilde{V}_2\right) \dot{x}_2 \tag{14}$$

$$z_2 = {\Lambda_{22}^{(i)}}^{-1} \left(\tilde{V}_2 \dot{x}_2 - \Lambda_{21}^{(i)} z_1 - \tilde{A}_2 x_2\right) \tag{15}$$

$$E_2 \dot{x}_2 = A_2^{(i)} x_2. \tag{16}$$

Since switched DAE $E_2\dot{x}_2 = A_2^{(i)} x_2$ is asymptotically stable $x_2(t)$ decays exponentially. As $x_2(t)$ is impulse free, $A_2^{(i)} x_2(t)$ is also impulse free and consequently $E_2\dot{x}_2(t)$ is also impulse free. Since $E_2\dot{x}_2(t)$ is impulse free it must be that the impulsive component of $\dot{x}_2(t)$ is contained in $ker\ E_2$. But $ker\ E_{12} \supseteq ker\ E_2$ and thus $S_1E_{12}\dot{x}_2(t)$ is impulse free. Hence $(\tilde{V}_1 - {\Lambda_{22}^{(i)}}^{-1} \tilde{V}_2)\dot{x}_2$ is impulse free and decays exponentially. Equation (14) can now be seen as a switched system with exponentially decaying input and using Lemma 3, $z_1(t)$ decays exponentially for any initial condition. From equation (15), it is obvious that $z_2(t)$ also decays exponentially.

Now we prove the Theorem for the case where $im\ E_{12} \subseteq im\ E_1$. Consider the dual of the switched DAE $E\dot{x} = A_i x$ as given below

$$\begin{bmatrix} E_1^T & 0 \\ E_{12}^T & E_2^T \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} {A_{11}^{(i)}}^T & 0 \\ {A_{12}^{(i)}}^T & {A_{22}^{(i)}}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{17}$$

Since $E_1\dot{x}_1 = A_1^{(i)} x_1$ and $E_2\dot{x}_2 = A_2^{(i)} x_2$ are GUES, using Theorem 24, it must be that $E_1^T \dot{y}_1 = {A_1^{(i)}}^T y_1$ and

$E_2^T \dot{y}_2 = A_2^{(i)^T} y_2$ are GUES. Since $im\ E_{12} \subseteq im\ E_1$, we have that $rowspan\ E_{12}^T \subseteq rowspan\ E_1^T$ and consequently $ker\ E_{12}^T \supseteq ker\ E_1^T$. Using part of the Theorem that was proved for the case where $kerE_{12} \supseteq kerE_2$, we have that switched DAE corresponding to (17) is GUES. Consequently dual of the switched DAE corresponding to (17) $E\dot{x} = A_i x$, where $im\ E_{12} \subseteq im\ E_1$, is also GUES.
(Only if) Obvious. $\square$

Next result is a straight forward generalization.

Theorem 26. Consider a switched DAE with block upper triangular structure as given below

$$E = \begin{bmatrix} E_1 & & \tilde{E}_1 & \\ 0 & E_2 & \tilde{E}_2 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & E_r \end{bmatrix}, \ A_i = \begin{bmatrix} A_1^{(i)} & A_{12}^{(i)} & \dots & A_{1r}^{(i)} \\ 0 & A_2^{(i)} & \dots & A_{2r}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & A_r^{(i)} \end{bmatrix},$$

where $E_j, A_j^{(i)} \in \mathbb{R}^{n_j \times n_j}$ for each $j \in \{1, \dots, r\}$ and $\tilde{E}_j \in \mathbb{R}^{n_j \times (\sum_{k=j+1}^r n_k)}$ for each $j \in \{1, \dots, r-1\}$. Let the following property be satisfied:

$$im\ \tilde{E}_j \subseteq im\ \tilde{E}_1 \text{ or } ker\ \tilde{E}_j \supseteq ker\ E_{j+1}^{aug}$$

for each $j \in \{1, \dots, r-1\}$, where

$$E_{j+1}^{aug} = \begin{bmatrix} E_{j+1} & & \tilde{E}_{j+1} & \\ 0 & E_{j+2} & \tilde{E}_{j+2} & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & E_r \end{bmatrix}.$$

Then the switched DAE $E\dot{x} = A_i x$ is GUES if and only if the switched DAEs corresponding to the blocks are individually GUES.

Proof. Using Theorem 25 inductively, it is easy to see that Theorem 26 is true. $\square$

Remark 27. If each off-diagonal block $(\tilde{E}_j)$ in block upper triangular E matrix is such that $im\ \tilde{E}_j \subseteq im\ E_j$ or $ker\ \tilde{E}_j \supseteq ker\ E_j$, then for simplicity we say that the off-diagonal blocks are consistent with the image or kernel condition.

3.2 Main results

Before we prove the Lie algebraic result for switched DAEs with common descriptor matrix, we present a few results used in proving the main result.

Definition 28. (Solvability). A Lie algebra $\mathfrak{g} \subseteq \mathbb{C}^{n\times n}$ is solvable if the derived Lie algebra $\mathfrak{g}^{(k)} = 0$ for some $k \in \mathbb{N}$, where $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(k)} := \{[A, B] : A, B \in \mathfrak{g}^{(k-1)}\}$.

Remark 29. Due to Lie's theorem we have that a Lie algebra $\mathfrak{g} \subset \mathbb{C}^{n\times n}$ is solvable if and only if there exists a basis for $\mathbb{C}^n$ in which $\mathfrak{g}$ is upper triangular.

Lemma 30. Consider $\{E_i, A_i : i \in \mathcal{P}\}$, where $E_i,\ A_i \in \mathbb{C}^{n\times n}$ and $\mathcal{P}$ is an index set. There exist invertible matrices $S,\ T \in \mathbb{C}^{n\times n}$ such that $\{SE_iT,\ SA_iT : i \in \mathcal{P}\}$ are upper triangular if and only if there exists an invertible matrix $N \in \mathbb{C}^{n\times n}$ such that the Lie algebra $\{NE_i,\ NA_i : i \in \mathcal{P}\}_{LA}$ is solvable.

Proof. *(If)* Consider that there exist invertible matrices $S,\ T \in \mathbb{C}^{n\times n}$ such that $\{SE_iT,\ SA_iT : i \in \mathcal{P}\}$ are upper triangular. Since S is invertible we can re-write $S = T^{-1}N$, where N is invertible. Thus $\{T^{-1}NE_iT,\ T^{-1}NA_iT : i \in \mathcal{P}\}$ are upper triangular or $\{NE_i,\ NA_i\}_{LA}$ is solvable.

(Only if) If $\{NE_i,\ NA_i : i \in \mathcal{P}\}_{LA}$ is solvable then there exists invertible matrix $T \in \mathbb{C}^{n\times n}$ such that $\{T^{-1}NE_iT,\ T^{-1}NA_iT : i \in \mathcal{P}\}$ are upper triangular. $\square$

Lemma 31. Consider a DAE $E\dot{x} = Ax$, where $E, A \in \mathbb{C}^{n\times n}$, (E, A) pair are regular and impulse free. Suppose $(E, A) = \left(\left[\begin{smallmatrix} e_1 & e_{12} \\ 0 & E_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_1 & a_{12} \\ 0 & A_2 \end{smallmatrix}\right]\right)$, where $e_1, a_1 \in \mathbb{C}$ and $E_2, A_2 \in \mathbb{C}^{(n-1)\times(n-1)}$. If e_1 is zero then $ker\ e_{12} \supseteq ker\ E_2$.

Proof. Since (E, A) pair is regular and impulse free it must be that (E_2, A_2) pair is also regular and impulse free and hence $rank\ E_2 = deg\,(det(sE_2 - A_2))$. As (E, A) pair is regular and impulse free, we have

$$\begin{aligned} rank\ E &= deg\,(det(sE - A)) = deg\,(det(sE_2 - A_2)) \\ &= rank\ E_2. \end{aligned}$$

Thus $e_{12}^T \in im\ E_2^T$ and consequently $ker\ e_{12} \supseteq ker\ E_2$. $\square$

Lemma 32. Consider a scalar switched DAE $e\dot{x} = a_i x$, where $e, a_i \in \mathbb{C}$ and $\{a_i : i \in \mathcal{P}\}$ is compact and each (e, a_i) pair is regular. The switched DAE is GUES if and only if the individual first order DAEs are GUES.

Proof. If $e = 0$, then the only solution is $x(t) = 0$ for all $t > 0$, for any initial condition and any switching sequence. Thus the result is trivially true if e is zero. If e is nonzero, then the switched DAE reduces to a conventional first order switched system $\dot{x} = \tilde{a}_i x$, where $\tilde{a}_i = \frac{a_i}{e}$. It is well known that if the underlying first order systems are individually stable and $\{\tilde{a}_i : i \in \mathcal{P}\}$ is compact, then the switched system is stable (Liberzon (2003)). $\square$

Theorem 33. Consider a switched DAE $E\dot{x} = A_i x$, where $i \in \mathcal{P}$, $\mathcal{P}$ is an index set, $\{A_i : i \in \mathcal{P}\}$ is compact, each (E, A_i) pair is regular, impulse free and stable. If there exists an invertible $N \in \mathbb{C}^{n\times n}$ such that the Lie algebra $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable, then the switched DAE is GUES.

Proof. Since there exists invertible $N \in \mathbb{C}^{n\times n}$ such that $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable, using Lemma 30, we have that there exist invertible $S, T \in \mathbb{C}^{n\times n}$ such that each (SET, SA_iT) pair is upper triangular as given below,

$$(SET, SA_iT) = \left(\begin{bmatrix} e_1 & & \tilde{E}_1 & \\ 0 & e_2 & \tilde{E}_2 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & e_n \end{bmatrix}, \begin{bmatrix} a_1^{(i)} & a_{12}^{(i)} & \dots & a_{1n}^{(i)} \\ 0 & a_2^{(i)} & \dots & a_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_n^{(i)} \end{bmatrix} \right),$$

where each off-diagonal row $\tilde{E}_j \in \mathbb{C}^{1\times(j-1)}$. Since switched DAE $SET\dot{z} = SA_iTz$ is equivalent to $E\dot{x} = A_i x$, it must be that each (SET, SA_iT) pair is also regular, impulse free and stable. If $e_j = 0$ for some j, then using Lemma 31, we have that $ker\ \tilde{E}_j \supseteq ker \begin{bmatrix} e_{j+1} & & \tilde{E}_{j+1} & \\ 0 & e_{j+2} & \tilde{E}_{j+2} & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & e_n \end{bmatrix}$. If e_j is not zero then $im\ \tilde{E}_j$ is trivially contained in $im\ e_j$. Hence the off-diagonal blocks are consistent with kernel or image conditions in Theorem 26. As $\{A_i : i \in \mathcal{P}\}$ is compact, so is $\{SA_iT : i \in \mathcal{P}\}$ and consequently $\{a_j^{(i)} : i \in \mathcal{P}\}$ is also compact for each $j \in \{1, \cdots, n\}$. Using Lemma 32, scalar switched DAEs corresponding to each diagonal entry are GUES. Finally using Theorem 26, we have that the switched DAE is GUES. $\square$

As an immediate implication of Theorem 33, we have the following corollary.

Corollary 34. Consider a switched DAE $E\dot{x} = A_i x$, here $i \in \mathcal{P}$, $\mathcal{P}$ is an index set, each (E, A_i) pair is regular, impulse free and stable. If the Lie algebra $\{E, A_i : i \in \mathcal{P}\}_{LA}$ is solvable, then the switched DAE is GUES.

Remark 35. Cartan's criterion for solvability can be used to determine the solvability of a Lie algebra.

4. DISCUSSION

In this section we discuss the results with some examples.

Remark 36. Consider a switched DAE $E\dot{x} = A_i x$, where the system matrices (including E) commute pairwise. Then the Lie algebra $\{E, A_i : i \in \mathcal{P}\}_{LA}$ is trivially solvable. Thus the Lie algebraic result generalizes commutativity based result in Zhai et al. (2009).

The following example demonstrates that $\{E, A_i : i \in \mathcal{P}\}_{LA}$ may not be solvable, but there could exist an invertible N such that $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable.

Example 37. Consider a switched DAE $E\dot{x} = A_i x$, where $i \in \{1,\ 2\}$, $E = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 0 \end{smallmatrix}\right]$, $A_1 = \left[\begin{smallmatrix} -1 & 1 \\ 2 & -20 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} -2 & 1 \\ 4 & 10 \end{smallmatrix}\right]$. It is easy to see that A_1 and A_2 don't commute. The Lie algebra $\{E, A_1, A_2\}_{LA}$ is not solvable as A_1 and A_2 don't share an eigenvector, but the Lie algebra $\{NE, NA_1, NA_2\}_{LA}$, where $N = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$, is solvable and due to Theorem 33 the switched DAE is GUES.

Remark 38. Existence of an invertible matrix N such that the Lie algebra $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable is still an unanswered question.

The following example demonstrates the fact that there exist switched DAEs whose stability can be inferred using Theorem 33, but the same can't be inferred using the commutativity based result in Liberzon et al. (2011).

Example 39. Consider a switched DAE with common descriptor matrix and two DAE subsystems. Let $E = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$, $A_1 = \left[\begin{smallmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} -2 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & -1 \end{smallmatrix}\right]$. Using Theorem 33, it is easy to see that the switched system is GUES. Using commutativity based result in Liberzon et al. (2011), we can't conclude the stability of switched DAE as the differential flows don't commute.

The Lie algebraic result that has been proved in this paper was for switched DAEs with common descriptor matrix. While the commutativity based result in Liberzon et al. (2011) was for switched DAE $E_\sigma \dot{x} = A_\sigma x$.

Remark 40. The proposed Lie algebraic result doesn't generalize the commutativity based result in Liberzon et al. (2011).

5. CONCLUSION AND FUTURE WORK

In conclusion, we have shown that switched DAE with common descriptor matrix is GUES if it is asymptotically stable. We have also shown that switched DAE with common descriptor matrix and a consistent block upper triangular structure is GUES if and only if switched DAEs corresponding to diagonal blocks are GUES. We also prove the main result that switched DAE with common descriptor matrix and stable DAE subsystems is GUES if there exists an invertible N such that the Lie algebra $\{NE, NA_i : i \in \mathcal{P}\}_{LA}$ is solvable. The Lie algebraic result generalizes the commutativity based result in Zhai et al. (2009) and doesn't generalize the commutativity based result in Liberzon et al. (2011). We would like to generalize the result for general switched DAE $E_\sigma \dot{x} = A_\sigma x$.

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