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Phase Margins in a Class of Nonlinear Systems: Lyapunov, Circle Criterion and Describing Function Approaches

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Abstract: In this work, Phase Margins are studied for a class of Nonlinear Systems specifically the Lur'e type. New definitions are proposed for Practical Phase Margins in such systems and the corresponding computational algorithms are developed via Describing Function, Circle Criterion and Lyapunov methods. The efficacy of the proposed approaches are illustrated on a tutorial example.

Keywords: Lyapunov methods, Stability of nonlinear systems, Convex optimization.

1. INTRODUCTION

In control theory, Phase and Gain Margins (PM and GM) are considered as ubiquitous stability indicators for Linear Time Invariant (LTI) feedback control systems in presence of dynamic uncertainties (regular or parasitic) [Dorf and Bishop, 2010]. However, the inability to define such margins for characterizing the robustness of Nonlinear (NL) Control Systems to unmodeled dynamics (UD) creates a huge issue in the design and modelling such systems. Therefore, there is a need to introduce new definitions of PM and GM for NL systems which are reliable, mathematically proven and practically measurable.

Nevertheless, there exists few results aligned with computing PM/GM for NL systems. In [Chang, Chang and Han, 1993], a rudimentary idea of PM and GM for NL control systems are presented. A phase margin like parameter named Singular Perturbation Margin (SPM) is introduced for LTI in [Yang and Zhu, 2012] and extended to Nonlinear Time Invariant (NLTI) systems in [Yang and Zhu, 2016]. However, PM/GM of NL systems are not clearly defined in these works and only applicable for a smaller set of NL systems. Another attempt has been made in defining of Practical Phase and Gain Margins (PPM/PGM) for Sliding Mode Controllers in [Shtessel, Foreman and Tournes, 2011] and for Finite Time Convergent Controllers in [Rosales, Shtessel and Fridman, 2018]. A novel concept of Stability Margin identification based on the Describing Function – Harmonic Balance (DF/HB) [Khalil, 2002] and Circle Criterion (CC) [Khalil, 2002] was introduced in the authors' previous work [Das, Shtessel and Plestan, 2018], derived for Lur'e type NLTI systems, which are a class of NL systems with a linear block connected by a feedback connection to a sector nonlinearity.

In this paper, PM is defined for Lur'e type NLTI systems as the systems' robustness to cascade UDs and corresponding algorithms are developed via DF/HB and CC based techniques. Note that unlike the algorithms presented in [Das, Shtessel and Plestan, 2018] which rely on an extended geometrical interpretation of the Nyquist Stability Criteria (NSC), the algorithms presented in this work are refined and simplified with no dependency on NSC. Also, a novel Lyapunov's Second Method (LSM) [Khalil, 2002] based technique is introduced for PM identification in NLTI system. In this method, the NLTI system is augmented with a UD and then a Quadratic Lyapunov Function (QLF) for this system is introduced that gives a sufficient proof of stability via LSM. This QLF can be transformed as convex or quasi-convex optimization problem involving Linear Matrix Inequalities (LMIs) [Boyd et al., 1994], [Kürşad and Mustafa, 2006], which can be solved to derive a PM like metric on the onset of instability. This procedure accommodates for a larger class of nonlinearities than in [Das, Shtessel and Plestan, 2018] with less restrictive conditions. Note again that definition and corresponding algorithms of GM for Lur'e type NLTI SISO systems based on LSM approach is proposed in [Das, Shtessel and Plestan, 2020].

The main contribution of this paper are listed below:

A. Introducing definitions for PM in Lur'e type NLTI SISO systems.

B. Developing computational algorithms for calculating Practical Phase Margin (PPM) in Lur'e type NLTI SISO systems. The PM in NLTI systems are as PPM since the computational algorithms results in a predicted PM (DF/HB based algorithm) and a conservatively estimated PM (CC and LSM – LMI based algorithms).

C. Validating the proposed algorithm on a tutorial example of Lur'e type NLTI SISO system.

The paper is divided into seven sections, including the introductory part. Sections 2 and 3 introduce the system dynamics and preliminaries respectively. Section 4 contains the main results, where the definition of PM in NLTI systems is introduced and corresponding algorithms used for the PM identification are proposed and discussed. In Section 5, a tutorial example is used to validate the efficacy of the proposed algorithms. The paper is concluded with comments on the obtained results in Section 6 and References.

2. MATHEMATICAL MODELING AND ASSUMPTIONS

Consider a Lur'e type NLTI system [Khalil, 2002] given by

$$\dot{x} = Ax + Bw,\ y = Cx \\ w = u + v,\ v = \psi(u),\ u = -Ky \tag{1}$$

where state vector $x \in \Re^n$, matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{r\times n}$, $K \in \Re^{m\times r}$, the control $w \in \Re^m$ and the output $y \in \Re^r$ with $r = m = 1$ (Fig. 1). The following assumptions are made for NLTI system (1):

A1. The triplet $\{A, B, C\}$ is completely controllable and observable.

A2. The following sector condition holds globally for the static memoryless nonlinearity $\psi(\cdot):[0,\infty)\times\Re^m \to \Re^m$, given by

$$k_1 u^2 \le u\psi(u) \le k_2 u^2 \tag{2}$$

such that $\psi(0) = 0$. Eqn. (2) can be expanded as

$$(v - k_2 u)(v - k_1 u) \le 0 \\ \to v^T v - u^T \delta v - v^T \delta u + \gamma u^T u \le 0 \tag{3}$$

where $\delta = \frac{(k_1 + k_2)}{2}$ and $\gamma = k_1 k_2$.

A3. The equilibrium point $x = 0$ is globally asymptotically stable in the sector $[k_1, k_2]$ for any $\psi(u)$.

The Lur'e type NLTI system (1) can be further reduced to

$$\dot{x} = \tilde{A}x + Bv,\ v = \psi(-KCx) \tag{4}$$

where $\tilde{A} = (A - BKC)$ is Hurwitz. The open loop transfer function (TF) of the linear part of system (4) is given by

$$G_{Lur'e}(s) = \frac{Y(s)}{V(s)} = C\left(sI - \tilde{A}\right)^{-1} B \tag{5}$$

where $\{A, B, C\}$ is a minimal realization of $G_{Lur'e}(s)$. A functional diagram of system (1) is shown in Fig. 1.

Note that, if $v = \psi(u) \equiv 0$, then NLTI system (1) transforms into an LTI system, given by

$$\dot{x} = \tilde{A}x \tag{6}$$

However, the matrix $\tilde{A} = (A - BKC)$ is Hurwitz if and only if the Kimura – Davison conditions [Kimura, 1994], [Davison and Chow, 1973] for output feedback pole assignment are met so that the matrix K exists. Specifically, the following Lemma can be cited:

Lemma: Given $\{A, B, C\}$ controllable and observable with $A \in \Re^{n\times n}$, $rank\ B = m$, $rank\ C = r$ then for almost all $\{B, C\}$ pairs, there exists an output gain matrix K so that $(A - BKC)$ has $\min(n, m + r - 1)$ eigenvalues assigned arbitrarily close to $\min(n, m + r - 1)$ specified symmetric eigenvalues.

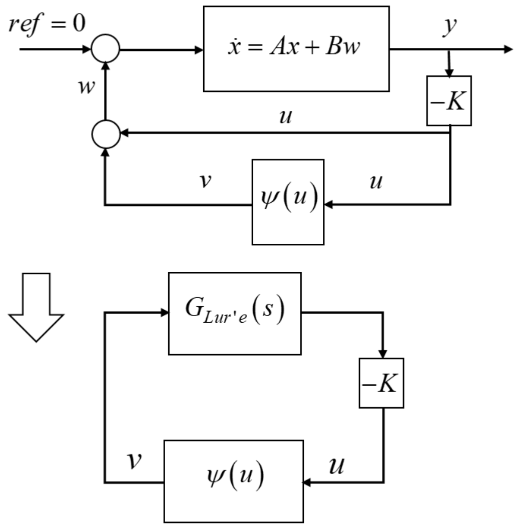

Fig. 1. Functional diagram of NLTI system (1)

3. PRELIMINARIES

3.1. Describing Functions/Harmonic Balance Technique:

The basic idea of the Describing Function approach, which can be used to study limit cycle behavior in NLTI system (4), is to replace a given nonlinear element $v = \psi(u)$ with a (quasi) linear descriptor or Describing Function, whose gain is a function of input amplitude [Khalil, 2002]. In order to use the DF/HB technique, the following assumptions should be valid for $v = \psi(u)$ in system (4):

A4. The nonlinearity $v = \psi(u)$ is odd.

A5. The nonlinear function is time invariant.

A6. The frequency response characteristic has low-pass filter properties, i.e. $|G_{Lur'e}(j\omega)| \gg |G_{Lur'e}(jk\omega)|,\ k = 1,2,3...$

The DF/HB based frequency domain analysis [Khalil, 2002] of NLTI systems (4) allows computing the parameters of a predicted limit cycle, specifically, the amplitude A^* and the frequency ω^* of the fundamental harmonic, via solving HB equation

$$G_{Lur'e}(j\omega) = -N(A, \omega)^{-1} \tag{7}$$

where $N(A, \omega)$ is a DF of the nonlinear element $v = \psi(u)$.

Since assumption (A3) holds for NLTI system (4), it can be further assumed that

A7. The solution to the DF/HB eqn. (7) does not exist.

3.2. Circle Criterion:

Assume that the assumptions (A1) – (A4) hold for NLTI system (4) such that $\{A,B,C\}$ is a minimal realization of $G_{Lur'e}(s)$, then system (4) is absolutely stable if one of the following conditions is satisfied, as appropriate [Khalil, 2002]:

B1. When $0<k_2<k_1$, the Nyquist plot of $G_{Lur'e}(j\omega)$ does not enter the disk $\Phi(k_1,k_2)$ (a circle with the line between $-k_2^{-1}$ and $-k_1^{-1}$ as diameter).

B2. When $0=k_2<k_1$, $G_{Lur'e}(s)$ is Hurwitz and the Nyquist plot of $G_{Lur'e}(j\omega)$ lies to the right of a vertical line defined by $\mathrm{Re}\left[G_{Lur'e}(j\omega)\right]=-k_1^{-1}$.

B3. When $k_2<0<k_1$, $G_{Lur'e}(s)$ is Hurwitz and the Nyquist plot of $G_{Lur'e}(j\omega)$ lies inside the disk $\Phi(k_1,k_2)$.

Note that assumption (A3) implies that NLTI system (4) is absolutely stable in the sense that the following assumption holds

A8. NLTI system (4) does not violate conditions (B1), (B2) or (B3), whichever is applicable.

4. MAIN RESULTS

4.1. Definition of PM in NLTI system (4):

Definition 1: The PM in NLTI system (4) is defined as the minimum negative phase shift

$$PM=-\phi_{\min} \tag{8}$$

added to the frequency characteristics $G_{Lur'e}(j\omega)$, so that a limit cycle starts emerging in system (4) or assumption (A8) starts becoming invalid.

4.2. Computational methods for PM identification in NLTI system (4):

The PM in dynamical systems can be characterized as the system's robustness to cascade UD that adds a phase shift to the system's frequency characteristic. In sense of Definition 1, we study the effects of the cascade UDs of the first and second order given by

$$H_1(s)=\frac{\omega_{01}}{s+\omega_{01}} \tag{9}$$

and

$$H_2(s)=\left(\frac{\omega_{02}}{s+\omega_{02}}\right)^2 \tag{10}$$

that add phase shifts to the frequency characteristic $G_{Lur'e}(j\omega)$ in NLTI system (4). Apparently, $G_{Lur'e}(j\omega)$ will experience an additional phase shift $\phi_1=\arg\left[H_1(j\omega)\right]=-tan^{-1}\frac{\omega}{\omega_{01}}$, of up to -90°, due to the cascade frequency characteristic $H_1(j\omega)$ and $\phi_2=\arg\left[H_2(j\omega)\right]=-2tan^{-1}\frac{\omega}{\omega_{02}}$ of up to -180°, due to $H_2(j\omega)$. It means that $H_1(j\omega)$ can be used to identify $PM\le 90^{\circ}$ in NLTI system (4). However, $H_1(j\omega)$ may not be sufficient $PM>90^{\circ}$ and in this case, $H_2(j\omega)$ should be employed. Therefore, the PM in NLTI system (4) is given by

$$PM=\begin{cases} tan^{-1}\frac{\omega_1^*}{\omega_{01}^*}, & \text{if } H_1(j\omega) \text{ considered} \\ 2tan^{-1}\frac{\omega_2^*}{\omega_{02}^*} & otherwise \end{cases} \tag{11}$$

such that the values $\omega_{01}=\omega_{01}^*,\ \omega=\omega_1^*$ and $\omega_{02}=\omega_{02}^*,\ \omega=\omega_2^*$ corresponds to cases on the onset of instability in sense of Definition 1.

Remark 1: Note that the sets of values ω_{01}^*,ω_1^* and ω_{02}^*,ω_2^* obtained via the DF/HB technique are predicted whereas via the CC and LSM – LMI techniques are conservative in nature. Hence, the PM in NLTI system (4) calculated using these techniques is named as *Practical* Phase Margin or PPM.

The following algorithms are proposed for PPM computation in NLTI system (4) in sense of Definition 1.

4.3. Algorithms for computing PPM based on DF/HB technique:

In sense of Definition 1, the set of values $\omega_{01}=\omega_{01}^*,\ \omega=\omega_1^*$ or $\omega_{02}=\omega_{02}^*,\ \omega=\omega_2^*$ can be computed such that assumption (A7) starts violating.

Firstly, considering the scenario when $H_1(j\omega)$ is selected, the values $\omega_{01}=\omega_{01}^*,\ \omega=\omega_1^*$ can be calculated as solutions to the HB eqn.

$$H_1(j\omega)G_{Lur'e}(j\omega)=-N(A,\omega)^{-1} \tag{12}$$

such that the parameters of a predicted limit cycle (A^*,ω_1^*) start emerging as solutions to eqn. (12) at $\omega_{01}=\omega_{01}^*$ and does not exist for $\omega_{01}>\omega_{01}^*$ (Fig. 2). Similarly, if $H_2(j\omega)$ is considered then the values $\omega_{02}=\omega_{02}^*,\ \omega=\omega_2^*$ can be calculated by replacing $H_1(j\omega)$ with $H_2(j\omega)$ in eqn. (14).

Following these conditions, the computational algorithms for calculating ω_{01}^*,ω_1^* or ω_{02}^*,ω_2^* are presented below:

DF/HB – Algorithm I for computing ω_{01}^* and ω_1^*:

Step I: Transforming the DF/HB eqn. in (14) into real and imaginary parts gives

$$\begin{aligned} \mathrm{Re}\left[H_1(j\omega)G_{Lur'e}(j\omega)\right] &= \mathrm{Re}\left[-N(A,\omega)^{-1}\right] \\ \mathrm{Im}\left[H_1(j\omega)G_{Lur'e}(j\omega)\right] &= \mathrm{Im}\left[-N(A,\omega)^{-1}\right] \end{aligned} \tag{13}$$

Step II: The parameter $\omega_{01} = \omega_{01}^*$ and frequency $\omega = \omega_1^*$ can be computed as the solutions of eqn. (13). ■

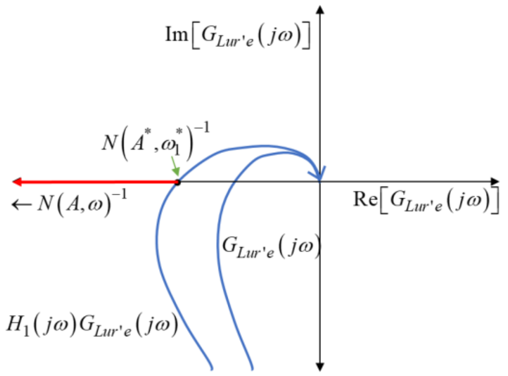

Fig. 2. PPM computation via DF/HB technique

DF/HB – Algorithm II for computing ω_{01}^* and ω_1^*:

Step I: Firstly, calculate the magnitude of both sides of DF/HB eqn. in (12)

$$\left|H_1(j\omega)G_{Lur'e}(j\omega)\right| = \left|N\left(A^*,\omega\right)^{-1}\right| \tag{14}$$

Step II: Next, taking argument on both sides of eqn. (12)

$$\arg\left[H_1(j\omega)\right] + \arg\left[G_{Lur'e}(j\omega)\right] = \arg\left[-N\left(A^*,\omega\right)^{-1}\right] \tag{15}$$

Or eqn. (15) can be rewritten as

$$\arg\left[G_{Lur'e}(j\omega)\right] - tan^{-1}\frac{\omega}{\omega_{01}} = \arg\left[-N\left(A^*,\omega\right)^{-1}\right] \tag{16}$$

Step III: The parameter $\omega_{01} = \omega_{01}^*$ and frequency $\omega = \omega_1^*$ can be computed as the solutions of eqn. (14) and (16). ■

DF/HB – Algorithms for computing ω_{02}^* and ω_2^*:

Step I: The parameter $\omega_{02} = \omega_{02}^*$ and frequency $\omega = \omega_2^*$ can be calculated by replacing $H_1(j\omega)$ with $H_2(j\omega)$ in eqn. (13) for *computing* ω_{01}^* *and* ω_1^* or in eqns. (14) – (16) for *computing* ω_{01}^* *and* ω_1^*, as appropriate.

Remark 2: Note that the PM identification in NLTI system (4) based on the DF/HB technique can only be accomplished for a particular set of nonlinearities whose DF can be computed in order to use the HB equation (7) or (12). In order to overcome these restrictions, in the next two sections, the CC and LSM – LMI based algorithms are proposed for computing PPM in system (4) for a class of nonlinear function with less restrictive conditions (2), (3).

4.4. Algorithms for computing PPM based on CC technique:

In sense of Definition 1, the set of values $\omega_{01} = \omega_{01}^*$, $\omega = \omega_1^*$ or $\omega_{02} = \omega_{02}^*$, $\omega = \omega_2^*$ can be computed such that Assumption (A8) starts violating.

Firstly, considering the scenario when $H_1(j\omega)$ is selected, the frequency characteristics $H_1(j\omega)G_{Lur'e}(j\omega)$ starts intersecting the vertical line defined by $\mathrm{Re}\left[G_{Lur'e}(j\omega)\right] = -k_1^{-1}$ at some point $\left(-k_1^{-1},b\right)$ (Fig. 3) provided $0 = k_2 < k_1$, which will make condition (B2) invalid, at $\omega_{01} = \omega_{01}^*$, $\omega = \omega_1^*$. Note that the coordinate parameter b is unknown and identified later on. This condition can be presented as the equation

$$H_1(j\omega)G_{Lur'e}(j\omega) = -k_1^{-1} + jb, \quad \text{when } \omega_{01} = \omega_{01}^* \tag{17}$$

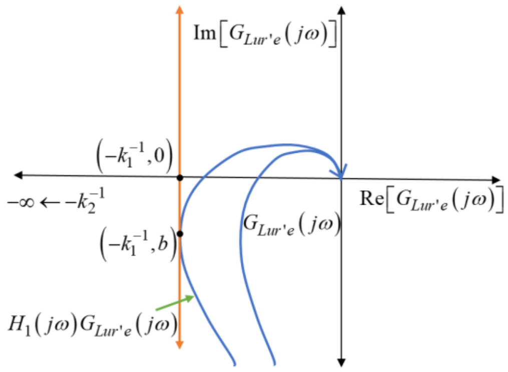

Fig. 3. PPM computation via CC technique

The same reasoning applies if $H_2(j\omega)$ is considered, with parameter ω_{02}^* and frequency ω_2^*. In this case, $H_2(j\omega)$ should be used in eqn. (17) instead of $H_1(j\omega)$.

Following these conditions, the computational algorithms for calculating the set of values $\omega_{01}^*, \omega_1^*$ or $\omega_{02}^*, \omega_2^*$ based on CC are presented:

CC – Algorithm for computing ω_{01}^* and ω_1^*:

Step I: Transforming eqn. (20) into real and imaginary parts gives

$$\begin{aligned} \mathrm{Re}\left[H_1(j\omega)G_{Lur'e}(j\omega)\right] &= -\frac{1}{k_1} \\ \mathrm{Im}\left[H_1(j\omega)G_{Lur'e}(j\omega)\right] &= b \end{aligned} \tag{18}$$

Step II: Taking magnitude of both sides of equation (17)

$$\left|H_1(j\omega)G_{Lur'e}(j\omega)\right| = \left|-\frac{1}{k_1} + jb\right| \tag{19}$$

Step III: The parameter $\omega_{01} = \omega_{01}^*$, corresponding frequency $\omega = \omega_1^*$ and b (to be excluded) can be computed as a solution to eqns. (18) and (19). ■

CC – Algorithm for computing ω_{02}^* and ω_2^* :

Step I: The parameter $\omega_{02} = \omega_{02}^*$ and frequency $\omega = \omega_2^*$ can be calculated by replacing $H_1(j\omega)$ with $H_2(j\omega)$ in eqns. (17) – (19) for *computing* ω_{01}^* *and* ω_1^*. ■

4.5. Algorithms for computing PPM based on LSM – LMI technique:

As mentioned earlier, the CC based results obtained in the previous section are conservative in nature. Another method based on the LSM – LMI technique which gives conservative results as well, is discussed in this section. In this case, the least conservative result obtained through either of these algorithms maybe used for PPM identification in NLTI system (4).

Remark 3: A limitation of the LSM – LMI algorithm is the inability to compute the frequencies ω_1^* and ω_2^* since it is a time domain technique. In this case, it can be assumed that these values are comparable to the frequencies derived via the CC algorithm since both algorithms have similar stability constraints and sector conditions. Hence ω_1^* and ω_2^* derived via CC algorithm may be used with the parameters ω_{01}^* and ω_{02}^* computed through LSM – LMI analysis to calculate PPM in NLTI system (4) in sense of Definition 1.

Next, considering the scenario when $H_1(j\omega)$ is selected, it is clear that NLTI system (4) augmented by $H_1(j\omega)$ is stable for $\forall \omega_{01} > \omega_{01}^*$ and becomes marginally stable at the parameter $\omega_{01} = \omega_{01}^*$ and corresponding frequency $\omega = \omega_1^*$.

In order to compute $\omega_{01} = \omega_{01}^*$, firstly the stability conditions for NLTI system (4) is analysed through the LSM – LMI technique, as presented in the following theorem:

Theorem 1: NLTI system (4) is globally asymptotically stable if there exists a positive definite matrix P_1 and a scalar $\tau \geq 0$ such that the matrix inequality

$$\begin{bmatrix} \tilde{A}^T P_1 + P_1\tilde{A} - \tau\gamma C^T K^T KC & P_1 B - \tau\delta C^T K^T \\ B^T P_1 - \tau\delta KC & -\tau \end{bmatrix} \preceq 0 \quad (20)$$

is feasible.

Proof: Firstly, introduce a QLF candidate $V_1(x) = x^T P_1 x,\ V_1(x) > 0$ for NLTI system (4) where P_1 is a positive definite matrix. The derivative of function $V_1(x)$ is given by

$$\begin{aligned} \dot{V}_1(x) &= x^T\left(\tilde{A}^T P_1 + P_1\tilde{A}\right)x + v^T B^T P_1 x + x^T P_1 B v \\ &= \begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} \tilde{A}^T P_1 + P_1\tilde{A} & P_1 B \\ B^T P_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} < 0 \end{aligned} \quad (21)$$

In accordance with LSM [Khalil, 2002], global asymptotic stability of the equilibrium point $x = 0$ can be established via eqn. (21), which can be rewritten in quadratic function form of (x, v) as

$$\begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} \tilde{A}^T P_1 + P_1\tilde{A} & P_1 B \\ B^T P_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} < 0 \quad (22)$$

Inequality (22) can be further simplified into LMI form [Boyd et al., 1994] as

$$\begin{bmatrix} \tilde{A}^T P_1 + P_1\tilde{A} & P_1 B \\ B^T P_1 & 0 \end{bmatrix} \prec 0 \quad (23)$$

Remark 2: A marginal stability case for system (4) can be analyzed by replacing the sign $\prec$ by $\preceq$ in eq. (23).

Next, if assumption (A1) is valid, then the sector condition in eqn. (3) can be expanded as a quadratic function of (x, v) as

$$\begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} \gamma C^T K^T KC & \delta C^T K^T \\ \delta KC & 1 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \leq 0 \quad (24)$$

The validity of the inequality (24) can be ascertained by the LMI

$$\begin{bmatrix} \gamma C^T K^T KC & \delta C^T K^T \\ \delta KC & 1 \end{bmatrix} \preceq 0 \quad (25)$$

Apparently, through the application of S-Lemma [Boyd et al., 1994], [Kürşad and Mustafa, 2006], a relation between the inequalities (22) and (24) can be established such that the LMI (25) is valid if and only if LMI (23) holds and there exists a scalar $\tau \geq 0$ so that the following inequality is feasible

$$\tau \begin{bmatrix} \gamma C^T K^T KC & \delta C^T K^T \\ \delta KC & 1 \end{bmatrix} \succeq \begin{bmatrix} \tilde{A}^T P_2 + P_2\tilde{A} & P_2 B \\ B^T P_2 & 0 \end{bmatrix} \quad (26)$$

Inequality (26) can finally be transformed into LMI form as given in (20) which can be solved to derive P_1 and τ such that NLTI system (4) is globally asymptotically stable.

The theorem is proven. ■

Next, the global asymptotic stability conditions for NLTI system (4) augmented by $H_1(j\omega)$, are studied in the following theorem:

Theorem 2: NLTI system (4) augmented by $H_1(j\omega)$ is globally asymptotically stable if there exists a positive definite matrix $P_2 > 0$ and a decay rate $\beta > 0$, scalars $\omega_{01} > \omega_{01}^*,\ \rho \geq 0$ such that the following LMI is satisfied

$$\begin{bmatrix} \tilde{A}^T P_1 + P_1\tilde{A} + \beta P_1 - \rho\gamma C^T K^T KC & P_1 B & -\rho\delta C^T K^T \\ B^T P_1 & -2P_2\omega_{01} + \beta P_2 & P_2\omega_{01} \\ -\rho\delta KC & P_2\omega_{01} & -\rho \end{bmatrix} \preceq 0 \quad (27)$$

where positive definite matrix P_1 is derived from LMI (20).

Proof: Firstly, a state variable model of NLTI system (4) cascaded with $H_1(j\omega)$ is obtained as

$$\dot{\mu} = N\mu + Sv, \quad \mu = \begin{bmatrix} x & p \end{bmatrix}^T \tag{28}$$

where $p \in \Re^1$, $N = \begin{bmatrix} \tilde{A} & B \\ 0 & -\omega_{01} \end{bmatrix}$ and $S = \begin{bmatrix} 0 & \omega_{01} \end{bmatrix}^T$. Next introduce a QLF candidate $V_3(\mu) = \mu^T P_3 \mu$ for system (28), where $P_3 = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ is a positive definite matrix. System (28) will be globally asymptotically stable if and only if

$$\dot{V}_3(\mu) \le -\beta V_3(\mu), \quad \beta > 0 \tag{29}$$

where β is the decay rate. Inequality (29) is equivalent to

$$\begin{aligned} &\dot{V}_3(\mu) + \beta V_3(\mu) \\ &= \mu^T (N^T P_3 + P_3 N + \beta P_3)\mu + v^T S^T P_3 \mu + \mu^T P_3 S v \le 0 \end{aligned} \tag{30}$$

which can be written as a quadratic function of (x, p, v) as,

$$\begin{bmatrix} x^T & p^T & v^T \end{bmatrix} \begin{bmatrix} \tilde{A}^T P_1 + P_1 \tilde{A} + \beta P_1 & P_1 B & 0 \\ B^T P_1 & -2P_2\omega_{01} + \beta P_2 & P_2\omega_{01} \\ 0 & P_2\omega_{01} & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ v \end{bmatrix} \le 0 \tag{31}$$

Condition (31) can be validated with the LMI

$$\begin{bmatrix} \tilde{A}^T P_1 + P_1 \tilde{A} + \beta P_1 & P_1 B & 0 \\ B^T P_1 & -2P_2\omega_{01} + \beta P_2 & P_2\omega_{01} \\ 0 & P_2\omega_{01} & 0 \end{bmatrix} \preceq 0 \tag{32}$$

Note that, if assumption (A2) holds for system (31), eqn. (3) can be written as a quadratic inequality of (x, p, v) as

$$\begin{bmatrix} x^T & p^T & v^T \end{bmatrix} \begin{bmatrix} \gamma C^T K^T K C & 0 & \delta C^T K^T \\ 0 & 0 & 0 \\ \delta K C & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p \\ v \end{bmatrix} \le 0 \tag{33}$$

Equation (33) can be further reduced to LMI form as

$$\begin{bmatrix} \gamma C^T K^T K C & 0 & \delta C^T K^T \\ 0 & 0 & 0 \\ \delta K C & 0 & 1 \end{bmatrix} \preceq 0 \tag{34}$$

A relation between the LMIs (32) and (34) can be derived via the lossless S – Lemma [Kürşad and Mustafa, 2006] as

$$\rho \begin{bmatrix} \gamma C^T K^T K C & 0 & \delta C^T K^T \\ 0 & 0 & 0 \\ \delta K C & 0 & 1 \end{bmatrix} \succeq \begin{bmatrix} \tilde{A}^T P_1 + P_1 \tilde{A} + \beta P_1 & P_1 B & 0 \\ B^T P_1 & -2P_2\omega_{01} + \beta P_2 & P_2\omega_{01} \\ 0 & P_2\omega_{01} & 0 \end{bmatrix} \tag{35}$$

for some scalar $\rho \ge 0$.

Inequality (35) can be further transformed into LMI form as given in eqn. (27) which can be solved to derive matrix P_2 and scalars ρ, β, $\omega_{01} > \omega_{01}^*$, such that NLTI system (28) is globally asymptotically stable.

The theorem is proven. ∎

In sense of Definition 2, the main goal is to compute the parameter $\omega_{01} = \omega_{01}^*$ such that LMI (27) is no longer satisfied and augmented NLTI system (4) starts violating assumption (A3). In this case, the following algorithm is presented for calculating the parameters ω_{01}^*:

LSM – LMI Algorithm for computing ω_{01}^*:

Step I: Compute P_1 and τ by solving LMI (20) as a Convex Optimization Problem (COP)

minimize τ

$$\text{subject to} \begin{bmatrix} \tilde{A}^T P_1 + P_1 \tilde{A} - \tau\gamma C^T K^T K C & P_1 B - \tau\delta C^T K^T \\ B^T P_1 - \tau\delta K C & -\tau \end{bmatrix} \preceq 0 \tag{36}$$

$P_1 > 0$

in MATLAB via an LMI solver.

Step II: If LMI (27) is valid then the following COP can be formulated

minimize β

$$\text{subject to } \tilde{A}^T P_1 + P_1 \tilde{A} + \beta P_1 - \rho\gamma C^T K^T K C \preceq 0 \tag{37}$$

$\rho \ge 0$

where P_1 is a solution of (36). COP (37) can be solved for β in MATLAB via an LMI solver.

Step III: Again, the following condition can be derived from LMI (27)

$$-2P_2\omega_{01} + \beta P_2 \le 0 \rightarrow \beta \le 2\omega_{01} \rightarrow \omega_{01} \ge \frac{\beta}{2}, \forall \omega_{01} \le \omega_{01}^* \tag{38}$$

From eqn. (28), it is clear that the parameter $\omega_{01}^* = \dfrac{\beta_{\min}}{2}$. ∎

Next, if $H_2(j\omega)$ is considered, then NLTI system (4) augmented by $H_2(j\omega)$ is stable for $\omega_{02} > \omega_{02}^*$ and becomes marginally stable at $\omega_{02} = \omega_{02}^*, \omega = \omega_2^*$. In this case, a global asymptotic stability condition, identical to Theorem 2, can be derived for the NLTI system (4) augmented by $H_2(j\omega)$, by following steps analogous to eqns. (28) – (38) albeit for $H_2(j\omega)$. The subsequent steps of calculations are not mentioned for reasons of brevity.

LSM – LMI Algorithm for computing ω_{02}^*:

Step I: The parameter ω_{02}^* is computed as $\omega_{02}^* = \dfrac{\beta_{\min}}{4}$. ∎

5. EXAMPLE

Consider a Lur'e type NLTI system (4) with parameters

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -7 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, K = 3 \tag{39}$$

and with the open loop transfer function

$$G_{Lur'e}(s) = \frac{3}{s(s+6)(s+1)} \tag{40}$$

It is assumed that system (29) is perturbed by a saturation nonlinearity $v = sat(u)$ which satisfies the sector conditions in eqn. (2) such that $0 = k_2 < k_1 = 2$. Note that the PM in system (39) when $v = sat(u) \equiv 0$ is computed via MATLAB as 49°. In this case, the PPM in NLTI system (4) can be computed as follows:

5.1. DF/HB method:

Note that, the saturation function $v = sat(u)$ has the describing function

$$N(A,\omega) = N(A) = \frac{2k}{\pi}\left[\sin^{-1}\left(\frac{a}{A}\right) + \left(\frac{a}{A}\right)\sqrt{1-\left(\frac{a^2}{A^2}\right)}\right] \tag{41}$$

In case of saturation, a limit cycle starts emerging at $A = a$ and the DF $N(A) = k = k_1 = 2$. Since, for the LTI system $PM < 90^\circ$, $H_1(j\omega)$ is selected and the parameters $\omega_{01}^*, \omega_1^*$ are calculated as $\omega_{01}^* = 0.135$, $\omega_1^* = 0.115$ via eqns. (14) – (16). Thus $PPM = 40.9^\circ$.

5.2. CC method:

Using eqns. (17) – (19), $\omega_{01}^* = 0.15, \omega_1^* = 0.19$ are computed and thus $PPM = 38.29^\circ$.

5.3. LSM/LMI method:

Following the steps in eqns. (36) – (38), the parameter ω_{01}^* is calculated as $\omega_{01}^* = 0.11$ and hence $PPM = 30.1^\circ$.

Remark 6: The PPMs obtained via CC and LSM/LMI methods are more conservative than the one obtained via DF/HB algorithm. It was expected, since the DF/HB algorithm is applied for a particular nonlinearity while CC and LSM/LMI methods are applied for a class of sector nonlinearities.

6. CONCLUSIONS

The main contribution of this work is introducing a novel concept of Phase Margin identification in Lur'e Type NLTI systems. The proposed algorithms based on DF/HB, CC and LSM – LMI techniques provide a sufficient PM for NLTI systems, however such margins are either predicted or conservative in nature. Keeping this in mind, future works may include developing algorithms for identification of PM in Sliding Mode Controllers and Higher Order Sliding Mode Controllers (HOSM).

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