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Conversion of Certain Stochastic Control Problems into Deterministic Control Problems ⋆

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Abstract: A class of nonlinear, stochastic staticization control problems (including minimization problems with smooth, convex, coercive payoffs) driven by diffusion dynamics with constant diffusion coefficient is considered. A fundamental solution form is obtained where the same solution can be used for a limited variety of terminal costs without re-solution of the problem. One may convert this fundamental solution form from a stochastic control problem form to a deterministic control problem form. This yields an equivalence between certain second-order (in space) Hamilton-Jacobi partial differential equations (HJ PDEs) and associated first-order HJ PDEs. This reformulation has substantial numerical implications.

Keywords: Stochastic control, nonlinear control, Hamilton-Jacobi, optimal control, dynamic programming.

1. INTRODUCTION

We consider nonlinear optimal stochastic control problems where the finite-dimensional dynamics are driven by Brownian motion processes, taking the form of stochastic differential equations (SDEs). These problems are typically converted into Hamilton-Jacobi partial differential equation (HJ PDE) problems. In the case of deterministic optimal control problems, the HJ PDEs are first-order equations, while in the stochastic case, these are second-order HJ PDEs. The dimension of the space over which these PDEs are defined is that of the state process of the control problem. Of course, realistic control problems typically have relatively high dimensional state processes (i.e., greater than dimension three), leading to PDEs over high dimensional spaces. The solution of such HJ PDE problems has long been hampered by the curse-of-dimensionality, which refers to the fact that with classical algorithms, the computational cost grows exponentially fast as a function of space dimension, and we note that this has limited the solvability of such problems by classical methods to state-space dimensions on the order of three to five. More recently, max-plus based curse-of-dimensionality-free methods have demonstrated computational tractability for certain classes of problems in significantly higher space dimension, and this approach have been quite effective in the case of first-order HJ PDEs Gaubert et al (2011); McEneaney (2006, 2009); Qu (2014); Sridharan et al (2014), with the caveat being a curse-of-complexity that grows rapidly with propagation.

Extensions of the max-plus based curse-of-dimensionality-free methods to second-order HJ PDEs and stochastic control problems has been less computationally successful, where this effectiveness reduction is due to the requirement that one employ a max-plus distributive property at each time step McEneaney and Kaise (2016); see also Akian and Fodjo (2016).

Here, we demonstrate that for certain classes of problems, one may convert the second-order (in space) HJ PDEs associated to stochastic control problems driven by Brownian motion into first-order HJ PDEs, thereby admitting the potential application of the rapid curse-of-dimensionality-free methods. Further, in this transition to first-order HJ PDEs, the solutions may be obtained as fundamental solutions, which implies that the same solution may be applied to varying terminal costs (within a certain class) without complete re-solution of the HJ PDE problems. In the most general case, one requires approximation of a series, but there are particular cases where this reduces to a closed-form solution.

2. DEFINITION OF THE PROBLEM CLASS

We consider a nonlinear stochastic control problem where the SDE dynamics and initial state are given by

$$d\xi_t = f(\xi_t, u_t)\,dt + \mu\,dB_t, \quad \xi_s = x \in \mathbb{R}^n, \tag{1}$$

where the underlying probability space is denoted as $(\Omega, \mathcal{F}_\infty, P)$ with Ω denoting the sample space, $\mathcal{F}_\infty$ denoting the σ-algebra and P denoting the probability measure. Also, B_t denotes an n-dimensional Brownian motion adapted to filtration $\mathcal{F}_t$. Assumptions on f will be indicated further below. We let $U \subseteq \mathbb{R}^k$, and suppose the controls take values in U. Fix $T \in (0, \infty)$, and for $s \in [0, T]$, let

$\mathcal{U}_s \doteq \{u : [s, T] \times \Omega \to \mathbb{R}^n \,|\, u$ is $\mathcal{F}_\cdot$-adapted, right-contin.

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$$\text{and such that } \mathbb{E}\int_s^T |u_t|^m\,dt < \infty\ \forall m \in I\!N\,\}. \quad (2)$$

We also define the state-process space

$$\mathcal{X}_s \doteq \{\xi : [s,T]\times\Omega \to \mathbb{R}^n \,|\, \xi \text{ is } \mathcal{F}_{\cdot}\text{-adapted, right-contin.} \text{ and such that } \mathbb{E}\sup_{t\in[s,T]} |\xi_t|^m < \infty\ \forall m \in I\!N\,\}. \quad (3)$$

The payoff will be given by

$$J(s,x,u;z) \doteq \mathbb{E}\Big\{\int_s^T L(\xi_t,u_t)\,dt + \psi(\xi_T;z)\Big\}, \quad (4)$$

$$\psi(x;z) \doteq \tfrac{1}{2}(x-z)^T\bar{M}(x-z) + \bar{\gamma}, \quad (5)$$

where $\bar{M}$ is positive-definite and symmetric, $\bar{\gamma} \in \mathbb{R}$ and $z \in \mathbb{R}^n$. In the more general case, one takes a terminal cost form

$$\Psi(x) \doteq \operatorname*{stat}_{z\in\mathbb{R}^n} \big\{\tfrac{1}{2}(x-z)^T\bar{M}(x-z) + \hat{\gamma}(z)\big\}, \quad (6)$$

with some specified function, $\hat{\gamma}(\cdot)$, where the definition of operator stat follows in Section 3. The terminal cost in (6) is a "stat-quad" representation, McEneaney and Dower (2018), of a general class of terminal costs that may be represented as such. Because of the already technical nature of the sequel, we will mainly not include the general form (6) in the analysis below, but see McEneaney and Dower (2015) for a deeper discussion of the usage of such a form. For $(s,x) \in [0,T]\times\mathbb{R}^n$, the value function is

$$\bar{W}(s,x;z) = \bar{W}(s,x;z,\bar{M},\bar{\gamma}) \doteq \operatorname*{stat}_{u\in\mathcal{U}_s} J(s,x,u;z). \quad (7)$$

We remark that in the case of a convex, coercive, C^1 payoff, stat is equivalent to minimization; that is

$$\bar{W}(s,x;z) = \bar{W}(s,x;z,\bar{M},\bar{\gamma}) = \min_{u\in\mathcal{U}_s} J(s,x,u;z). \quad (8)$$

Hence, in such cases, all results obtained for staticization problems hold for minimization problems.

Lastly, in the case of the more general terminal payoff of (6), the value becomes

$$\mathcal{W}(s,x) \doteq \operatorname*{stat}_{z\in\mathbb{R}^n}\operatorname*{stat}_{u\in\mathcal{U}_s} J(s,x,u;z) = \operatorname*{stat}_{z\in\mathbb{R}^n} \bar{W}(s,x;z), \quad (9)$$

with terminal cost (6) replacing (5).

3. STATICIZATION DEFINITIONS

"Staticization" has recently proven to be quite useful. Specifically, the principle of stationary action can be used to generate fundamental solutions of conservative dynamical systems and to obtain stochastic control representations for solutions of Schrödinger initial value problems (IVPs) Dower and McEneaney (2017); McEneaney and Dower (2015); McEneaney and Zhao (2019); Zhao and McEneaney (2019); McEneaney and Dower (2019); McEneaney (2019) (also cf. Doss (2011); Fleming (1983) among others). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization", with these points being statica, in analogy with minima/maxima. We make the following definitions. Let $\mathcal{F}$ denote either the real or complex field. Suppose $\mathcal{U}$ is a normed vector space (over $\mathcal{F}$) with $\mathcal{A} \subseteq \mathcal{U}$, and suppose $G : \mathcal{A} \to \mathcal{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$ and either $\limsup_{u\to\bar{u},u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0$, or there exists $\delta > 0$ such that $\mathcal{A}\cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). If $\operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\operatorname*{stat}^s_{u\in\mathcal{A}} G(u) \doteq \big\{G(\bar{u}) \,\big|\, \bar{u} \in \operatorname{argstat}\{G(u)\,|\,u\in\mathcal{A}\}\,\big\}. \quad (10)$$

If $\operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\} = \emptyset$, then $\operatorname{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. We are mainly interested in a single-valued stat operation. In particular, if there exists $a \in \mathcal{F}$ such that $\operatorname{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathcal{A}} G(u)$ is undefined.

In the case where $\mathcal{U}$ is a Banach space and $\mathcal{A} \subseteq \mathcal{U}$ is an open set, $G : \mathcal{A} \to \mathcal{F}$ is Fréchet differentiable at $\bar{u} \in \mathcal{A}$ with derivative $DG(\bar{u}) \in \mathcal{L}(\mathcal{U};\mathcal{F})$ if

$$\lim_{w\to 0,\,\bar{u}+w\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(\bar{u}+w)-G(\bar{u})-[DG(\bar{u})]w|}{|w|} = 0. \quad (11)$$

The following is immediate from the above definitions.

Lemma 1. Suppose $\mathcal{U}$ is a Banach space, with open set $\mathcal{A} \subseteq \mathcal{U}$, and that G is Fréchet differentiable at $\bar{u} \in \mathcal{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y)\,|\,y \in \mathcal{A}\}$ if and only if $DG(\bar{u}) = 0$.

4. RECOLLECTION OF RESULTS

We will proceed through several steps that will eventually lead to formulation as a deterministic control problem. The first step is to obtain the equivalence between the value function and the solution of the associated HJ PDE problem. This equivalence is standard in the optimization and game cases (i.e., minimization, maximization and/or minimax), and less so in staticization cases that do not correspond to these. Hence, we only recall some results here, so as to ground the sequel. In particular, in this first presentation of the approach, we work under strong conditions so as to avoid excessively technical proofs, and more clearly indicate the structure of the approach. Let $\mathcal{Z} \doteq (0,T)\times\mathbb{R}^n\times\mathbb{R}^n$ and $\bar{\mathcal{Z}} \doteq (0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, and consider

$$\begin{aligned}0 &= W_t + \operatorname*{stat}_{v\in U}\big\{f(x,v)^T W_x + L(x,v)\big\} + \tfrac{1}{2}\operatorname{tr}[AW_{xx}]\\ &\doteq W_t + H_0(x,W_x) + \mathcal{Q}_0(x,W_x) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}],\\ &\doteq W_t + \tilde{H}_0(x,W_x) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}], \quad (t,x,z)\in\mathcal{Z}, \quad (12)\end{aligned}$$

$$W(T,x;z) = \psi(x;z), \quad (x,z)\in\mathbb{R}^n\times\mathbb{R}^n, \quad (13)$$

where $\mathcal{Q}_0$ is a quadratic function of its arguments, and the non-quadratic components of the Hamiltonian are isolated within H_0 (where we note that the diffusion coefficient in (1) is constant). The second set of assumptions, which are for the more general staticization case, are as follows.

Assume that for $z \in \mathbb{R}^n$, there exists $W = W(\cdot,\cdot;z) \in C^{1,4}(\mathcal{Y}) \cap C_p(\bar{\mathcal{Y}})$ satisfying (12)–(13), and that there exists $\bar{C}_0 < \infty$ and $q \in I\!N$ such that $|W_x(s,x)| \le \bar{C}_0(1+|x|^{2q})$ and $|W_{xx}(s,x)| \le \bar{C}_0(1+|x|^{2q})$ for all $(s,x) \in \bar{\mathcal{Y}}$. Assume $U = \mathbb{R}^k$; $f, L \in C^3(\mathbb{R}^n\times U)$; $\exists \bar{C}_1 < \infty$ such that $|f_x(x,v)|, |f_v(x,v)| \le \bar{C}_1$, $|f_{xx}(x,v)|, |f_{xv}(x,v)|, |f_{vv}(x,v)| \le \bar{C}_1$ and $|L_{xx}(x,v)|, |L_{xv}(x,v)|, |L_{vv}(x,v)|, \le \bar{C}_1$. Assume that for each $z \in \mathbb{R}^n$, there exists $\bar{u} \in C(\bar{\mathcal{Y}})$ such that $f(x,\bar{u}(t,x))$ is globally Lipschitz in x on $\mathcal{Y}$ and such that $\bar{u}(t,x) \in \operatorname{argstat}_{v\in U}\{f(x,v)^T W_x(t,x) + L(x,v)\}$ for all $(t,x) \in \bar{\mathcal{Y}}$. $(\mathcal{A}.1)$

Theorem 2. Assume $(\mathcal{A}.1)$. Then $W = \bar{W}$ on $\bar{\mathcal{Z}}$, and $\bar{u}$ is a stationary control yielding payoff $\bar{W}$.

The proof of Theorem 2 appears in McEneaney and Dower (2020), and is quite similar to that of (McEneaney, 2019, Th 4.1), but with a real-valued (rather than complex-valued) system and less specific dynamics and cost.

5. FUNDAMENTAL-SOLUTION FORM

We now proceed through several steps that will lead to a fundamental solution form, and then further, to a deterministic-control, fundamental solution form. We remark that the term "fundamental solution form" is being employed here to indicate that modifications of the terminal cost, within a certain class, will not require re-solution of the problem. For $x, p, \alpha, \beta \in \mathbb{R}^n$, let

$$\mathcal{Q}(x,p,\alpha,\beta) \doteq \tfrac{c_1}{2}|x-\alpha|^2 + \tfrac{c_2}{2}|p-\beta|^2,$$

where $c_1, c_2 \in \mathbb{R}$. Analogous to semiconvex duality in the convex-duality framework, we have the following variant in the stationarity framework, which is referred to as "stat-quad" duality (McEneaney and Dower, 2018, Th. 4).

Lemma 3. Suppose C is nonsingular, and $\phi \in C^1(\mathbb{R}^n;\mathbb{R})$. Letting $\eta(w) \doteq D\phi(w) - Cw$ for all $w \in \mathbb{R}^n$, suppose $\eta^{-1} \in C^1(\mathbb{R}^n;\mathbb{R}^n)$. Then,

$$\phi(w) = \operatorname*{stat}_{y\in\mathbb{R}^n} \left[a(y) + \tfrac{1}{2}(y-w)^T C(y-w)\right] \quad \forall\, w \in \mathbb{R}^n,$$
$$a(y) = \operatorname*{stat}_{w\in\mathbb{R}^n} \left[\phi(w) - \tfrac{1}{2}(y-w)^T C(y-w)\right] \quad \forall\, y \in \mathbb{R}^n.$$

We will say that a generic function, $G : \mathcal{M} \to \mathbb{R}$ with $\mathcal{M}$ being an open subset of a Hilbert space is *uniformly Morse in* $\eta \in \mathcal{M}$ if there exists $\tilde{K} < \infty$ such that for all $\hat{\eta} \in \mathcal{M}$ such that $G_\eta(\hat{\eta}) = 0$, $G_{\eta\eta}(\hat{\eta})$ is invertible with $\left|[G_{\eta\eta}(\hat{\eta})]^{-1}\right| \le \tilde{K}$. Further, we will say that a generic function, $G : \mathcal{M}\times\mathcal{N} \to \mathbb{R}$ with $\mathcal{M},\mathcal{N}$ being open subsets of their respective Hilbert spaces is *uniformly Morse in* $\eta \in \mathcal{M}$ *over* $\zeta \in \mathcal{N}$ if there exists $\tilde{K} < \infty$ such that for all $(\hat{\eta},\hat{\zeta}) \in \mathcal{M}\times\mathcal{N}$ such that $G_\eta(\hat{\eta},\hat{\zeta}) = 0$, $G_{\eta\eta}(\hat{\eta},\hat{\zeta})$ is invertible with $\left|[G_{\eta\eta}(\hat{\eta},\hat{\zeta})]^{-1}\right| \le \tilde{K}$. We make the following assumption, which will be sufficient to guarantee existence of all the relevant duality objects to follow.

We assume that $H_0 \in C^3(\mathbb{R}^{2n})$, that the first, second and third derivatives of H_0 are uniformly bounded, and that H_0 is uniformly Morse in $(x,p) \in \mathbb{R}^{2n}$. $(\mathcal{A}.2)$

By Lemma 3, Assumption $(\mathcal{A}.2)$ and straightforward calculations, one obtains the following.

Lemma 4. Let $|c_1|, |c_2|$ be sufficiently large. Then,

$$H_0(x,p) = \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \left[G_0(\alpha,\beta) + \mathcal{Q}(x,p,\alpha,\beta)\right] \ \forall (x,p) \in \mathbb{R}^{2n},$$
$$G_0(\alpha,\beta) = \operatorname*{stat}_{(x,p)\in\mathbb{R}^{2n}} \left[H_0(x,p) - \mathcal{Q}(x,p,\alpha,\beta)\right] \ \forall (\alpha,\beta) \in \mathbb{R}^{2n},$$

where $\operatorname{argstat}_{(\alpha,\beta)\in\mathbb{R}^{2n}}\left[G_0(\alpha,\beta) + \mathcal{Q}(x,p,\alpha,\beta)\right]$ is single-valued for all $(x,p) \in \mathbb{R}^{2n}$, and $\operatorname{argstat}_{(x,p)\in\mathbb{R}^{2n}}\left[H_0(x,p) - \mathcal{Q}(x,p,\alpha,\beta)\right]$ is single-valued for all $(\alpha,\beta) \in \mathbb{R}^{2n}$. Further, $G_0 \in C^3(\mathbb{R}^{2n})$, with bounded second and third derivatives. Lastly, denoting the consitutent argstat functions as $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{x}$ and $\tilde{p}$, one has $\tilde{\alpha}, \tilde{\beta}, \tilde{x}, \tilde{p} \in C^1(\mathbb{R}^{2n})$, with $\tilde{\alpha}, \tilde{\beta}$ globally Lipschitz.

Combining Theorem 2 and Lemma 4, one obtains the following.

Lemma 5. Let $|c_1|, |c_2|$ be sufficiently large. Then, for each $z \in \mathbb{R}^n$, the value function given by (1)–(7) is the unique, classical solution of

$$0 = W_t + \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \left\{G_0(\alpha,\beta) + \mathcal{Q}(x, W_x, \alpha, \beta)\right\} + \mathcal{Q}_0(x, W_x) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}], \quad (t,x) \in \mathcal{Y}, \tag{14}$$
$$W(T,x;z) = \psi(x;z), \quad x \in \mathbb{R}^n. \tag{15}$$

Now, we let $\mathcal{Q}_0$ take the specific form

$$\mathcal{Q}_0(x,p) = \tfrac{1}{2}\left[x^T D_{1,1}x + 2x^T D_{1,2}p + p^T D_{2,2}p\right] + d_1^T x + d_2^T p \tag{16}$$

where $D_{1,1}, D_{2,2}$ are symmetric. Note that with $\mathcal{I}_m$ generically denoting the $m\times m$ identity matrix throughout,

$$\begin{aligned} &G_0(\alpha,\beta) + \mathcal{Q}_0(x,p) + \mathcal{Q}(x,p,\alpha,\beta) = G_0(\alpha,\beta) \\ &+ \tfrac{1}{2}x^T D_{1,1}x + \tfrac{c_1}{2}|x-\alpha|^2 + \tfrac{c_2}{2}|\beta|^2 + d_1^T x, \\ &+ [D_{1,2}^T x + d_2 - c_2\beta]^T p + \tfrac{1}{2}p^T(c_2\mathcal{I}_n + D_{2,2})p, \end{aligned} \tag{17}$$

which, with $|c_2|$ sufficiently large being a sufficient condition, is equivalently given by,

$$= \operatorname*{stat}_{v\in\mathbb{R}^n} \left\{[D_{1,2}^T x + d_2 - c_2\beta + v]^T p + H_1(x,\alpha,\beta,v)\right\}, \tag{18}$$
$$\begin{aligned} &H_1(x,\alpha,\beta,v) \doteq G_0(\alpha,\beta) + \tfrac{1}{2}x^T D_{1,1}x + \tfrac{c_1}{2}|x-\alpha|^2 \\ &+ \tfrac{c_2}{2}|\beta|^2 + d_1^T x + \tfrac{1}{2}v^T\Gamma v, \\ &\Gamma \doteq -(c_2\mathcal{I}_n + D_{2,2})^{-1}. \end{aligned}$$

6. PREPARATORY ITEMS

Prior to the main development, we need some minor results and objects.

6.1 Iterated Staticization

One may note that by (12), Lemma 4 and (18),

$$\begin{aligned} \tilde{H}_0(x,p) &= \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \left\{G_0(\alpha,\beta) + \mathcal{Q}_0(x,p) + \mathcal{Q}(x,p,\alpha,\beta)\right\} \\ &= \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \left\{H_1(x,\alpha,\beta,v) \right. \\ &\quad \left. + \operatorname*{stat}_{v\in\mathbb{R}^n} \left[(D_{1,2}^T x + d_2 - c_2\beta + v)^T p\right]\right\}. \end{aligned} \tag{19}$$

Lemma 6. Let $|c_2|$ be sufficiently large. Then, for all $(x,p) \in \mathbb{R}^{2n}$,

$$\begin{aligned} \tilde{H}_0(x,p) &= \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \operatorname*{stat}_{v\in\mathbb{R}^n} \left\{H_1(x,\alpha,\beta,v) \right. \\ &\quad \left. + [D_{1,2}^T x + d_2 - c_2\beta + v]^T p\right\} \\ &= \operatorname*{stat}_{(\alpha,\beta,v)\in\mathbb{R}^{3n}} \left\{H_1(x,\alpha,\beta,v) \right. \\ &\quad \left. + [D_{1,2}^T x + d_2 - c_2\beta + v]^T p\right\}. \end{aligned}$$

By (18) and Lemma 5, one easily obtains the following.

Lemma 7. Let $|c_1|, |c_2|$ be sufficiently large. Then, for each $z \in \mathbb{R}^n$, the value function given by (1)–(7) is the unique, classical solution of

$$\begin{aligned} 0 = &W_t + \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \operatorname*{stat}_{v\in\mathbb{R}^n} \left\{H_1(x,\alpha,\beta,v) \right. \\ &\left. + [D_{1,2}^T x + d_2 - c_2\beta + v]^T W_x + \tfrac{1}{2}\operatorname{tr}[AW_{xx}]\right\}, \end{aligned} \tag{20}$$
$$\begin{aligned} = &W_t + \operatorname*{stat}_{(\alpha,\beta,v)\in\mathbb{R}^{3n}} \left\{H_1(x,\alpha,\beta,v) \right. \\ &\left. + [D_{1,2}^T x + d_2 - c_2\beta + v]^T W_x\right\} + \tfrac{1}{2}\operatorname{tr}[AW_{xx}], \end{aligned} \tag{21}$$

for $(t,x)\in\mathcal{Y}$, with terminal condition (15).

Consider the following stationarity control problem. Let the dynamics be given by

$$d\xi_t = f'(\xi_t,\bar\beta_t,u_t)\,dt + \mu\,dB_t \doteq (D_{1,2}^T\xi_t + d_2 - c_2\bar\beta_t + u_t)\,dt + \mu\,dB_t, \quad \xi_s = x. \tag{22}$$

where $u.\in\mathcal{U}_{s,T}$ and $\bar\beta.\in\mathcal{O}_{s,T}$, with

$$\mathcal{O}_{s,T} \doteq \{\nu : [s,T]\times\Omega\to\mathbb{R}^n \mid \nu \text{ is } \mathcal{F}.\text{-adapted, right-contin., and s.t. } \mathbb{E}\textstyle\int_s^T|\nu_t|^2\,dt<\infty\}.$$

Let the payoff and stationary value be given by

$$J'(s,x,u,\bar\alpha,\bar\beta;z) \doteq \mathbb{E}\Big\{\int_s^T H_1(\xi_t,\bar\alpha_t,\bar\beta_t,u_t)\,dt + \psi(\xi_T;z)\Big\}, \tag{23}$$

$$W'(s,x;z) \doteq \operatorname*{stat}_{(u.,\bar\alpha.,\bar\beta.)\in\mathcal{U}_{s,T}\times[\mathcal{O}_{s,T}]^2} J'(s,x,u,\bar\alpha,\bar\beta;z), \tag{24}$$

Using Theorem 2 and Lemmas 4 and 7, one obtains the following.

Lemma 8. Let $|c_1|,|c_2|$ be sufficiently large. Then, for each $z\in\mathbb{R}^n$, the value function, $\bar W$, given by (1)–(7) is identical to the value function, W', given by (22)–(24). Further, there exists unique $(\tilde u,\tilde\alpha,\tilde\beta):\mathcal{Y}\to\mathbb{R}^{3n}$ such that

$$[\tilde u,\tilde\alpha,\tilde\beta](t,x)\in \operatorname*{argstat}_{(\alpha,\beta,u)\in\mathbb{R}^{3n}}\big\{[f'(x,\beta,u)]^T\bar W_x(t,x) + H_1(x,\alpha,\beta,u)\big\}, \tag{25}$$

and $[\tilde\alpha,\tilde\beta,\tilde u](t,\xi_t)$ is a staticizing control.

Consider the iterated form of (24) given by

$$\hat W'(s,x;z) \doteq \operatorname*{stat}_{\bar\alpha,\bar\beta\in[\mathcal{O}_{s,T}]^2}\ \operatorname*{stat}_{u.\in\mathcal{U}_{s,T}} J'(s,x,u,\bar\alpha,\bar\beta;z). \tag{26}$$

Note that the inner staticization of (26) is a set of linear-quadratic Gaussian control problems, indexed by the $\bar\alpha,\bar\beta$ that motivates the following.

6.2 Relevant Differential Riccati Equations

Consider the dynamics, driven by stochastic processes $\bar\alpha_t,\bar\beta_t$, given by

$$\dot\Pi_t = -\bar F_1(\Pi_t) \doteq -\big\{\Pi_tK_2\Pi_t + K_3^T\Pi_t + \Pi_tK_3 + K_1\big\} \tag{27}$$

$$\dot\pi_t = -\bar F_2(\Pi_t,\pi_t,\bar\alpha_t,\bar\beta_t) \doteq -\big\{\Pi_tK_2\pi_t + \Pi_t\hat{\mathcal{I}}^{1,1}V_t^2 + K_3\pi_t + V_t^1\big\}, \tag{28}$$

$$\dot\gamma_t = -\bar F_3(\Pi_t,\pi_t,\bar\alpha_t,\bar\beta_t) \doteq -\big\{G_0(\bar\alpha_t,\bar\beta_t) + \tfrac{c_1}{2}|\bar\alpha_t|^2 + \tfrac{c_2}{2}|\bar\beta_t|^2 + \tfrac12\pi_t^TK_2\pi_t + (V_t^2)^T\pi_t + \tfrac12\operatorname{tr}(K_4\Pi_tK_5)\big\}, \tag{29}$$

with terminal conditions, following from (5), given by

$$\Pi_T = \bar\Pi \doteq \begin{pmatrix}\bar M & -\bar M\\ -\bar M & \bar M\end{pmatrix},\ \pi_T = \bar\pi \doteq 0,\ \gamma_T = \bar\gamma, \tag{30}$$

where

$$K_1 \doteq \begin{pmatrix}(c_1\mathcal{I}_n + D_{1,1}) & 0\\ 0 & 0\end{pmatrix},\quad K_2 \doteq \begin{pmatrix}(c_2\mathcal{I}_n + D_{2,2}) & 0\\ 0 & 0\end{pmatrix},$$

$$K_3 \doteq \begin{pmatrix}D_{1,2}^T & 0\\ 0 & 0\end{pmatrix},\quad K_4 \doteq (A\ 0),\quad V_t^1 \doteq \begin{pmatrix}d_1 - c_1\bar\alpha_t\\ 0\end{pmatrix},$$

$$\hat{\mathcal{I}}^{1,1} \doteq \begin{pmatrix}\mathcal{I}_n & 0\\ 0 & 0\end{pmatrix},\quad K_5 \doteq \begin{pmatrix}\mathcal{I}_n\\ 0\end{pmatrix},\quad V_t^2 \doteq \begin{pmatrix}d_2 - c_2\bar\beta_t\\ 0\end{pmatrix}. \tag{31}$$

One should note that $\bar F_1$ is independent of $\bar\alpha,\bar\beta$.

7. THE FIRST-ORDER HJ PDE

We define notation that will be used throughout the section. Let $-\infty<s<T<\infty$. Recall the dynamics

$$\dot\Pi_t = \bar F_1(\Pi_t),\quad t\in(s,T),\quad \Pi_0 = \bar\Pi, \tag{32}$$

where $\bar\Pi$ is given in (30).

Assume there exists $\Pi\in C^1\big((0,T);\mathbb{R}^{2n\times 2n}\big)\cap C\big([0,T];\mathbb{R}^{2n\times 2n}\big)$ satisfying (32), (30). $(\mathcal{A}.4)$

Let $\mathcal{N}_s \doteq L_2\big((s,T);\mathbb{R}^n\big)$, and let $\tilde\alpha,\tilde\beta\in\mathcal{N}_0$. Let $\pi\in C^1\big((0,T);\mathbb{R}^{2n}\big)\cap C\big([0,T];\mathbb{R}^{2n}\big)$ satisfy

$$\dot\pi_t = \bar F_2(\Pi_t,\pi_t,\tilde\alpha_t,\tilde\beta_t),\quad t\in(0,T),\quad \pi_0 = \bar\pi \doteq 0. \tag{33}$$

Note that we may write (33) as

$$\dot\pi_t = B(t)\pi_t + b(t,\tilde\alpha_t,\tilde\beta_t),\quad k_2 = c_2\mathcal{I}_n + D_{2,2},$$

$$B(t) \doteq \Pi_tK_2 + K_3 = \begin{pmatrix}P_tk_2 + D_{1,2}^T & 0\\ Q_t^Tk_2 & 0\end{pmatrix}, \tag{34}$$

$$b(t,\tilde\alpha_t,\tilde\beta_t) = \begin{pmatrix}P_t(d_2 - c_2\tilde\beta_t) + (d_1 - c_1\tilde\alpha_t)\\ Q_t^T(d_2 - c_2\tilde\beta_t)\end{pmatrix}. \tag{35}$$

Let the state-transition matrix associated to $B(\cdot)$ be denoted by $\Phi(t,s) = \bar\psi_t\bar\psi_s^{-1}$ where we recall fundamental matrix $\bar\psi.$ satisfies $\dot{\bar\psi}_t = B(t)\bar\psi_t$ for all $t\in(s,T)$. One has the following standard result.

Lemma 9. Let $(\tilde\alpha,\tilde\beta)\in\mathcal{N}_0^2$. For all $t\in[0,T]$, $\pi_t = \Phi(t,0)\bar\pi + \int_0^t\Phi(t,r)b(r,\tilde\alpha_r,\tilde\beta_r)\,dr$.

Let $\gamma\in C^1\big((0,T);\mathbb{R}\big)\cap C\big([0,T];\mathbb{R}\big)$ satisfy

$$\dot\gamma_t = \bar F_3(\Pi_t,\pi_t,\tilde\alpha_t,\tilde\beta_t) + \mathcal{C}(t,\pi_t,\tilde\alpha_t,\tilde\beta_t),\quad \gamma_0 = \bar\gamma, \tag{36}$$

where $\mathcal{C}\in C([0,T]\times\mathbb{R}^{2n}\times\mathbb{R}^n\times\mathbb{R}^n)$ will be specified further below. It should be noted that DRE (32) is independent of the control processes, and that existence and uniqueness for (33) and (36) is immediate.

7.1 The first representation

For compactness of notation, we hereafter let y denote $\begin{pmatrix}x\\ z\end{pmatrix}$. For $t\in[0,T]$, let $\bar G(t,x,z;\bar\Pi) \doteq \frac12 y^T\Pi_ty$. Also let

$$\tilde W(t,x;z,\bar\Pi,\bar\pi,\bar\gamma) \doteq \operatorname*{stat}_{(\tilde\alpha,\tilde\beta)\in\mathcal{N}_0}\big\{y^T\pi_t + \gamma_t\big\}. \tag{37}$$

Let $\mathcal{X} \doteq (s,T)\times\mathbb{R}^{2n}$. Let $|c_1|,|c_2|<\infty$ be sufficiently large. For $(t,\pi)\in\mathcal{X}$, let

$$(\alpha^*(t,\pi),\beta^*(t,\pi)) = \operatorname*{argstat}_{(\alpha,\beta)\in\mathbb{R}^{2n}}\big\{y^T\bar F_2(\Pi_t,\pi,\alpha,\beta) + \bar F_3(\Pi_t,\pi,\alpha,\beta) + \mathcal{C}(t,\pi,\alpha,\beta)\big\} \tag{38}$$

where Lemma 4 implies existence and that α^*,β^* are globally Lipschitz in π. Then, for $t\in[0,T)$, let π^*, $(\tilde\alpha_t^*,\tilde\beta_t^*)$ and γ^* be given by

$$\dot\pi_t^* = \bar F_2(\Pi_t,\pi_t^*,\alpha^*(t,\pi_t^*),\beta^*(t,\pi_t^*));\quad \pi_0^* = \bar\pi, \tag{39}$$

$$(\tilde{\alpha}_t^*, \tilde{\beta}_t^*) \doteq (\alpha^*(t, \pi_t^*), \beta^*(t, \pi_t^*)) \quad \forall\, t \in [0, T], \tag{40}$$

$$\gamma_t^* \doteq \bar{\gamma} + \int_0^t \bar{F}_3(\Pi_t, \pi_t^*, \tilde{\alpha}_t^*, \tilde{\beta}_t^*) + \mathcal{C}(t, \pi_t^*, \tilde{\alpha}_t^*, \tilde{\beta}_t^*)\, dt. \tag{41}$$

Lemma 10. Letting $\tilde{\alpha}^*_{[0,t]}, \tilde{\beta}^*_{[0,t]}$ denote the restrictions of $\tilde{\alpha}^*, \tilde{\beta}^*$ to domain $[0, t]$, $(\tilde{\alpha}^*_{[0,t]}, \tilde{\beta}^*_{[0,t]}) \in \operatorname{argstat}_{(\tilde{\alpha},\tilde{\beta})} \{y^T \pi_t + \gamma_t\}$ and $\tilde{W}(t, x; z, \bar{\Pi}, \bar{\pi}, \bar{\gamma}) = \{y^T \pi_t^* + \gamma_t^*\}$ for all $t \in [0, T]$.

For $(t, x, z) \in \mathcal{X}$, let

$$W^f(t, x; z, \bar{\Pi}, \bar{\pi}, \bar{\gamma}) \doteq \bar{G}(t, x, z; \bar{\Pi}) + \tilde{W}(t, x; z, \bar{\Pi}, \bar{\pi}, \bar{\gamma}).$$

Theorem 11. Let $0 \le s \le T < \infty$, $x, z \in \mathbb{R}^n$ and $\bar{\pi} = 0$. Let $|c_1|, |c_2| < \infty$ be sufficiently large. Assume

$$\begin{aligned}
&\operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big\{\pi^T \bar{F}_2(\Pi_t, \pi_t^*, \alpha, \beta) + \bar{F}_3(\Pi_t, \pi_t^*, \alpha, \beta) \\
&\qquad\qquad + \mathcal{C}(t, \pi_t^*, \alpha, \beta)\big\} \\
&= \operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big\{G_0(\alpha, \beta) + \tfrac{c_1}{2}|\alpha - x|^2 \\
&\quad + \tfrac{c_2}{2}|\beta - (P_t x + Q_t z + \rho_t^*)|^2\big\} + \tfrac{1}{2}\operatorname{tr}[A\tilde{W}_{xx}(t, \pi_t^*)]
\end{aligned} \tag{42}$$

for all $t \in (0, T)$. Then

$$W^f(T - s, x; z, \bar{\Pi}, \bar{\pi}, \bar{\gamma}) = \bar{W}(s, x; z, \bar{M}, \bar{\gamma}, T).$$

We also obtain a first-order HJ PDE problem representation for $\tilde{W}$. Fix any $\bar{\pi} \in \mathbb{R}^{2n}$, and let $(\alpha^*, \beta^*), \pi^*, (\tilde{\alpha}^*, \tilde{\beta}^*)$ be given by (38)–(40). Note that Lemma 9 and the globally Lipschitz aspect of (α^*, β^*) as a function of π (from Lemma 4) imply that, for each $t \in [0, T]$, there exists a globally Lipschitz bijection from $\bar{\pi} \in \mathbb{R}^{2n}$ to $\pi_t^* \in \mathbb{R}^{2n}$, say $\pi_t^* = \mathcal{F}_1(\bar{\pi}; t)$, or alternatively, $\bar{\pi} = \mathcal{F}_1^{-1}(\pi_t^*; t)$. For $t \in [0, T]$, let $\tilde{\tilde{W}}(t, \pi_t^*; x, z, \bar{\Pi}, \bar{\gamma}) \doteq \tilde{W}(t, x; z, \bar{\Pi}, \bar{\pi}, \bar{\gamma})$, where $\pi_t^* = \mathcal{F}_1(\bar{\pi}; t)$. Equivalently,

$$\tilde{\tilde{W}}(t, \pi; x, z, \bar{\Pi}, \bar{\gamma}) \doteq \tilde{W}\big(t, x; z, \bar{\Pi}, \mathcal{F}_1^{-1}(\pi; t), \bar{\gamma}\big). \tag{43}$$

Theorem 12. Let $x, z \in \mathbb{R}^n$ and $|c_1|, |c_2| > 0$. Suppose (42) holds for all $t \in (0, T)$. Then, $\tilde{\tilde{W}}(\cdot, \cdot; x, z, \bar{\Pi}, \bar{\gamma})$ satisfies

$$\begin{aligned}
0 = -\tilde{\tilde{W}}_t + &\operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big\{\tilde{\tilde{W}}_\pi \cdot \bar{F}_2(\Pi_t, \pi, \alpha, \beta) + \bar{F}_3(\Pi_t, \pi, \alpha, \beta) \\
&+ \tfrac{1}{2}\operatorname{tr}[A\tilde{\tilde{W}}_{xx}(t, \pi_t^*)]\big\}, \quad (t, \pi) \in \mathcal{X},
\end{aligned} \tag{44}$$

$$\tilde{\tilde{W}}(0, \pi) = y^T \pi + \bar{\gamma}, \quad \pi \in \mathbb{R}^{2n}. \tag{45}$$

Consider the control problem with payoff and value

$$\begin{aligned}
\tilde{\bar{J}}(s, \bar{\pi}, \tilde{\alpha}, \tilde{\beta}; \bar{\Pi}, x, z, T) &\doteq y^T \pi_T + \bar{\gamma} \\
&+ \int_{T-s}^T \bar{F}_3(\Pi_{T-t}, \pi_t, \tilde{\alpha}_t, \tilde{\beta}_t) + \mathcal{C}(t, \pi_t, \tilde{\alpha}_t, \tilde{\beta}_t)\, dt,
\end{aligned} \tag{46}$$

$$\tilde{\bar{W}}(s, \bar{\pi}; \bar{\Pi}, x, z, T) \doteq \operatorname*{stat}_{(\tilde{\alpha},\tilde{\beta})\in[\mathcal{N}_s]^2} \tilde{\bar{J}}(s, \bar{\pi}, \tilde{\alpha}, \tilde{\beta}; \bar{\Pi}, x, z, T), \tag{47}$$

where π satisfies $\dot{\pi} = -\bar{F}_2(\Pi_{T-t}, \pi_t, \tilde{\alpha}_t, \tilde{\beta}_t)$ with $\pi_{T-s} = \bar{\pi}$, and we suppose $\mathcal{C} \in C^3(\mathcal{X})$ with bounded second and third derivatives. Note also that

$$\begin{aligned}
\bar{G}(s, x, z; \bar{\Pi}) + \tilde{\bar{J}}(s, \bar{\pi}, \tilde{\alpha}, \tilde{\beta}; \bar{\Pi}, x, z, T) &= \tfrac{1}{2} y^T \Pi_{T-s} y + y^T \pi_T \\
&+ \gamma_T + \int_{T-s}^T \mathcal{C}(t, \pi_t, \tilde{\alpha}_t, \tilde{\beta}_t)\, dt.
\end{aligned} \tag{48}$$

The HJ PDE problem associated to value $\tilde{\bar{W}}$ is (44)–(45) with $\frac{1}{2}\operatorname{tr}[A\tilde{\tilde{W}}_{xx}(t, \pi_t^*)]$ replaced by $\mathcal{C}(t, \pi_t^*, \tilde{\alpha}_t^*, \tilde{\beta}_t^*)$, that is

$$\begin{aligned}
0 = -W_t + &\operatorname*{stat}_{(\alpha,\beta)\in\mathbb{R}^{2n}} \big\{W_\pi \cdot \bar{F}_2(\Pi_t, \pi, \alpha, \beta) \\
&+ \bar{F}_3(\Pi_t, \pi, \alpha, \beta) + \mathcal{C}(t, \pi, \alpha, \beta)\big\},
\end{aligned} \tag{49}$$

$$W(0, \pi; \bar{\Pi}, x, z, T) = y^T \pi + \bar{\gamma}, \quad \pi \in \mathbb{R}^{2n}. \tag{50}$$

Note that the HJ PDE is first-order and over $\mathcal{X}$.

Theorem 13. Fix x, z. Let $|c_1|, |c_2|$ be sufficiently large, and suppose there exists a solution to (32) on $[s, T]$. Suppose $W(\cdot, \cdot; \bar{\Pi}, x, z, T) \in C^{1,4}(\mathcal{X}) \cap C_p(\bar{\mathcal{X}})$ satisfies (49)–(50). Suppose $W_{\pi\pi}$ and the second derivatives of $\mathcal{C}$ with respect to α, β are uniformly bounded. Then, $W(\cdot, \cdot; \bar{\Pi}, x, z, T) = \tilde{\bar{W}}(\cdot, \cdot; \bar{\Pi}, x, z, T)$ for all $(t, \pi) \in \bar{\mathcal{X}})$, and there exist unique feedback controls $\alpha^*(t, \pi), \beta^*(t, \pi)$ satisfying (49) such that there exists a unique solution to (33), and that yield the stationary value.

7.2 Obtaining the correction term

Consider (49)–(50). Reversing time, differentiating twice with respect to x, and taking the linear combination corresponding to $\frac{1}{2}\sum_{j=1}^n [AW_{xx}]_{j,j}$ to obtain the HJ PDE for $\frac{1}{2}\operatorname{tr}[A\tilde{\tilde{W}}_{xx}(t, \pi_t^*)]$, yields

$$\begin{aligned}
0 = J_t^{n+1} &+ J_\pi^{n+1} \cdot \bar{F}_2(\Pi_t, \pi, \alpha^*(t, \pi), \beta^*(t, \pi)) \\
&+ \tfrac{1}{2}\sum_{j,k=1}^n \sum_{\ell,m=1}^{2n} A_{j,k} J_{\pi_\ell}^k \big[M_t^T \hat{C}[\hat{C} + G_0''(\alpha^*(t, \pi), \beta^*(t, \pi)) \\
&+ \mathcal{C}''(t, \pi, \alpha^*(t, \pi), \beta^*(t, \pi))]^{-1} \hat{C} M_t\big]_{\ell,m} J_{\pi_m}^j,
\end{aligned} \tag{51}$$

$$J^{n+1}(T, \pi; \alpha^*, \beta^*, \bar{\Pi}, x, z, T) = 0, \quad \pi \in \mathbb{R}^{2n}. \tag{52}$$

where $J_{\pi_k}^j$ generically corresponds to W_{x_j,π_k}, and we note that one similarly generates, but do not include, the HJ PDEs for W_{x_j} for $j \in]1, n[$.

Using the method of characteristics, one eventually obtains

$$J^{n+1}(t; \tilde{\alpha}^*, \tilde{\beta}^*) = \int_t^T G_1(r, \tilde{\alpha}_r^*, \tilde{\beta}_r^*)\, dr, \tag{53}$$

$$\begin{aligned}
G_1(t, \tilde{\alpha}_t^*, \tilde{\beta}_t^*) = \tfrac{1}{2}\operatorname{tr}\Big\{&A\big[(\Phi^-(T, t))^T M_t^T \hat{C}[\hat{C} + G_0'' + \mathcal{C}'']^{-1} \\
&\hat{C} M_t \Phi^-(T, t)\big]\Big\},
\end{aligned}$$

where $\Phi^-(\cdot, \cdot)$ denotes the state-transition matrix associated to $-B^T(\cdot)$.

Lemma 14. Fix $\bar{\Pi}, x, z, T$. Suppose $\tilde{W}(t, \pi; \bar{\Pi}, x, z, T)$ is twice differentiable in x for all $(t, \pi, x) \in \bar{\mathcal{X}} \times \mathbb{R}^n$. Let $J^{n+1}(\cdot, \cdot; \tilde{\alpha}^*, \tilde{\beta}^*)$ be the solution of HJ PDE problem (51)–(52) with $(\tilde{\alpha}, \tilde{\beta}) = (\tilde{\alpha}^*, \tilde{\beta}^*)$. Then,

$$J^{n+1}(t; \tilde{\alpha}^*, \tilde{\beta}^*) = \tfrac{1}{2}\operatorname{tr}[A\tilde{W}_{xx}(t, \pi); \bar{\Pi}, x, z, T)]\ \forall (t, \pi) \in \bar{\mathcal{X}}.$$

By (53) and Lemma 14, we see that in order for the assumption of Theorems 11 and 12 to hold, we must have

$$\mathcal{C}(t, \pi_t^*, \tilde{\alpha}_t^*, \tilde{\beta}_t^*) = \int_t^T G_1(r, \tilde{\alpha}_r^*, \tilde{\beta}_r^*)\, dr \quad \forall\, t \in [s, T]. \tag{54}$$

Differentiating this, we have the requirement that

$$\begin{aligned}
0 &= \mathcal{C}_t(t, \pi_t^*, \tilde{\alpha}_t^*, \tilde{\beta}_t^*) + G_1(t, \tilde{\alpha}_t^*, \tilde{\beta}_t^*) \\
&= \mathcal{C}_t + \tfrac{1}{2}\operatorname{tr}\Big\{\Lambda_t A \Lambda_t^T \big[\hat{C} + [G_0]_{(\alpha,\beta),(\alpha,\beta)} + \mathcal{C}_{(\alpha,\beta),(\alpha,\beta)}\big]^{-1}\Big\}.
\end{aligned}$$

Assuming existence and using a standard expansion, this is equivalently given as

$$0 = \mathcal{C}_t + \tfrac{1}{2}\operatorname{tr}\Big\{(\hat{C} + [G_0]_{(\alpha,\beta),(\alpha,\beta)})^{-1} \Lambda_t A \Lambda_t^T$$

$$\cdot \sum_{k=0}^{\infty}[-(\hat{C}+[G_0]_{(\alpha,\beta),(\alpha,\beta)})^{-1}\mathcal{C}_{(\alpha,\beta),(\alpha,\beta)}]^k\Big\}.$$

Employing only the first term in the series, and integrating, generates the approximation

$$\mathcal{C}(t,\pi_t^*,\tilde{\alpha}_t^*,\tilde{\beta}_t^*) = \int_t^T \tfrac{1}{2}\,\mathrm{tr}\left[A\Lambda_t^T(\hat{C}+[G_0]_{(\alpha,\beta),(\alpha,\beta)})^{-1}\Lambda_t\right]dt.$$

Employing the first two terms in the series yields an approximation in the form of a linear, second-order PDE.

A closed-form solution case In some cases, one can explicitly compute $\mathcal{C}$, and this is true in the class of problems with linear dynamics and nonlinear payoff of a gravitational/Coulomb potential form. Consider the second-order HJ PDE

$$0 = W_t - \tfrac{1}{2m}|W_x|^2 - V(x) + \tfrac{\bar{k}}{2}\,\mathrm{tr}[W_{xx}], \quad -V(x) \doteq \tfrac{\mu}{|x|}.$$

There are multiple useful stat-dual forms for such potentials. In addition to stat-dual forms such as those used above, there is a scalar-α form given by $-V(x) = (3/2)^{3/2}\mu\,\mathrm{stat}_{\alpha>0}[\alpha - \frac{\alpha^3|x|}{2}]$ McEneaney and Dower (2019, 2015). Here, we use a simple stat-dual analogous to the Legendre-Fenchel transform. Specifically, one has

$$-V(x) = \operatorname*{stat}_{\alpha\in\mathbb{R}^n}[G_0(\alpha)+\alpha^T x],$$
$$G_0(\alpha) = \operatorname*{stat}_{x\in\mathbb{R}^n}[-V(x)-\alpha^T x],$$

yielding $G_0(\alpha) = 2\sqrt{c|\alpha|}$, and in particular,

$$\alpha^* = \alpha^*(x) \doteq \operatorname*{argstat}_{\alpha\in\mathbb{R}^n}[G_0(\alpha)+\alpha^T x] = \tfrac{-\mu}{|x|^3}x. \quad (55)$$

One obtains

$$0 = J_t^{n+1} + J_\pi^{n+1}\cdot \bar{F}_2(\Pi_t,\pi,\alpha^*(t,\pi),\beta^*(t,\pi))$$
$$+\,\bar{k}\sum_{j,k=1}^{n} J^j_{\pi_k}\frac{\mu}{|x|^5}\left[3|x|^2\mathcal{I}_n - xx^T\right]_{j,k} \quad (t,\pi)\in\mathcal{X}, \quad (56)$$

still with terminal condition (52). Also, suppose we take $\bar{M} = \bar{m}\mathcal{I}_n$ yielding $\psi(x;z,\bar{M},\bar{\gamma}) = \frac{\bar{m}}{2}|x-z|^2$, one has $Q_t = -P_t = \frac{\lambda_t}{2}\mathcal{I}_n$ for all t where $\dot{\lambda} = \frac{-1}{m}\lambda^2$, $\lambda_T = \bar{m}$. Employing apply the characteristics approach, one eventually finds

$$\mathcal{C}(t,\pi_t^*,\tilde{\alpha}_t^*,\tilde{\beta}_t^*) = \tfrac{1}{2}\,\mathrm{tr}[A\tilde{\tilde{W}}_{xx}(t,\pi_t^*)] = J^{n+1}(t;\tilde{\alpha}^*,\tilde{\beta}^*)$$
$$= \frac{\bar{k}\mu}{|x|^5}\,\mathrm{tr}[3|x|^2\mathcal{I}_n - xx^T]\int_t^T \exp[\tfrac{-1}{m}\int_t^T \lambda_r\,dr]\,dt. \quad (57)$$

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