Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Shifting H_∞ Linear Parameter Varying State-Feedback Controllers Subject to Time-Varying Input Saturations ⋆

Adrián Ruiz *, **Damiano Rotondo** **,*** and **Bernardo Morcego** *

* *Research Center for Supervision, Safety and Automatic Control, UPC, Rambla Sant Nebridi, 22, 08222 Terrassa, Spain (e-mail: adrian.ruiz.royo@upc.edu)*
** *Department of Electrical and Computer Engineering (IDE), University of Stavanger, 4009, Norway*
*** *Institut de Robòtica i Informàtica Industrial, CSIC-UPC, Llorens i Artigas 4-6, 08028 Barcelona, Spain*

Abstract: This paper establishes a methodology based on linear matrix inequalities (LMIs) to design a shifting H_∞ linear parameter varying (LPV) state-feedback controller for systems affected by time-varying input saturations. By means of the shifting paradigm, the instantaneous saturation values are linked to a scheduling parameter vector. Then, the disturbance rejection is dealt with the quadratic boundedness concept and the shifting H_∞ methodology. The design conditions are obtained within the LPV framework using ellipsoidal invariant sets, thus obtaining an LMI-based feasibility problem that can be solved via available solvers. Finally, the main characteristics of the proposed approach are validated by means of an illustrative example.

Keywords: Linear parameter varying (LPV), Saturation, Disturbance rejection, Linear matrix inequalities (LMIs)

1. INTRODUCTION

In Ruiz et al. (2019), a methodology for designing state-feedback controllers taking into account time-varying input saturations was developed. In addition to saturation, real-world systems can be also affected by unknown disturbances. These disturbances may contribute to saturate the control inputs, emphasize the actuator degradation and even make the system unstable. This paper is about disturbance rejection under time-varying saturation limits.

In order to deal with disturbances, quadratic boundedness (QB) received attention in the literature (Brockman and Corless, 1998). The QB approach consists in enforcing all the state trajectories to converge inside an ellipsoidal region described by a quadratic Lyapunov function, in spite of the disturbances. On the other hand, the H_∞ methodology is a well-known approach to consider disturbances, which allows to design a controller that minimizes the energy transmission from the input to the output. For example, Boyd et al. (1994) proposed a linear matrix inequality (LMI)-based problem in order to obtain a controller that satisfies an L_2 gain constraint for a constant specification. Furthermore, the H_∞ methodology was extended to deal with linear parameter varying (LPV) systems in Apkarian et al. (1995).

The works addressing the saturation phenomenon in combination with disturbance rejection assume that the saturation limits are constant. For instance, Köse and Jabbari (2003) used the H_∞ methodology and the QB concept through a parameter-dependent Lyapunov function to design two gain-scheduled controllers. These controllers are subject to saturations with constant limits and provide a guaranteed L_2 gain. Moreover, the controllers adapt their gains based on the distance from the origin, providing high-gains when the states are close to the origin, thus increasing the system's performance. Sajjadi-Kia and Jabbari (2013) extended the previous work by adding an anti-windup to handle the saturation under worst-case disturbance. In Ping et al. (2017), the assumption of the input saturation limits being constant in time is maintained with the objective of approximating the region of attraction by means of off-line optimization algorithms. Also, this approach allows to design a saturated dynamic output-feedback controller for an LPV system with bounded disturbances using the QB concept. On the other hand, Ding (2009) presented an output-feedback controller based on

⋆ This work has been partially funded by the Spanish State Research Agency (AEI) and the European Regional Development Fund (ERFD) through the project SCAV (ref. MINECO DPI2017-88403-R) and the SMART project (ref. nº EFA 153/16 Interreg Cooperation Program POCTEFA 2014-2020). This work has also been partially funded by AGAUR of Generalitat de Catalunya through the Advanced Control Systems (SAC) group grant (2017 SGR 482) and by the Spanish State Research Agency through the Maria de Maeztu Seal of Excellence to IRI (MDM-2016-0656). A. Ruiz is also supported by Formació Personal Investigador a la Universitat Politècnica de Catalunya under an FPI-UPC 2018 grant (ref. 20 FPI-UPC 2018). D. Rotondo has been also supported by AEI through the Juan de la Cierva - Formacion grant (FJCI-2016-29019) and by the University of Stavanger through the project IN-12267.

a Takagi-Sugeno formulation, with input constraints and bounded noise.

However, from a practical viewpoint, it makes sense to consider the case in which the saturation limits are time-varying due, e.g., to the degradation of the actuators over time. Obviously, such degradation affects the disturbance rejection performance provided by the controller. The main contribution of this work lies in proposing an LMI-based methodology to design a shifting H_∞ and quadratically bounded LPV state-feedback controller, which ensures rejection against bounded, unknown exogenous disturbances while maintaining the control inputs inside the actuator linearity zone. To achieve this objective, the shifting paradigm is applied (Rotondo et al., 2015), following some ideas developed recently in Ruiz et al. (2019). The overall design approach is developed using a constant quadratic Lyapunov matrix, thus introducing some conservativeness. Nevertheless, such restriction can be relaxed by considering results available in the literature which use parameter-dependent Lyapunov functions, at the cost of inceasing the computational burden.

This paper is structured as follows. In Section 2, the problem statement is introduced. In Section 3, the LMI-based procedure for controller design is presented. Section 4 presents an illustrative example with simulation results. Finally, Section 5 summarizes the main conclusions and discusses possible future work.

2. PROBLEM STATEMENT

Let us consider the following continuous-time LPV system

$$\begin{cases} \dot{x}(t) &= A(\theta(t))x(t) + B_w w(t) + B_u sat(u(t)) \\ z(t) &= C_z x(t) \end{cases}, \quad (1)$$

where: $x(t) \in \mathbb{R}^{n_x}$ is the state vector; $w(t) \in \mathbb{R}^{n_w}$ is the unknown external disturbance, such that $\|w(t)\|_2 \leq 1$; $u(t) \in \mathbb{R}^{n_u}$ is the control input; and $z(t) \in \mathbb{R}^{n_z}$ denotes the performance output. The matrices B_w, B_u and C_z are the disturbance, input and performance output matrices, respectively [1]. On the other hand, the parameter-dependent state matrix $A(\theta(t))$ depends on a scheduling parameter vector $\theta(t) \in \Theta \subset \mathbb{R}^{n_\theta}$, and it can be represented as the convex hull of a finite set of N vertex matrices: $A(\theta) \in \mathcal{C}o\{A_i, i = 1, \ldots, N\}$.

The polytopic representation of system (1) is used throughout this paper, which means that the following holds for matrices A_i

$$\mathcal{C}o\{A_i\} := \left\{ \sum_{i=1}^{N} \mu_i(\theta(t))A_i, \mu_i(\theta(t)) \geq 0, \sum_{i=1}^{N} \mu_i(\theta(t)) = 1 \right\}. \quad (2)$$

Note that the input $u(t)$ in (1) is affected by symmetric saturations with time-varying limits

$$sat(u_h(t)) = \begin{cases} sign(u_h(t))\sigma_h(t) & \text{if } \mid u_h(t) \mid > \sigma_h(t) \\ u_h(t) & \text{if } \mid u_h(t) \mid \leq \sigma_h(t) \end{cases}, \quad (3)$$

where $h = 1, \ldots, n_u$, the symbols $>$ and $\leq$ denote an element-wise comparison and $\sigma_h(t) \in \mathbb{R}_+^{n_u}$ is the instantaneous saturation limit value.

Following the shifting paradigm (Rotondo et al., 2015), the time-varying saturation is described in polytopic form by introducing a new scheduling vector $\phi(t) \in \Phi \subset \mathbb{R}^{n_u}$, linked to the instantaneous saturation limits $\sigma(t) \in [\underline{\sigma}, \overline{\sigma}]$ by (Ruiz et al., 2019)

$$\phi_h(t) = \frac{\overline{\sigma}_h{}^2 - \sigma_h(t)^2}{\overline{\sigma}_h{}^2 - \underline{\sigma}_h{}^2}. \quad (4)$$

The time dependency of x, w, u, θ and ϕ is dropped from now on and it will only be made explicit when necessary.

2.1 Background

Let us introduce the definitions of quadratic boundedness (Brockman and Corless, 1998) and the shifting H_∞ performance in order to address the unknown external disturbances:

Definition 1. (Quadratic boundedness). Given $w(t) \in \Omega$ with $\Omega \subset \mathbb{R}^{n_w}$ closed and bounded, the system $\dot{x}(t) = Ax(t) + B_w w(t)$ is quadratically bounded with Lyapunov symmetric matrix $P \succ 0$, if $x^T P x > 1 \implies x^T P(Ax + B_w w) < 0, \forall w(t) \in \Omega$.

Definition 2. (Shifting H_∞ performance). An LPV system is said to achieve shifting H_∞ performance if the L_2 gain of the input/output map is bounded by $\gamma(\phi(t))$ $\forall \phi(t) \in \Phi \subset \mathbb{R}^{n_u}$

$$\left\| \gamma(\phi(t))^{-\frac{1}{2}} z(t)^T z(t) \right\|_2 < \left\| \gamma(\phi(t))^{\frac{1}{2}} w(t)^T w(t) \right\|_2. \quad (5)$$

3. CONTROLLER DESIGN PROCEDURE

Let us consider the following control law for (1)

$$u(t) = K(\theta(t), \phi(t))x(t) = \sum_{i=1}^{N} \mu_i(\theta(t)) \sum_{j=1}^{M} \eta_j(\phi(t))K_{ij}x(t), \quad (6)$$

where $K(\theta(t), \phi(t)) \in \mathbb{R}^{n_u \times n_x}$ is a parameter-dependent gain and $K_{ij} \in \mathbb{R}^{n_u \times n_x}$ denotes the polytopic vertex gain matrix for the pair (i, j).

Then, by introducing the control law (6) in the system's equation (1), the polytopic expression of the system under control is obtained

$$\begin{cases} \dot{x}_{cl}(t) &= \sum_{i=1}^{N} \mu_i(\theta(t)) \sum_{j=1}^{M} \eta_j(\phi(t))(A_i + B_u K_{ij})x(t) + B_w w(t) \\ z_{cl}(t) &= C_z x(t) \end{cases}. \quad (7)$$

3.1 Quadratic Boundedness

Theorem 1 introduces a set of LMIs that guarantee quadratic boundedness of (7), thus ensuring that all the closed-loop system trajectories evolve towards an ellipsoid $\mathcal{V}$ defined by the unit level curve of the Lyapunov function $V(x) = x^T P x$, in spite of the external disturbance $w(t)$.

[1] The results obtained throughout this paper can be extended to the case where $B_u(\theta(t))$, $B_w(\theta(t))$ and $C_z(\theta(t))$ are parameter-dependent matrices. This extension is possible by applying Polya's theorems for definiteness of double sums, although at the cost of increasing the overall complexity of the solution. The interested reader is referred to Sala and Arino (2007).

Theorem 1. Consider the continuous-time closed-loop LPV system (7) with $\|w(t)\|_2 \leq 1$ and, assume that there exists a symmetric matrix $Q \succ 0$, a constant parameter $\alpha \in \mathbb{R}_+$ and matrices Γ_{ij} such that

$$Q \succ 0, \tag{8}$$

$$\begin{bmatrix} A_iQ + B_u\Gamma_{ij} + QA_i^T + \Gamma_{ij}^T B_u^T + \alpha Q & B_w \\ B_w^T & -\alpha I \end{bmatrix} \preceq 0. \tag{9}$$

If the vertex gains of the LPV controller (6) are calculated as $K_{ij} = \Gamma_{ij}Q^{-1}$, then the closed-loop system is quadratically bounded with Q as a Lyapunov matrix.

Proof. This proof follows the reasoning in Brockman and Corless (1998). The LMIs

$$\begin{bmatrix} PA_i + A_i^T P + \alpha P & PB_w \\ B_w^T P & -\alpha I \end{bmatrix} \preceq 0 \tag{10}$$

can be applied to each vertex for assessing whether an autonomous polytopic LPV system described by matrix A_i is quadratically bounded. Then, by pre and post multiplying (10) by $diag(Q, I)$, where $Q = P^{-1}$ and replacing A_i by the closed-loop vertex matrices of (7), the following bilinear matrix inequalities are obtained for design

$$\begin{bmatrix} A_iQ + B_uK_{ij}Q + QA_i^T + QK_{ij}^T B_u^T + Q\alpha & B_w \\ B_w^T & -\alpha I \end{bmatrix} \preceq 0, \tag{11}$$

which can be converted into LMIs (note that α is fixed beforehand) by means of the change of variable $\Gamma_{ij} = K_{ij}Q$, thus obtaining (9). ■

3.2 Shifting disturbance rejection

By means of the shifting H_∞ performance, the controller will modify online its performance whenever the input saturation limits undergo variations. Consequently, the controller will reject larger disturbances when larger control actions are available and, conversely, smaller disturbances when the saturation limits are smaller.

Theorem 2 provides the conditions for the vertex gains K_{ij} to ensure the robustness of (7) against external disturbances and time-varying saturation limits.

Theorem 2. Consider the continuous-time closed-loop LPV system (7), and assume that there exisits a symmetric matrix $Q \succ 0$, a set of M values $\gamma_j > 0$ and matrices Γ_{ij} such that the following set of LMIs is feasible for $i = 1, \ldots, N$ and $j = 1, \ldots, M$

$$Q \succ 0, \tag{12}$$

$$\begin{bmatrix} A_iQ + B_u\Gamma_{ij} + QA_i^T + \Gamma_{ij}^T B_u^T & B_w & QC_z^T \\ B_w^T & -\gamma_j I & 0 \\ C_zQ & 0 & -\gamma_j I \end{bmatrix} \prec 0. \tag{13}$$

If the vertex gains of the LPV controller (6) are calculated as $K_{ij} = \Gamma_{ij}Q^{-1}$, then the closed-loop system is robust against external disturbances with a guaranteed shifting H_∞ gain performance

$$\gamma(\phi) = \sum_{j=1}^{M} \eta_j(\phi)\gamma_j. \tag{14}$$

Proof. Let us consider the following system

$$\begin{cases} \dot{x}(t) &= A(\theta)x(t) + B_w w(t) \\ z(t) &= C_z x(t) \end{cases}. \tag{15}$$

Assume that the Lyapunov function $V(x) = x^T Px$, with $P \succ 0$, is such that for all $t > 0$,

$$\dot{V}(x) + \gamma(\phi(t))^{-1} z(t)^T z(t) - \gamma(\phi(t)) w(t)^T w(t) < 0, \tag{16}$$

where $\gamma(\phi) > 0$ and

$$\dot{V}(x) = x^T(A(\theta)^T P + PA(\theta))x + 2x^T PB_w w \tag{17}$$

Then, the L_2 gain is bounded in the interval described by $\gamma(\phi)$ for all x and w. To show this, let us integrate (16) from t=0 to t=∞ with $x(0) = 0$, thus obtaining

$$\begin{aligned} V(x(\infty)) + \int_0^\infty \gamma(\phi(t))^{-1} z(t)^T z(t)\,dt \\ - \int_0^\infty \gamma(\phi(t)) w(t)^T w(t)\,dt < 0. \end{aligned} \tag{18}$$

$V(x(\infty)) \geq 0$ by construction, implying that

$$\int_0^\infty \gamma(\phi(t))^{-1} z(t)^T z(t)\,dt < \int_0^\infty \gamma(\phi(t)) w(t)^T w(t)\,dt, \tag{19}$$

thus obtaining a similar expression to the one in Köse and Jabbari (2003). After that, by recalling the definition of L_2 *norm* (Boyd et al., 1994), (5) is obtained.

In order to obtain (13) by means of appropriate manipulations of (16), the inequality

$$\begin{bmatrix} A(\theta)^T P + PA(\theta) + \gamma(\phi)^{-1} C_z^T C_z & PB_w \\ B_w^T P & -\gamma(\phi) I \end{bmatrix} \prec 0 \tag{20}$$

is obtained. Then, applying the Schur complement to (20) and after some calculations, the following LMI

$$\begin{bmatrix} A(\theta)Q + QA(\theta)^T & B_w & QC_z^T \\ B_w^T & -\gamma(\phi) I & 0 \\ C_zQ & 0 & -\gamma(\phi) I \end{bmatrix} \prec 0. \tag{21}$$

is derived. Finally, the LMI (21) is replaced by the closed-loop vertex matrices of (7) obtaining (13). ■

Note that (19) can be compared with the traditional H_∞ performance with a constant γ, since replacing $\gamma(\phi)$ by γ, the following L_2 gain is obtained

$$\sup_{\|w(t)\|_2 \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma, \tag{22}$$

demonstrating that the shifting H_∞ performance defined in *Definition 2* is an extension of the above concept.

Let us define $\eta_j(\phi(t))$ in order to calculate the polytopic weights of (14) for any number of vertices as follows

$$\eta_j(\phi(t)) = \prod_{h=1}^{n_u} \gamma_{jh}(\phi_h(t)), \tag{23}$$

where

$$\gamma_{jh}(\phi_h(t)) = \begin{cases} \phi_h(t) & \text{if} \quad mod(j, 2^h) \in \{1, \ldots, 2^{h-1}\} \\ 1 - \phi_h(t) & \text{else} \end{cases}. \tag{24}$$

Furthermore, with the goal of casting an optimization problem which involves fewer parameters than M, let us define each value γ_j as follows

$$\gamma_j = \frac{\overline{\gamma}\mathcal{C}_j + \underline{\gamma}(n_u - \mathcal{C}_j)}{n_u}, \tag{25}$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are the lower and upper limits of $\gamma(\phi(t))$, respectively. $\mathcal{C}_j = |\{h \in \{1, \ldots, n_u\} : mod(j, 2^h) \in \{1, \ldots, 2^{h-1}\}\}|$ and $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$.

Remark 1. Note that the interval limits $[\underline{\gamma}, \overline{\gamma}]$ in (25) are introduced as symbolic decision variables. These variables can be obtained by minimizing the following

$$min\ J = \frac{1}{2}(\underline{\gamma} + \overline{\gamma}) \tag{26}$$

under the set of LMIs showed in Theorem 2, thus obtaining a gain-scheduled controller with an optimal shifting H_∞ gain performance $\gamma(\phi)$.

3.3 Overall controller design

The design of the shifting H_∞ LPV state-feedback controller that takes into account (3) and quadratic boundedness is ensured as follows. Following Ruiz et al. (2019), let us establish three ellipsoidal regions in the state space domain that satisfy

$$\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{U}(\phi(t)) \subseteq \mathcal{L}, \tag{27}$$

where $\mathcal{E}$ contains the set of allowed initial conditions of the system

$$\mathcal{E} = \{x \in \mathbb{R}^{n_x} : x^T R x \leq 1\}, \tag{28}$$

$\mathcal{V}$ corresponds to the unit level curve defined by $V(x)$ as

$$\mathcal{V} = \{x \in \mathbb{R}^{n_x} : x^T P x \leq 1\}, \tag{29}$$

and the control input region $\mathcal{U}(\phi)$

$$\mathcal{U}(\phi) = \{u \in \mathbb{R}^{n_u} : u^T S(\phi) u \leq 1\}, \tag{30}$$

where $S(\phi)$ contains the orientations and magnitudes of the control input ellipsoids in the following way

$$S(\phi) = diag\left(\sigma_1(\phi_1)^2,\ \ldots,\ \sigma_h(\phi_h)^2\right)^{-1}. \tag{31}$$

$\sigma_h(\phi_h)^2$, with $h = 1, \ldots, n_u$, can be obtained from (4).

Then, replacing the control law (6) in (30), the region $\mathcal{U}(\phi)$ can be mapped onto a state-space region $\mathcal{U}(\theta, \phi)$, which corresponds to an ellipsoidal subset of $\mathcal{L}$, where the inputs are not saturated

$$\mathcal{U}(\theta, \phi) = \{x \in \mathbb{R}^{n_x} : x^T K(\theta, \phi)^T S(\phi) K(\theta, \phi) x \leq 1\}. \tag{32}$$

Note that $\mathcal{E} \subseteq \mathcal{V}$ allows guaranteeing that $x(0) \in \mathcal{E} \implies x(t) \in \mathcal{V}, \forall t$ as long as the system works in the linear region described by (3). Also, by means of the inclusion $\mathcal{V} \subseteq \mathcal{U}(\theta, \phi)$ any state trajectory will converge inside $\mathcal{V}$, in spite of the external disturbance $w(t)$ thanks to the QB and, maintaining $u(t)$ in the region of linearity of the actuators, such that no saturation occurs.

Finally, combining Theorems 1 and 2 with the abovementioned ellipsoidal regions, Corollary 1 provides the conditions to obtain the vertex gains of the controller K_{ij} that ensure the quadratic boundedness and the rejection of unknown exogenous disturbances, as in Sections 3.1 and 3.2. Moreover, this controller is able to adapt its rejection performance according to the instantaneous saturation limit values, considering that the input signal $u \in U(\phi)$.

Corollary 1. Consider the closed-loop LPV system (7), a constant parameter $\alpha \in \mathbb{R}_+$, the regions $\mathcal{E}$ and $\mathcal{U}(\theta, \phi)$ of the state space described by (28) and (32), respectively, with given matrices $R \succ 0$ and $S(\phi) \succ 0$ and the function $\gamma(\phi) \in \mathbb{R}_+$ that varies within the interval $[\underline{\gamma}, \overline{\gamma}]$. Assume that both $\gamma(\phi)$ and $S(\phi)^{-1}$ can be expressed in polytopic form as in (14) and (33), respectively.

$$S(\phi)^{-1} = \sum_{j=1}^{M} \eta_j(\phi) S_j^{-1} \tag{33}$$

Let a symmetric matrix Q and matrices Γ_{ij} be solution of the minimization problem (26), such that

$$Q \succ 0, \tag{34}$$

$$\begin{bmatrix} Q & I \\ I^T & R \end{bmatrix} \succeq 0, \tag{35}$$

$$\begin{bmatrix} S_j^{-1} & \Gamma_{ij} \\ \Gamma_{ij}^T & Q \end{bmatrix} \succeq 0, \tag{36}$$

and the LMIs (9) and (13) are feasible. If the vertex gains of the LPV state-feedback controller (6) are calculated as $K_{ij} = \Gamma_{ij} Q^{-1}$, then the closed-loop system (7) is quadratically bounded with Lyapunov matrix Q and robust against external disturbances with shifting H_∞ performance $\gamma(\phi(t))$. Moreover, if $x(0) \in \mathcal{E}$, then $x(t) \in \mathcal{V}$ and the control input $u(t)$ is such that $u(t) \in \mathcal{U}(\phi(t))$.

Proof. Part of this proof is developed in Ruiz et al. (2019), where the LMIs (35) and (36) are obtained through the inclusions $\mathcal{E} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \mathcal{U}(\theta, \phi)$, respectively. Similarly, the LMIs (9) and (13) are proven in the proof of Theorems 1 and 2 of this paper, respectively. ∎

4. ILLUSTRATIVE EXAMPLE

Consider a numerical continuous-time LPV system with the following matrices

$$\begin{aligned} A(\theta(t)) &= \begin{bmatrix} 4.25 + 3.5\theta(t) & 3.8971 \\ 3.8971 & 8.75 - 5.5\theta(t) \end{bmatrix}, \\ B_u &= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_z = [1\ 0], \end{aligned} \tag{37}$$

where $\theta(t) \in [0, 1]$. Note that the system is open-loop unstable for every frozen value of $\theta(t)$.

Let us introduce the additional scheduling parameter vector $\phi(t)$, which is linked to the time-varying input saturation limits of $u_1(t)$ and $u_2(t)$ as follows

$$\phi_1(t) = \frac{\overline{\sigma}_1{}^2 - \sigma_1(t)^2}{\overline{\sigma}_1{}^2 - \underline{\sigma}_1{}^2} \text{ and } \phi_2(t) = \frac{\overline{\sigma}_2{}^2 - \sigma_2(t)^2}{\overline{\sigma}_2{}^2 - \underline{\sigma}_2{}^2}. \tag{38}$$

$\phi_1(t)$ and $\phi_2(t)$ vary within the interval $[0, 1]$ and the saturation limits $\sigma_1(t), \sigma_2(t) \in [5, 10]$ for the inputs $u_1(t)$ and $u_2(t)$, respectively.

The controller is obtained applying Corollary 1. In this case, the resulting 26 LMIs are as follows

$$\begin{cases} Q \succ 0 \\ \begin{bmatrix} A_i Q + B_u \Gamma_{ij} + Q A_i^T + \Gamma_{ij}^T B_u^T + \alpha Q & B_w \\ B_w^T & -\alpha I \end{bmatrix} \preceq 0 \\ \begin{bmatrix} A_i Q + B_u \Gamma_{ij} + Q A_i^T + \Gamma_{ij}^T B_u^T & B_w & Q C_z^T \\ B_w^T & -\gamma_j I & 0 \\ C_z Q & 0 & -\gamma_j I \end{bmatrix} \prec 0 \\ \begin{bmatrix} Q\ I;\ I^T\ R \end{bmatrix} \succeq 0 \\ \begin{bmatrix} S_j^{-1}\ \Gamma_{ij};\ \Gamma_{ij}^T\ Q \end{bmatrix} \succeq 0 \end{cases}, \tag{39}$$

where $i = 1, 2,\ j = 1, 2, 3, 4$. The parameter α and the matrix R have been chosen as

$$\alpha = 1.2975,\ R = diag\,(100,\ 100)\,. \tag{40}$$

Which means that the expected initial conditions for the system lie in a 0.1 radius circle centred in the origin of the state-space. The polytopical expression of (14) for $M = 4$ is

$$\begin{aligned}\gamma(\phi) = \phi_1\phi_2\gamma_1 + (1-\phi_1)\phi_2\gamma_2 \\ + \phi_1(1-\phi_2)\gamma_3 + (1-\phi_1)(1-\phi_2)\gamma_4\end{aligned}, \tag{41}$$

and $\gamma_1, \ldots, \gamma_4$ are obtained by means of (25) as follows

$$\gamma_1 = \overline{\gamma},\ \gamma_2 = \gamma_3 = \frac{\overline{\gamma} + \underline{\gamma}}{2}, \gamma_4 = \underline{\gamma}, \tag{42}$$

which are introduced in the LMI-based problem as symbolic decision variables.

Finally, the vertex matrices of S_j are as follows

$$\begin{aligned} S_1 = diag\left(10^2,\ 10^2\right)^{-1}, \quad S_2 = diag\left(5^2,\ 10^2\right)^{-1}, \\ S_3 = diag\left(10^2,\ 5^2\right)^{-1}, \quad S_4 = diag\left(5^2,\ 5^2\right)^{-1}, \end{aligned} \tag{43}$$

taking into account the extreme values of $\sigma_1(t)$ and $\sigma_2(t)$.

The solution to (39), minimizing (26), was found using SeDuMi solver (Sturm, 1999) and the YALMIP toolbox (Löfberg, 2004). The resulting interval of $\gamma(\phi)$ is $[0.2410, 1.0623]$. Accordingly, the eight controller vertex gains are calculated as $K_{ij} = \Gamma_{ij}Q^{-1}$.

4.1 Scenario I: constant saturation limits

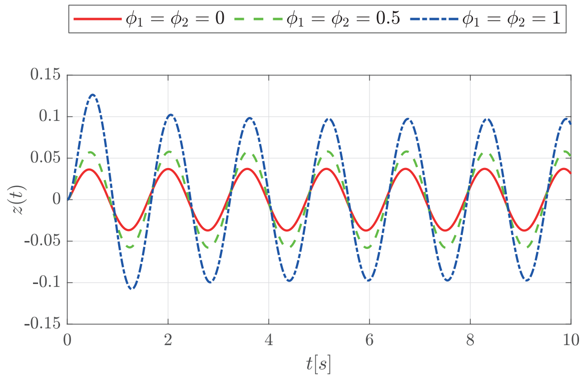
The controller is tested in an scenario with external disturbance, $w(t) = \sin(4t)$, subject to three different saturation limit values kept constant throughout the simulation, $\sigma_1 = \sigma_2 = \{10, 7.5, 5\}$ that lead to $\phi_1 = \phi_2 = \{0, 0.5, 1\}$ by means of (38). The controlled system is simulated with an initial state $x(0) = [0, 0]^T$ and $\theta(t) = 1 - e^{-t}$.

Fig. 1 (a) shows the disturbance rejection for the three values of σ_1 and σ_2. Note the controller rejects the disturbance the most when $\phi_1 = \phi_2 = 0$, which corresponds to the largest saturation limit and to the obtained value $\underline{\gamma} = 0.2410$. Conversely, the output is more affected by the disturbance when $\phi_1 = \phi_2 = 1$, showing that the performance of the controller depends on the instantaneous saturation limits. Fig. 1 (b) and (c) show the evolution of $u_1(t)$ and $u_2(t)$, where it can be seen that they both remain inside the boundaries determined by all the values of $\sigma_1(t)$ and $\sigma_2(t)$ that were mentioned.

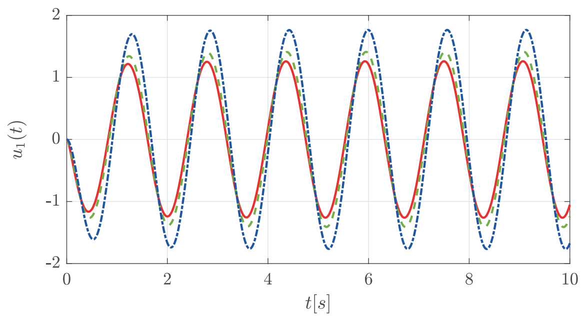
Fig. 2 and Fig. 3 show the state space phase portrait with the established ellipsoidal regions $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{U}(\theta, \phi)$. For illustrative purposes, the region $\mathcal{U}(\theta, \phi)$ is drawn only on the vertex values of $\theta(t)$ and $\phi(t)$. It can be seen that the state trajectories in the worst case scenario, which correspond to $\phi_1 = \phi_2 = 1$ and $\overline{\gamma} = 1.0623$, remain inside $\mathcal{V}$, demonstrating the effectiveness of the QB. Moreover, it is guaranteed that the control inputs do not saturate because the states do not exceed the boundaries that are established by the multiple regions of $\mathcal{U}(\theta, \phi)$.

4.2 Scenario II: time-varying saturation limit

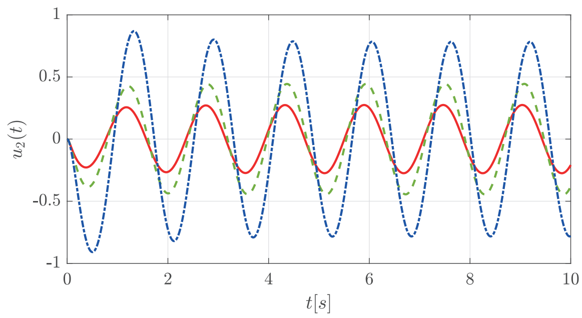
Scenario II shows the adaptability of the designed controller to time variations of the saturation limit of $u_1(t)$. The controller is tested in the same conditions as in the previous scenario, except for the limit of $u_2(t)$, which is fixed to $\sigma_2(t) = 10$, while $\sigma_1(t)$ varies within the interval $[5, 10]$ depending on $\phi_1(t)$.



(a) Plot of $z(t)$ with different frozen values of $\phi(t)$.



(b) Plot of $u_1(t)$ with different frozen values of $\phi(t)$.



(c) Plot of $u_2(t)$ with different frozen values of $\phi(t)$.

Fig. 1. Scenario I: results of $z(t)$, $u_1(t)$ and $u_2(t)$.

Fig. 4 shows the adaptability of the control performance output signal $z(t)$, for the different values of $\sigma_1(t)$. It can be seen that $z(t)$ is less affected by the external disturbance when $\sigma_1(t) = 10$, which corresponds to $\phi_1(t) = 0$ and, conversely, it is more affected when $\sigma_1(t) = 5$ $(\phi_1(t) = 1)$. Therefore, it is shown that the controller is able to adapt when the limits of $u_1(t)$ change.

5. CONCLUSION

In this paper, the problem of designing a shifting H_∞ LPV state-feedback controller that takes into account the time-varying input saturation limits subject to unknown external disturbances has been investigated. The quadratic boundedness concept has been added in order to ensure that all the state trajectories converge inside the ellipsoidal region described by the quadratic Lyapunov function, in spite of external disturbances. Moreover, the shifting H_∞ paradigm has been incorporated to adapt the rejection

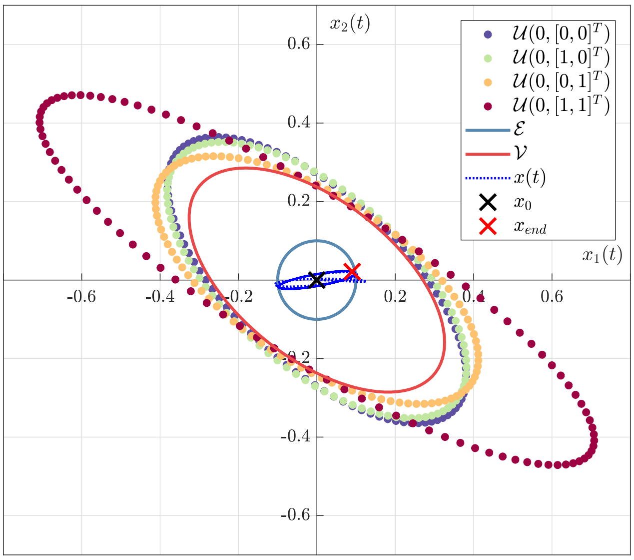


Fig. 2. Scenario I: plot of ellipsoidal regions for $\theta(t) = 0$.

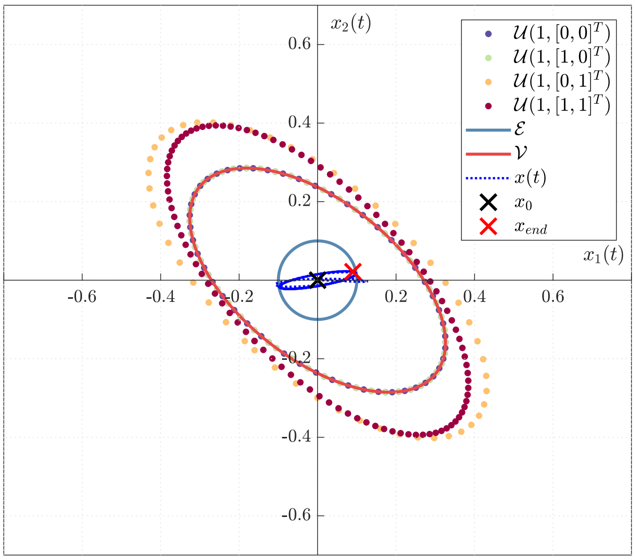


Fig. 3. Scenario I: plot of ellipsoidal regions for $\theta(t) = 1$.

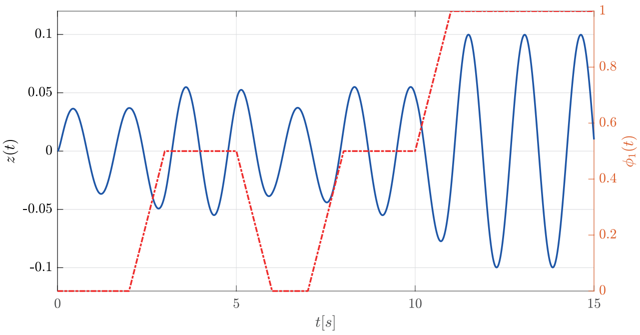


Fig. 4. Scenario II: evolution of $z(t)$ vs $\phi_1(t)$.

performance depending on the instantaneous saturation limits. This overall design corresponds to a minimization problem involving LMI-based feasibility constraints, which can be solved using the available solvers.

Finally, the results obtained in the illustrative example with an LPV numerical system have shown the effectiveness of the proposed approach. The designed controller has the capability of adapting its rejection performance based on the instantaneous saturation limit values of the actuators. However, the results appear to be conservative, probably due to the assumption of considering a constant Lyapunov matrix. For this reason, with the aim of applying the proposed method to more complex systems, future work will focus on developing a design procedure that uses a parameter-dependent Lyapunov matrix.

REFERENCES

Apkarian, P., Gahinet, P., and Becker, G. (1995). Self-scheduled H_∞ control of linear parameter-varying systems: a design example. *Automatica*, 31(9), 1251 – 1261.

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics.

Brockman, M.L. and Corless, M. (1998). Quadratic boundedness of nominally linear systems. *International Journal of Control*, 71(6), 1105–1117.

Ding, B. (2009). Quadratic boundedness via dynamic output feedback for constrained nonlinear systems in takagi–sugeno's form. *Automatica*, 45(9), 2093 – 2098.

Köse, İ.E. and Jabbari, F. (2003). Scheduled controllers for linear systems with bounded actuators. *Automatica*, 39(8), 1377 – 1387.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, volume 3. Taipei, Taiwan.

Ping, X., Wang, P., and Li, Z. (2017). Quadratic boundedness of lpv systems via saturated dynamic output feedback controller. *Optimal Control Applications and Methods*, 38(6), 1239–1248.

Rotondo, D., Nejjari, F., and Puig, V. (2015). Design of parameter-scheduled state-feedback controllers using shifting specifications. *Journal of the Franklin Institute*, 352(1), 93–116.

Ruiz, A., Rotondo, D., and Morcego, B. (2019). Design of State-Feedback Controllers for Linear Parameter Varying Systems Subject to Time-Varying Input Saturation. *Applied Sciences*, 9(17), 3606.

Sajjadi-Kia, S. and Jabbari, F. (2013). Controllers for linear systems with bounded actuators: Slab scheduling and anti-windup. *Automatica*, 49(3), 762 – 769.

Sala, A. and Arino, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of polya's theorem. *Fuzzy Sets and Systems*, 158(24), 2671–2686.

Sturm, J.F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization methods and software*, 11(1-4), 625–653.