Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Structural Controllability of Switching Max-Plus Linear Systems

Abhimanyu Gupta * **Ton van den Boom** *
Jacob van der Woude ** **Bart De Schutter** *

* *Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands*
** *DIAM, EWI, Delft University of Technology, Van Mourik Broekmanweg 6, 2628 XE, Delft, The Netherlands*
e-mail:
{a.gupta-3,a.j.j.vandenboom,j.w.vanderwoude,b.deschutter}@tudelft.nl

Abstract: We introduce a framework for studying controllability properties of discrete-event systems modelled as switching max-plus linear systems. In this framework, we generalise the notion of structural controllability to include the switching phenomenon. Such models provide an additional discrete input to change the synchronisation and/or ordering constraints of the system. In this paper, we solve the problem of assigning the throughput of the system by suitable controller configurations. In particular, we formulate structural conditions for the existence of controllers achieving stable stationary behaviour. We also classify the achievable throughput under different controller configurations.

Keywords: Discrete event systems, max-plus linear systems, switching, structural properties, controllability.

1. INTRODUCTION

A Switching Max-Plus Linear (SMPL) system models a discrete-event system that can switch between different modes of operations modelled by max-plus linear state space equations (van den Boom and De Schutter, 2006). These equations describe the time evolution of event occurrences and the synchronisation between them (Baccelli et al., 1992). An SMPL model extends the decision making capabilities of max-plus linear systems to discrete-event systems with event-varying synchronisation and ordering structures. These models have applications in control and scheduling of transportation networks and production systems with varying ordering and routing mechanisms (Kersbergen et al., 2016; van den Boom and De Schutter, 2012). The modelling framework has also found applications in performance analysis of networked conflicting timed-event graphs (Boussahel et al., 2016).

We revisit the problem of controllability for SMPL systems where the underlying timed-event graph has a varying structure. The aim is to address the additional complexity due to the switching phenomenon, especially when it is controlled. This offers an interesting area of research that serves as a building block for further analyses and controller (or scheduler) design.

The concept of controllability in dynamical systems concerns the possibility of driving a certain state to a desired set of states. We consider the case when these desired states are related to an optimal throughput of the discrete-event system (Commault, 1998). The optimal throughput information usually comes from a reference signal or timetable. This problem of controllability that ensures boundedness of the discrete-event system can be related to the structural properties of the underlying timed-event graph (Baccelli et al., 1992; Commault, 1998).

The problem of structural controllability, in discrete-event system literature, has been studied for timed continuous Petri nets (Vázquez and Ramírez-Treviño, 2012, and references therein). Such models approximate the behaviour of the discrete-event system by relaxing the integrality condition on markings in the underlying net. The approach leads to sufficient conditions for certain quantitative properties, like boundedness, and bounds on qualitative properties, like throughput, of the underlying discrete model. The control actions are, however, restricted to reducing the firing flow of transitions. The logical aspect of controllability of Petri nets is usually studied separately in a supervisory control framework (Giua and DiCesare, 1994).

In the above context, SMPL framework preserves the discreteness of the discrete-event system and takes advantage of max-plus algebra for explicit analysis of its timing properties. In addition, the logical control by disabling event occurrences is also explicitly modelled as a discrete control variable (van den Boom and De Schutter, 2012).

In this paper, we focus on the problem of stabilisability of SMPL systems. More specifically, we first consider the range of achievable growth rates of the states for a given SMPL system. Then we formulate structural conditions that allow synchronisation of these growth rates by using control actions that are allowed by the system.

The contributions of this paper are threefold. We propose a framework for studying the controllability problem for SMPL systems. We generalise the notion of structural controllability to include switching behaviour. We also characterise the achievable throughput of the system under different controller configurations.

The paper is organised as follows. In Section 2 we present some background on the max-plus algebra and the associated graph theory. In Section 3 we introduce the framework for studying controllability of SMPL systems, and the notion of stability associated to structural controllability. In Section 4 we present results on structural controllability. We end the paper with concluding remarks in Section 5.

2. PRELIMINARIES

The set of all positive integers up to n is denoted by $\underline{n} = \{l \in \mathbb{N} \,|\, l \leq n\}$ where $\mathbb{N} = \{1, 2, 3, \dots\}$.

The *max-plus algebra*, $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, consists of the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ endowed with the maximisation and the addition operations (Baccelli et al., 1992). These are called max-plus addition ($a \oplus b = \max(a, b)$) and max-plus multiplication ($a \otimes b = a + b$) respectively. The zero element is denoted as $\varepsilon = -\infty$ and the unit element as $\mathbb{1} = 0$. These elements are identities with respect to $\oplus$ and $\otimes$ respectively, and ε is absorbing for $\otimes$. The unit element $\mathbb{1}$ will sometimes be used to represent a vector, of appropriate dimension, with all entries 0. The max-plus vector and matrix operations can be defined analogously to the conventional algebra. The max-plus zero and identity matrices are denoted as $\mathcal{E}$ and $\mathcal{I}$ respectively. The max-plus powers of a matrix are defined recursively as $A^{\otimes^{k+1}} = A^{\otimes^{k}} \otimes A$ for $k \in \mathbb{N}$. For scalars $\gamma, c \in \mathbb{R}$, we have $\gamma^{\otimes^c} = c \cdot \gamma$. The partial order $\leq$ for vectors $x, y \in \mathbb{R}^n_\varepsilon$ is defined such that $x \leq y \Leftrightarrow x \oplus y = y \Leftrightarrow x_i \leq y_i, \ \forall i \in \underline{n}$.

Graph theory. (Heidergott et al., 2014) A directed graph $\mathcal{G}(A) = (V(A), E(A))$ can be associated to a matrix $A \in \mathbb{R}^{n \times n}_\varepsilon$ by defining the vertex set $V(A) = \underline{n}$ and letting the pair $(i, j) \in E(A)$, the edge set, whenever $a_{ji} \neq \varepsilon$. The matrix A is called irreducible if $\mathcal{G}(A)$ is strongly connected i.e., for each $i, j \in V(A)$, there is a path that starts in i and ends in j.

A reducible matrix can be transformed into the Frobenius normal form by a suitable max-plus permutation matrix:

$$P \otimes A \otimes P^{\otimes^{-1}} = \tilde{A} = \begin{pmatrix} A_{11} & \varepsilon & \dots & \varepsilon \\ A_{21} & A_{22} & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix} \tag{1}$$

where $A_{11}, \dots, A_{rr}$ are irreducible submatrices of $\tilde{A}$. The corresponding partition of the subset of vertices (classes) $V(A)$ is denoted as $V_1, \dots, V_r$. An arc from a vertex in V_i to a vertex in V_j exists only if $i \leq j$. The classes of A with no incoming arc are called the *initial classes* and those with no outgoing arcs are called the *final classes.*

Eigenvalue problem. The max-plus eigenvalue $\lambda(A) \in \mathbb{R}_\varepsilon$ of a matrix $A \in \mathbb{R}^{n \times n}_\varepsilon$ is defined as the solution to the following eigenproblem (Cuninghame-Green, 1979):

$$\begin{aligned} &\exists\, z \in \mathbb{R}^n_\varepsilon, \ z \neq \varepsilon \quad \text{s.t.} \\ &z^\top \otimes A = \lambda(A) \otimes z^\top. \end{aligned} \tag{2}$$

Then z is known as the left max-plus eigenvector of A corresponding to the max-plus eigenvalue $\lambda(A)$. This eigenvalue is unique if the matrix A is irreducible. The eigenvalue of a class j in (1) is denoted as $\lambda(A_{jj})$. We define $\overline{\lambda}(A)$ as the largest eigenvalue of A. All eigenvectors of an irreducible matrix are finite, $z \in \mathbb{R}^n$. A reducible matrix A in the Frobenius normal form (1) has a finite eigenvector if *i)* $\lambda(A_{11}) = \overline{\lambda}(A) \neq \varepsilon$, and *ii)* there is an arc from a vertex in V_1 to a vertex in V_j for all $j \in \{2, \dots, r\}$ (Cuninghame-Green, 1979).

Matrix semigroup. A matrix A is *regular* if it has at least one finite element in every row. A set of regular square matrices in the max-plus algebra of dimension n, denoted as $\mathcal{A} \subseteq \mathcal{M}^{n \times n}(\mathbb{R}_\varepsilon)$, forms a multiplicative semigroup [1] :

$$\Psi(\mathcal{A}) := \left\{ A^{(i_1)} \otimes \cdots \otimes A^{(i_k)} \mid A^{(i_j)} \in \mathcal{A}, \ j \in \underline{k}, \ k \in \mathbb{N} \right\}. \tag{3}$$

The max-plus convex hull of $m \in \mathbb{N}$ matrices in $\mathcal{A}$ is defined as

$$\begin{aligned} \mathrm{conv}_\otimes(\mathcal{A}) = \Bigg\{ \bigoplus_{j=1}^{m} \alpha_j \otimes A^{(j)} \ \Bigg| \ & j \in \underline{m}, \ A^{(j)} \in \mathcal{A}, \\ & \alpha_j \in \mathbb{R}_\varepsilon, \ \bigoplus_{j=1}^{m} \alpha_j = \mathbb{1} \Bigg\}. \end{aligned} \tag{4}$$

The semigroup is said to be irreducible if there exists an irreducible matrix in the max-plus convex hull of the matrices (Guglielmi et al., 2018). Then there also exist irreducible matrices $S \in \Psi(\mathcal{A})$. Equivalently, we have

$$\begin{aligned} &\exists\, c \in \mathbb{N}, \\ &S = A^{(i_c)} \otimes A^{(i_{c-1})} \otimes \cdots \otimes A^{(i_1)}, \ A^{(i_j)} \in \mathcal{A}, \ j \in \underline{c}, \end{aligned} \tag{5}$$

such that S is irreducible.

The maximum of the matrix set is defined as

$$\mathcal{S}_A = \bigoplus_{A \in \mathcal{A}} A. \tag{6}$$

The max-plus joint spectral radius of a given set of matrices $\mathcal{A}$ is defined analogously to the largest max-plus eigenvalue of a single matrix A (Guglielmi et al., 2018). It represents the maximum growth rate of the system under autonomous evolution. It can be calculated as the largest max-plus eigenvalue of the matrix $\mathcal{S}_A$ and is denoted as $\rho(\mathcal{A})$ (Gaubert, 1995; Guglielmi et al., 2018).

3. SMPL SYSTEMS

In this section we give a description of an SMPL system along with a framework for studying its controllability properties. Then we introduce the concepts of stability pertaining to structural controllability of such systems.

3.1 System description

The dynamics of a *non-autonomous* SMPL system with p modes is defined for event cycle k as (van den Boom and De Schutter, 2006)

[1] A semigroup consists of a set together with an associative binary operation without requiring the existence of an identity element or inverses.

$$\begin{aligned} x(k) &= A^{(l(k))} \otimes x(k-1) \oplus B^{(l(k))} \otimes u(k), \\ y(k) &= C^{(l(k))} \otimes x(k) \\ l(k) &= \phi(l(k-1), x(k-1), v(k), u(k)) \\ l(k) &\in \mathcal{L} \triangleq \underline{p}, \quad k \in \mathbb{N}. \end{aligned} \tag{7}$$

Here, the system matrices for the l-th mode are $A^{(l)} \in \mathbb{R}_\varepsilon^{n\times n}$, $B^{(l)} \in \mathbb{R}_\varepsilon^{n\times m}$ and $C^{(l)} \in \mathbb{R}_\varepsilon^{n_y\times n}$. The function $\phi(\cdot)$ specifies the switching rule, which may depend on the previous state $x(k-1)$, the previous mode $l(k-1)$, the discrete input $v(k)$, and the continuous input $u(k)$.

We assume that all the system matrices $A^{(l)}$ for each mode are regular. This is an acceptable assumption ensuring that the states do not become ε for finite initial states. The set of all such $A^{(l)}$ matrices is denoted as $\mathcal{A} \subseteq \mathcal{M}^{n\times n}(\mathbb{R}_\varepsilon)$. Similarly, the set of all $B^{(l)}$ matrices is denoted as $\mathcal{B}$. The dynamics with $B^{(l)} = \mathcal{E}$ for all $l \in \mathcal{L}$ is called an *autonomous* SMPL system.

The components of the state $x_i(k)$, for $i \in \underline{n}$, denote the occurrence time of the i-th event of the system. The entries of the matrix $A^{(l(k))}$ denote the minimum time duration offsets between event occurrences in consecutive event cycles. The same holds via the matrix $B^{(l)}$ for the time durations between the events and input delays. The state (dater) trajectory $x(k)$, for $k \in \mathbb{N}$, is called *realisable* if it is component-wise non-decreasing when the finite elements of the system matrices are non-negative.

A control input u delays the occurrence times of certain events in the discrete-event system with respect to the autonomous evolution. The discrete input v specifies the mode sequence via the function $\phi(\cdot)$ and can be used for scheduling. We restrict the scope of this paper to the cases where the mode sequence from $\phi(\cdot)$ is either generated arbitrarily or is controlled via the discrete signal v.

Thus, an SMPL system offers an additional degree of freedom for control as compared to max-plus linear systems. In this regard, we classify the controllability problem for SMPL systems as follows:

- *Discrete control*: Controllability using a discrete control input v in absence of continuous input u.
- *Continuous control*: Controllability using a continuous control input u under arbitrary switching.
- *Hybrid control*: Controllability using both discrete (v) and continuous control inputs (u).

Some notations and definitions are noted before discussing the problem of structural controllability.

We denote by $\sigma : \underline{k} \to \mathcal{L}^k$, a mode sequence of length $|\sigma| = k$ with $l(j) = \sigma(j)$ for $j \in \underline{k}$.

The state equation in (7) can be written recursively for $k > k_0 \geq 0$ and a mode sequence $\sigma : \underline{k-k_0} \to \mathcal{L}^{k-k_0}$ as

$$\begin{aligned} x(k) &= \Phi(k, k_0; \sigma) \otimes x(k_0) \oplus \bigoplus_{j=k_0+1}^{k} \Phi(k, j; \sigma) \otimes B^{(l(j))} u(j), \\ \Phi(k, j; \sigma) &= A^{(l(k))} \otimes \cdots \otimes A^{(l(j+1))}, \quad j < k. \end{aligned} \tag{8}$$

It is convenient to assume that the input sequence has the following dynamics (Baccelli et al., 1992, §7):

$$u(k+1) = U(k) \otimes u(k) \tag{9}$$

where $U(\cdot) \in \mathbb{R}_\varepsilon^{m\times m}$. It is assumed that $U(k)$ is a max-plus diagonal matrix with bounded elements and average values ρ_u. Then ρ_u defines the asymptotic growth rate of the input u.

The maximum of the system matrices are denoted as

$$\mathcal{S}_A = \bigoplus_{l\in\mathcal{L}} A^{(l)}, \quad \mathcal{S}_B = \bigoplus_{l\in\mathcal{L}} B^{(l)}. \tag{10}$$

3.2 Problem formulation

For discrete-event systems in max-plus algebra, the *cycle time* vector is defined as (Heidergott et al., 2014):

$$\xi_i = \lim_{k\to\infty} \frac{x_i(k)}{k}. \tag{11}$$

If the limits exist, $\xi \in \mathbb{R}^n$ signifies the asymptotic average delay between event occurrences. The values ξ_i's represent the eigenvalues $\lambda(A)$ of an autonomous max-plus linear system. The largest of these values represents bottleneck in the system and serves as a measure of performance (Commault, 1998).

The *throughput* of a discrete-event system is defined as the inverse of the average occurrence times of the events over k (Baccelli et al., 1992). It is therefore inversely proportional to the maximum ξ_i value.

Definition 3.1. A discrete-event system is *stabilisable* if there exist control input(s) such that the asymptotic average growth rates of the state trajectories attain a common value in (11):

$$\begin{aligned} &\exists\, \mu \in \mathbb{R} \quad \text{s.t.} \\ &\xi_i = \xi_j = \mu, \quad \forall\, i, j \in \underline{n}. \end{aligned} \tag{12}$$

We define such a system evolution as *synchronised.* ♦

The synchronisation of the growth rate of state trajectories is necessary for boundedness of a timed-event graph (Commault, 1998).

The throughput of a max-plus linear system may only be decreased from its autonomous value by the application of the control input u (Baccelli et al., 1992, §6). However, it is sometimes possible to increase the throughput of an SMPL system by suitably switching between different subsystems. Therefore, an interesting question is to study how we can control such a system via signals u and/or v to obtain a desired throughput of the system, usually specified by a due-date signal or a timetable.

We refer to properties depending only on the structure of the underlying timed-event graph of an SMPL system as *structural.* We study the problem of synchronisation (12) by assignment of the throughput ($1/\mu$) of the SMPL system as its *structural controllability* property. This value μ depends on the mode sequence via v and the time delay to the autonomous evolution from u.

Under the framework for studying controllability, this problem can be classified as:

- *Problem I*: Formulate conditions for synchronised evolution via discrete control.
- *Problem II*: Formulate conditions for synchronised evolution via continuous control.

- *Problem III*: Formulate conditions for synchronised evolution via hybrid control.

As will become clear later, the three different cases enforce different operating conditions with respect to the achievable throughput of the system.

3.3 Stability

We introduce a notion of stability related to the controllability problem under study. This notion concerns the bounded growth rate of the state of a discrete-event system modelled in max-plus algebra. The bounds then determine the achievable throughput of the system. This notion of stability, however, does not enforce the boundedness of event separation times in the same event cycle k.

Definition 3.2. A discrete-event system is said to be max-plus Lipschitz stable, in the given state space $\mathcal{X}$, if there exists an upper and lower bound on the first-order difference of the state trajectories, i.e.

$$\begin{aligned}&\exists\, \alpha, \beta \in \mathbb{R} \text{ with } \alpha \leq \beta \quad \text{s.t.}\\ &\alpha \otimes \mathbb{1} \leq \Delta\, x(k) = x(k) - x(k-1) \leq \beta \otimes \mathbb{1},\ \forall k \in \mathbb{N}.\end{aligned} \tag{13}$$

♦

The right inequality suggests maximal duration requirement between two consecutive events. The left inequality is similarly a minimum duration requirement. This condition holds naturally for max-plus linear systems over finite state evolution if the state matrix A is regular (Menguy et al., 2000).

For an autonomous evolution of the SMPL system (7), the smallest β that satisfies (13), for all trajectories, is the max-plus joint spectral radius, $\rho(\mathcal{A})$. Let us assume that there are r_l classes (see (1)) for matrices $A^{(l)}$ in $\mathcal{A}$. Then the largest α for which (13) is satisfied for all trajectories (Butkovič, 2016) is similarly defined as

$$\rho^*(\mathcal{A}) = \min_{l \in \mathcal{L}} \min_{j \in \underline{r_l}} \{\lambda(A_{jj}^{(l)}) \mid V_j \text{ is an initial class}\}. \tag{14}$$

We now provide a restriction for the existence of the cycle time vector (see (11)) for SMPL systems.

Definition 3.3. A discrete-event system is said to be asymptotically max-plus Lipschitz stable, in the given state space $\mathcal{X}$, if the average growth rates of all the states become constant, i.e.

$$\begin{aligned}&\forall i \in \underline{n},\ \exists\, c_i \in \mathbb{N} \quad \text{s.t.}\\ &\lim_{k\to\infty} \left[\Delta_c^2\, x(k) = x(k+c_i) - 2x(k) + x(k-c_i)\right]_i = 0.\end{aligned} \tag{15}$$

This asymptotic average delay between event occurrences is the cycle time vector (11) of the system (Gunawardena, 2003):

$$\lim_{k\to\infty} \frac{1}{c_i}\ (x_i(k+c_i) - x_i(k)) = \xi_i, \quad \forall i \in \underline{n}. \tag{16}$$

The smallest c satisfying this property for all $i \in \underline{n}$ is defined as the *cyclicity* of the underlying communication graph. ♦

For a regular max-plus linear system, the cycle time vector exists and is finite in practice (Gunawardena, 2003). This in turn means that a max-plus linear system is always asymptotically max-plus Lipschitz stable. This, however, is not the case for SMPL systems if the mode sequence is arbitrary and not known a priori.

Remark 3.1. We note that an SMPL systems with periodic mode sequences can be reduced to a max-plus linear system by appropriate change in event counter description. Such system trajectories achieve asymptotic max-plus Lipschitz stability.

Remark 3.2. It is worthwhile to note that the constraint of non-decreasing dater trajectories is also a structural property of the underlying timed-event graphs. This feasibility can be ensured by adding self-loops of unit $\mathbb{1}$ weight. This is equivalent to replacing indefinite ε diagonal elements of matrices in $\mathcal{A}$ by unity $\mathbb{1}$ without altering the system description while enforcing realisable evolution of the states.

4. STRUCTURAL CONTROLLABILITY

An input sequence $u : \underline{k} \to \mathbb{R}^{m\times k}$ is said to be *realisable* if it is non-decreasing over the event counter k (or, $u(j+1) \geq u(j)$ for all $j \in \underline{k}$). The asymptotic growth rate of this signal is denoted as ρ_{u} .

This section looks at the structural requirements on the system description such that all state trajectories are synchronised (12) in a, possibly infinite, number of events by an appropriate choice of control inputs u and v.

We also recall that the notion of max-plus Lipschitz stability for SMPL systems ensures the existence of bounds on the first-order difference of state trajectories, $\Delta x(k) \in [\alpha, \beta]$. For an autonomous evolution, this can be reduced to $\Delta x(k) \in [\rho^*(\mathcal{A}), \rho(\mathcal{A})]$.

The cases differentiating synchronisation properties via u and via v in Section 3.2 are formalised, and sufficient structural conditions are provided in the following subsections.

4.1 Problem I: Discrete control

We first formulate structural conditions for the existence of synchronised trajectories for autonomous SMPL systems.

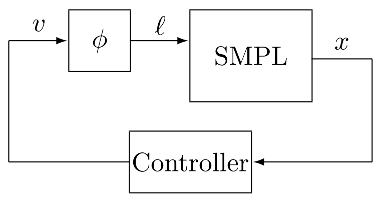

Figure 1. *Problem I*: Controllability via discrete control v.

Definition 4.1. A max-plus Lipschitz stable discrete-event system is said to be synchronised via discrete control v if there exists some $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, and a mode sequence $\sigma : \underline{k} \to \mathcal{L}^k$ such that (12) is satisfied for $\mu \in [\rho^*(\mathcal{A}), \rho(\mathcal{A})]$. ♦

The following theorem provides sufficient conditions for weak synchronisation via control input v.

Theorem 4.1. A max-plus Lipschitz stable autonomous SMPL system can be synchronised by discrete control input v for *some* $\mu \in [\rho^*(\mathcal{A}), \rho(\mathcal{A})]$ if the semigroup generated by matrices in $\mathcal{A}$ is irreducible.

Proof: For an irreducible semigroup, there exists an irreducible matrix $S \in \Psi(\mathcal{A})$ for a finite mode sequence $\sigma : \underline{c} \to \mathcal{L}^c$, with $l_j = \sigma(j)$ for $j \in \underline{c}$ (see Section 2):

$$S = A^{(l_c)} \otimes A^{(l_{c-1})} \otimes \cdots \otimes A^{(l_1)}. \tag{17}$$

The periodic mode sequence σ achieves asymptotic max-plus Lipschitz stability in (15) due to remark 3.1.

Moreover, the irreducibility implies existence of a finite max-plus eigenvector (see Section 2) such that

$$\exists z \in \mathbb{R}^n,\ z^\top S = \mu^{\otimes^c} \otimes z^\top. \tag{18}$$

This ensures weak synchronisation via v in (12). The value(s) μ then belongs to the interval $[\rho^*(\mathcal{A}), \rho(\mathcal{A})]$. ■

The theorem presented here for synchronisation via v only provides a sufficient condition. The irreducibility of the matrix S is only a sufficient condition for the existence of a finite eigenvector in (18) (see Section 2). If the semigroup is reducible, it is still possible to apply Theorem 4.1. The approach is to first break down the matrix $\mathcal{S}_A$ in (10) into irreducible submatrices (1) and then to study their connections (Mairesse, 1997).

It is, however, difficult to characterise the set of achievable growth rates μ for autonomous SMPL systems. This is because there can exist multiple matrices S in the semigroup $\Psi(\mathcal{A})$ that are irreducible. This also allows for an event-varying growth rate μ for which synchronisation is achieved via v for finite durations of event counter k. The system, however, is not guaranteed to be asymptotically max-plus Lipschitz stable.

4.2 Problem II: Continuous control

The problem of structural controllability via u can be studied as an extension of the findings in (van den Boom and De Schutter, 2012).

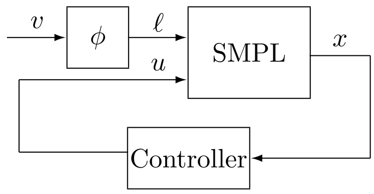

Figure 2. *Problem II*: Controllability via continuous control u with arbitrary v.

Definition 4.2. A max-plus Lipschitz stable discrete-event system is said to be synchronised via continuous control u if there exists a realisable input sequence $u : \underline{k} \to \mathbb{R}^{m\times k}$, $k \in \mathbb{N}$, with an asymptotic growth rate $\rho_{\mathrm{u}} \geq \rho(\mathcal{A})$ such that (12) is satisfied for all $\sigma : \underline{k} \to \mathcal{L}^k$ for $\mu = \rho_{\mathrm{u}}$. ♦

The following result is first recalled from literature to find conditions for synchronisation via u.

Definition 4.3. (van den Boom and De Schutter (2012)).

The SMPL system (7) is said to be *structurally controllable* if there exists a finite integer N for all mode sequences generated by w, $\sigma : \underline{N} \to \mathcal{L}^N$, with $l_j = \sigma(j)$ for all $j \in \underline{N}$, such that the reachability matrices

$$\Gamma_N(\sigma) = [\Phi(N,1;\sigma) \otimes B^{(l_1)} \ \cdots \ \Phi(N,N-2;\sigma) \otimes B^{(l_{N-2})} \\ \cdots \ \Phi(N,N-1;\sigma) \otimes B^{(l_{N-1})} \ B^{(l_N)}] \tag{19}$$

have a finite element in every row. ♦

Remark 4.1. The structural controllability of all constituent subsystems is a necessary condition for structural controllability of the SMPL system under arbitrary switching.

The following theorem gives a necessary and sufficient condition for synchronisation of an SMPL system by control input u alone under arbitrary switching.

Theorem 4.2. A max-plus Lipschitz stable non-autonomous SMPL system (7) can be synchronised by a continuous control input u for all mode sequences σ, and for all $\mu = \rho_{\mathrm{u}} \geq \rho(\mathcal{A})$ if and only if the system is structurally controllable.

Proof: Sufficiency: (van den Boom and De Schutter, 2012).

Necessity: We first note that the requirement of synchronisation via u has to hold for all switching sequences. This restricts the achievable growth rate μ to be larger than or equal to $\rho(\mathcal{A})$ (see Section 3.3).

Now assume that the system is not structurally controllable as in Definition 4.3. This implies that there exists a subset of states $J \subseteq \underline{n}$ that may never be delayed by input u. The growth rate of these states can not be increased from its largest possible autonomous value $\rho(\mathcal{A})$. So the system can not synchronised by u for all $\mu \geq \rho(\mathcal{A})$. ■

The preceding theorem suggests that in presence of uncontrollable switching, synchronised system evolution can be ensured only for growth rates larger than or equal to the spectral radius of the matrices in $\mathcal{A}$. The stated structural condition then guarantees the existence of continuous control input to achieve this synchronisation.

4.3 Problem III: Hybrid control

We finally deal with the problem of controllability via both u and v. In essence, we prove the existence of a lower bound such that any arbitrary growth rate above it can be achieved by an appropriate combination of u and v.

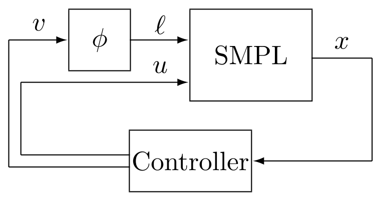

Figure 3. *Problem III*: Controllability via hybrid control u and v.

Definition 4.4. A max-plus Lipschitz stable discrete-event system is said to be synchronised via hybrid control u and v if there exists a real $\alpha^* \in [\rho^*(\mathcal{A}), \rho(\mathcal{A})]$, a mode sequence $\sigma : \underline{k} \to \mathcal{L}^k$, $k \in \mathbb{N}$, and a realisable input sequence $u : \underline{k} \to \mathcal{U}^k \subset \mathbb{R}^k$ with $\rho_{\mathrm{u}} \leq \rho(\mathcal{A})$ such that (12) is satisfied for an arbitrary $\mu \in [\alpha^*, \rho(\mathcal{A})]$. ♦

The following definition is necessary for presenting the next set of results.

Definition 4.5. A given SMPL system (7) is said to be *weakly structurally controllable* if for a finite positive integer $k \leq n$ there exists a mode sequence $\sigma : \underline{k} \to \mathcal{L}^k$, with $l_j = \sigma(j)$ for all $j \in \underline{k}$, such that the reachability matrix

$$\begin{aligned}\Gamma_k(\sigma) = [\Phi(k,1;\sigma) \otimes B^{(l_1)} \ \cdots \ \Phi(k,k-2;\sigma) \otimes B^{(l_{k-2})} \\ \cdots \ \Phi(k,k-1;\sigma) \otimes B^{(l_{k-1})} \ B^{(l_k)}]\end{aligned} \tag{20}$$

has a finite element in every row. ♦

Remark 4.2. Structural controllability of a subsystem in $(\mathcal{A}, \mathcal{B})$ implies weak structural controllability of the entire system but the converse is not true.

Theorem 4.3. A max-plus Lipschitz stable non-autonomous SMPL system (7) can be synchronised by a suitable combination of control inputs u and v as presented in Definition 4.4 if one of the following equivalent conditions hold:

(i) The system is weakly structurally controllable;
(ii) The max-plus linear system $(\mathcal{S}_A, \mathcal{S}_B)$ in (10) is structurally controllable. The matrix
$$\overline{\Gamma}_n = [\mathcal{S}_A{}^{\otimes^{n-1}} \otimes \mathcal{S}_B \ \cdots \ \mathcal{S}_A \otimes \mathcal{S}_B \ \ \mathcal{S}_B] \tag{21}$$
has a finite element in every row.

Proof: Without a loss of generality, we assume the system is driven by a single input ($u \in \mathbb{R}_\varepsilon$).

(i)⇒(ii) Weak structural controllability implies that for every $i \in \underline{n}$ there exists a mode sequence $\sigma : \underline{k} \to \mathcal{L}^k$, with $l_j = \sigma(j)$ for all $j \in \underline{k}$, such that we have

$$\left(\Phi(k,1;\sigma) \otimes B^{(l_1)}\right)_i \neq \varepsilon. \tag{22}$$

Now, the maximum of the first column of $\Gamma_k(\sigma)$ in (20) over all mode sequences σ of length k is equal to the first column of the reachability matrix $\overline{\Gamma}_k$ in (21) such that we have

$$\left(\mathcal{S}_A{}^{\otimes^{k-1}} \otimes \mathcal{S}_B\right)_i \neq \varepsilon. \tag{23}$$

Here, the mode sequence for every $i \in \underline{n}$ can be of varying length $k \in \underline{n}$. This ensures that the reachability matrix in (21) has at least one finite element in every row. Thus, the system $(\mathcal{S}_A, \mathcal{S}_B)$ is structurally controllable.

(ii)⇒(i) Conversely, condition (ii) implies that for every $i \in \underline{n}$, we have

$$\exists k \in \underline{n}, \ \text{s.t.} \quad \left(\mathcal{S}_A{}^{\otimes^{k-1}} \otimes \mathcal{S}_B\right)_i \neq \varepsilon. \tag{24}$$

Therefore, there exits a mode sequence $\sigma : k \to \mathcal{L}^k$, with $l_j = \sigma(j)$ for all $j \in \underline{k}$, such that

$$\left(\Phi(k,1;\sigma) \otimes B^{(l_1)}\right)_i \neq \varepsilon. \tag{25}$$

Again, stacking the columns for $k \in \underline{n}$, we obtain weak structural controllability of the SMPL system. The requirement $k \leq n$ comes from the assumption on regularity of matrices in $(\mathcal{A}, \mathcal{B})$ and hence matrices $(\mathcal{S}_A, \mathcal{S}_B)$.

Consider the set of all mode sequences σ of length $k \leq n$ that satisfy the property of weak structural controllability. Define the smallest average max-plus eigenvalue over this set as α^*,

$$\alpha^* = \overline{\lambda}\left(\Phi(k,1;\sigma)\right)^{\otimes^{1/k}}. \tag{26}$$

This value $\alpha^* \in [\rho^*(\mathcal{A}), \rho(\mathcal{A})]$ serves as the guaranteed lower bound for the growth rate μ in Definition 4.4.

Invoking the result of Theorem 4.2, the synchronisation can be achieved with a suitable control input u for all μ greater than and equal to α^*. ■

The conditions for synchronisation via u and v can thus be deduced from that of the system $(\mathcal{S}_A, \mathcal{S}_B)$ in (10).

The results of the preceding theorem are *conservative* in the sense that a smaller α^* can be achieved by mode sequences of length k greater than n. However, such a sequence can not be guaranteed to satisfy the condition of structural controllability (Definition 4.5) required to place the growth rate arbitrarily in a set. It is also important to note that it is computationally difficult to evaluate the lowest achievable growth rate for a set of max-plus matrices (Blondel et al., 2000).

The preceding theorem also suggest that a higher throughput can be achieved from the system via a hybrid control strategy as compared to its continuous control counterpart. Moreover, the operating region in terms of achievable throughput take arbitrary values in a set as opposed to the discrete control case.

4.4 Illustrations

In this subsection, we present a examples to illustrate the notions of structural controllability of SMPL systems.

Example 4.1. Consider a bimodal SMPL system with system matrices:

$$A^{(1)} = \begin{pmatrix} 2 & \varepsilon \\ \varepsilon & 2 \end{pmatrix}, \qquad B^{(1)} = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}$$
$$A^{(3)} = \begin{pmatrix} 1 & \varepsilon \\ 1 & \varepsilon \end{pmatrix}, \qquad B^{(3)} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}.$$

The max-plus joint spectral radius and the upper bound on the lower spectral radius are found to be $\rho(\mathcal{A}) = 2$ and $\rho^*(\mathcal{A}) = 1$ respectively.

We first note that $A^{(1)}$ possesses a finite eigenvector $z_1 = (0 \ 0)^\top$. The semigroup generated by $A^{(1)}$ and $A^{(2)}$ is reducible (as in (27)). Therefore, structural guarantee for synchronised evolution of the system via discrete control can not be provided. However, we can still achieve synchronised evolution for $\mu = 2$ by operating in mode 1. This shows that the structural condition presented in Theorem 4.1 is not necessary for synchronised evolution via discrete control.

It can be checked that the constituent subsystems are not structurally controllable and so is the SMPL system (Definition 4.3). Therefore, continuous control is ineffectual for synchronised evolution. Nevertheless, the max-plus linear system $(\mathcal{S}_A, \mathcal{S}_B)$ is structurally controllable:

$$\mathcal{S}_A = \begin{pmatrix} 2 & \varepsilon \\ 1 & 2 \end{pmatrix}, \quad \mathcal{S}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{27}$$
$$\overline{\Gamma}_n = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}.$$

Therefore, weak structurally controllability implies the existence of a lower bound $\alpha^* \in [1, 2]$ and a hybrid control allowing synchronised evolution of the system with arbitrary growth rates in $[\alpha^*, +\infty)$. For instance, switching sequences $\sigma \in \{(1,2),(2,1)\}$ satisfy weak structural controllability condition and provide $\alpha^* = 1.5$. Then a continuous control input with $\rho_{\mathrm{u}} \in [1.5, 2]$ can be used to achieve synchronised growth rates arbitrarily from the set $\mu \in [1.5, 2]$.

Example 4.2. Consider another bimodal SMPL system with system matrices:

$$A^{(1)} = \begin{pmatrix} 2 & \varepsilon \\ \varepsilon & 2 \end{pmatrix}, \qquad B^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$A^{(2)} = \begin{pmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{pmatrix}, \qquad B^{(2)} = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}.$$

The main difference with the preceding example is that now the subsystems are structurally controllable. It can be verified by enumeration that the system is also structurally controllable for $N = 2$ in Definition 4.3. Therefore, continuous control, with $\rho_{\mathrm{u}} \geq 2$, can be used for arbitrary switching to achieve synchronised growth rate of $\mu \geq 2$.

Moreover, the semigroup generated by $A^{(1)}$ and $A^{(2)}$ is irreducible:

$$\mathcal{S}_A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathcal{S}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that there exist discrete control signals that allow synchronised growth rates for multiple $\mu \in [1, 2]$. For example, a sequence $\sigma = (2, 1, 2)$ achieves $\mu_\sigma = 1.33$. This supplemented by weak structural controllability property implies that a hybrid control will allow arbitrary growth rate $\mu \geq 1.33$.

5. CONCLUDING REMARKS

In this paper, we have proposed a framework for studying controllability problems for switching max-plus linear systems. The control inputs in such systems appear as continuous event delays to autonomous evolutions and as discrete mode changes in the system. We have extended the property of structural controllability to such systems to guarantee throughputs that achieve bounded trajectories of the system. Unlike max-plus linear systems, the presence of a discrete control provides a greater flexibility in assigning the throughput of the system. We found that the suggested controller configurations offer advantages in different operating regions with respect to achievable throughput of the system. Finally, we have presented structural conditions that guarantee the existence of control inputs to achieve a desired throughput. The optimality of a certain control action then depends on other factors of the specific control problem like the due-date reference, and constraints.

In future, we will extend the framework to study structural controllability and the dual notion of structural observability for such discrete-event systems in presence of continuous and discrete state restrictions. We intend to study graph-based algorithms to verify the structural controllability properties. It will also be interesting to derive finite-path dependent controllers that achieve the desired synchronised growth rates for different control configurations.

REFERENCES

Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, J.P. (1992). *Synchronization and Linearity: An Algebra for Discrete Event Systems*. John Wiley & Sons.

Blondel, V.D., Gaubert, S., and Tsitsiklis, J.N. (2000). Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard. *IEEE Transactions on Automatic Control*, 45(9), 1762–1765.

Boussahel, W.M., Amari, S., and Kara, R. (2016). Analytic evaluation of the cycle time on networked conflicting timed event graphs in the (max,+) algebra. *Discrete Event Dynamic Systems: Theory and Applications*, 26(4), 561–581.

Butkovič, P. (2016). On tropical supereigenvectors. *Linear Algebra and Its Applications*, 498, 574–591.

Commault, C. (1998). Feedback stabilization of some event graph models. *IEEE Transactions on Automatic Control*, 43(10), 1419–1423.

Cuninghame-Green, R.A. (1979). *Minimax Algebra*. Springer-Verlag, Berlin Heidelberg.

Gaubert, S. (1995). Performance evaluation of (max,+) automata. *IEEE Transactions on Automatic Control*, 40(12), 2014–2025.

Giua, A. and DiCesare, F. (1994). Blocking and controllability of Petri nets in supervisory control. *IEEE Transactions on Automatic Control*, 39(4), 818–823.

Guglielmi, N., Mason, O., and Wirth, F. (2018). Barabanov norms, Lipschitz continuity and monotonicity for the max algebraic joint spectral radius. *Linear Algebra and its Applications*, 550, 37–58.

Gunawardena, J. (2003). From max-plus algebra to nonexpansive mappings: A nonlinear theory for discrete event systems. *Theoretical Computer Science*, 293(1), 141–167.

Heidergott, B., Olsder, G.J., and van der Woude, J. (2014). *Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and its Applications*. Princeton University Press.

Kersbergen, B., Rudan, J., van den Boom, T.J., and De Schutter, B. (2016). Towards railway traffic management using switching max-plus-linear systems. *Discrete Event Dynamic Systems*, 26(2), 183–223.

Mairesse, J. (1997). Products of irreducible random matrices in the (max, +) algebra. *Advances in Applied Probability*, 29(2), 444–477.

Menguy, E., Boimond, J.L., Hardouin, L., and Ferrier, J.L. (2000). Just-in-time control of timed event graphs: Update of reference input, presence of uncontrollable input. *IEEE Transactions on Automatic Control*, 45(11), 2155–2159.

van den Boom, T.J. and De Schutter, B. (2006). Modelling and control of discrete event systems using switching max-plus-linear systems. *Control Engineering Practice*, 14, 1199–1211.

van den Boom, T.J. and De Schutter, B. (2012). Modeling and control of switching max-plus-linear systems with random and deterministic switching. *Discrete Event Dynamic Systems: Theory and Applications*, 22(3), 293–332.

Vázquez, C.R. and Ramírez-Treviño, A. (2012). Structural and generic conditions for controllability of timed continuous Petri nets. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 45(29), 41–46.