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On constructing boundaries for boundary value problems defined by continuous 2-D autonomous systems ⋆

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Abstract: In this paper, we provide a constructive way of specifying initial/boundary data for a given continuous 2-D autonomous system described by a set of linear partial differential equations (PDEs) with real constant coefficients. One of the ways of specifying initial/boundary data is by specifying the values of various derivatives of the solution trajectories at the origin; the derivatives correspond to a standard monomial set obtained using Gröbner basis. However, such an initial/boundary data often lacks physical interpretation. In this paper, we consider subsets of the domain having some algebraic structure (in the form of subspaces and strips of finite width around such subspaces) such that trajectories restricted to these subsets, often called characteristic sets, serve as initial/boundary conditions for the given autonomous system of linear PDEs. We provide a systematic way to construct such characteristic sets with the help of Gröbner bases and Oberst-Riquier algorithm. Thus we bridge the gap between initial/boundary conditions involving standard monomials and more conventional initial/boundary conditions in the form of restrictions on characteristic sets. We also show that every scalar system of PDEs admits such a characteristic set given by a rectangular strip of finite width around a subspace whose dimension equals the Krull dimension of the system's quotient ring.

Keywords: Distributed parameter systems, boundary control, linear control system.

1. INTRODUCTION

Solving a system of partial differential equations (PDEs) requires initial and/or boundary conditions. In this paper, we consider systems of PDEs with two independent variables having no *inputs/free variables*. Such systems are also called continuous 2-D autonomous systems in the literature. It is known from the literature that exponential solutions for a given continuous n-D autonomous system can be computed using the Oberst-Riquier algorithm (see (Pal and Pillai, 2014, Algorithm 22)). The Oberst-Riquier algorithm uses Gröbner bases for determining a *standard monomial* set (Cox et al. (2007)), and then, the initial/boundary data is provided by specifying the values at the origin of various derivatives – corresponding to these standard monomials – of the trajectory to be evaluated. However, such initial/boundary data, based on standard monomials, often lacks a physical interpretation, which might be of crucial importance for problems arising in engineering. This is because the shape of a standard monomial set is often arbitrary, depending on the given system of equations and the term ordering used for computing the requisite Gröbner basis.

A more useful initial/boundary data comes in the form of restrictions of trajectories to proper subsets of the domain having some algebraic structure. For example, such restrictions of trajectories to a subset of the domain play an important role in systems theory, namely, boundary control (Krstic and Smyshlyaev (2008)), causality (Fornasini and Marchesini (1976); Fornasini et al. (1993)), stability analysis (Pillai and Shankar (1998); Valcher (2000); Oberst (2006)), controller design (Shankar (2000)) etc. In this paper, we attempt to bridge this gap by considering subsets of the domain having some algebraic structure (in the form of subspaces and strips of finite width around such subspaces) such that trajectories restricted to these subsets serve as initial/boundary conditions for a given 2-D autonomous system. Such sets are known as *characteristic sets* in the literature. A characteristic set is a proper subset of the domain (here $\mathbb{R}^n$) with the defining property that trajectories restricted to this subset allows to uniquely extend the trajectory over the whole domain. We provide, in this paper, a systematic way to construct characteristic sets (of the form as mentioned above) for continuous scalar 2-D autonomous systems with the help of Gröbner bases and Oberst-Riquier algorithm.

The notion of characteristic sets was originally developed for discrete 2-D systems (that is, for systems of partial difference equations having two independent variables) in Valcher (2000), and was later extended to arbitrary dimensions in Mukherjee and Pal (2016, 2017, 2019). It was shown in Pal (2017) that every dicrete autonomous 2-D system admits a characteristic set given by a union of finitely many parallel lines. This rightly generalizes the 1-D case where a characteristic set is always a collection of finitely many points of the domain (Willems (1991)). In this paper, we prove the continuous analogue for 2-D

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autonomous systems. It is important to note, here, that the extension to the continuous case requires significant developments over the discrete case. This is primarily because of the difference in action of a discrete and a continuous operator on a trajectory – in the discrete case the operator action is a shift action while for a continuous case the operator acts on a trajectory by differentiation.

An autonomous *scalar* system of PDEs having two independent variables can either be finite dimensional or be infinite dimensional depending on the *Krull dimension* of the system. It is known that in the finite dimensional case, that is when the system has Krull dimension equal to zero, a characteristic set for the system is a collection of finitely many points of the domain $\mathbb{R}^2$ (see Fornasini et al. (1993)). For the case when the Krull dimension of the system is equal to one, it is not clear what kind of subsets of $\mathbb{R}^2$ can then qualify as a characteristic set. We answer this question in this paper. We show that a rectangular strip of finite width containing a 1-D subspace, having the special property of being *free* with respect to the system, is a characteristic set for the system. We also show that *every* scalar 2-D autonomous system admits a characteristic set given by a rectangular strip of finite width around a subspace.

The paper is organized as follows. Section 2 discusses the preliminaries and sets the notation to be used for the rest of the paper. In Section 3 we characterize initial/boundary data using the notion of characteristic sets. Section 4 shows that every scalar system of PDEs admits a characteristic set given by a rectangular strip of finite width containing a free subspace.

2. NOTATION AND PRELIMINARIES

2.1 Notation

We use the symbols $\mathbb{N}$ and $\mathbb{R}$ to denote the set of natural numbers and the field of real numbers, respectively. We use the shorthand ∂_i to denote differentiation with respect to the independent variable x_i, that is $\partial_i := \frac{\partial}{\partial x_i}$, for $i = 1, 2$. The polynomial ring in two indeterminates is denoted by $\mathbb{R}[\partial]$, where $\partial = (\partial_1, \partial_2)$. We use the symbol $\langle , \rangle$ to denote the standard inner product in $\mathbb{R}^2$.

2.2 System Description

In this paper, we consider linear systems of partial differential equations (PDEs) with real constant coefficients having one dependent variable (i.e., a scalar system) and evolving over two independent variables. Consider the system of PDEs given by

$$\begin{bmatrix} f_1(\partial) \\ f_2(\partial) \\ \vdots \\ f_r(\partial) \end{bmatrix} w = 0, \tag{1}$$

where $f_i \in \mathbb{R}[\partial]$ for $i \in \{1, 2, \dots, r\}$. The solution to this system of PDEs, w, is called a *trajectory.* A trajectory w is a real valued function from the *domain* $\mathbb{R}^2$ to $\mathbb{R}$, that is $w : \mathbb{R}^2 \to \mathbb{R}$. Such a function can belong to the space of analytic functions, the space of exponential functions and so on. It is important to choose a *solution space* for our analysis. We consider real entire analytic solutions of the exponential type as defined below.

Definition 2.1. We denote by $\mathfrak{Exp}(\mathbb{R}^2, \mathbb{R})$ the set of all formal power series in two variables

$$w(x) = \sum_{\nu \in \mathbb{N}^2} \frac{w_\nu}{\nu!} x^\nu,$$

where $\nu := (\nu_1, \nu_2)$ is a 2-tuple, $x^\nu := x_1^{\nu_1} x_2^{\nu_2}$ and $\nu! := \nu_1! \nu_2!$ with the sequence of real numbers $\{w_\nu\}_{\nu \in \mathbb{N}^2}$ being such that w is convergent at all points of the domain, that is $w(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^2$.

Following Willems (Willems (1991)), the collection of all trajectories that satisfy a given system of PDEs is called the *behavior*, $\mathfrak{B}$, of the system. That is,

$$\mathfrak{B} = \left\{ w \in \mathfrak{Exp}(\mathbb{R}^2, \mathbb{R}) \,\middle|\, f_1(\partial) w = \ldots = f_r(\partial) w = 0 \right\}. \tag{2}$$

It has been shown in Oberst (1990) that the set of exponential trajectories, $\mathfrak{Exp}(\mathbb{R}^2, \mathbb{R})$, is a *large injective cogenerator.* Therefore, the one-to-one correspondence between ideals in $\mathbb{R}[\partial]$ and scalar autonomous behaviors can be utilized here. In other words, given a system of PDEs as in equation (1), let $\mathfrak{a} := \langle f_1, f_2, \dots, f_r \rangle \subseteq \mathbb{R}[\partial]$ be the *equation ideal* generated by the describing PDEs. Then

$$\mathfrak{B}(\mathfrak{a}) := \left\{ w \in \mathfrak{Exp}(\mathbb{R}^2, \mathbb{R}) \,\middle|\, f(\partial) w = 0 \ \forall\ f \in \mathfrak{a} \right\} = \mathfrak{B}. \tag{3}$$

Given the equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$, define the *quotient ring* $\mathcal{M} := \mathbb{R}[\partial]/\mathfrak{a}$, as the set of all equivalence classes defined by the following equivalence relation: $f_1, f_2 \in \mathbb{R}[\partial]$ are related if and only if $f_1 - f_2 \in \mathfrak{a}$. For an element $f \in \mathbb{R}[\partial]$, the equivalence class of f is denoted by $\overline{f}$. This gives the *canonical surjection* $\mathbb{R}[\partial] \twoheadrightarrow \mathcal{M}$, where every element is mapped to its equivalence class.

For an $\mathcal{A}$-module $\mathcal{M}$, the *annihilator ideal* is defined as

$$\operatorname{ann}_{\mathcal{A}} \mathcal{M} := \{ f \in \mathcal{A} \mid fm = 0 \text{ for all } m \in \mathcal{M} \}. \tag{4}$$

Definition 2.2. An $\mathcal{A}$-module $\mathcal{M}$ is said to be a *faithful module* over $\mathcal{A}$ if $\operatorname{ann}_{\mathcal{A}} \mathcal{M} = \{0\}$.

In this paper, we consider *autonomous* systems. Such systems have been characterized using various equivalent conditions in the literature (see Pillai and Shankar (1998); Rocha and Willems (1989), Pommaret and Quadrat (1999), Zerz (2000) among others). It follows from the equivalent characterizations that a scalar 2-D system $\mathfrak{B}$ is autonomous if and only if the equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ is non-zero.

The *Krull dimension* of the system plays a crucial role in this paper. By Krull dimension of a system $\mathfrak{B}$, we mean the Krull dimension of the associated quotient ring $\mathcal{M}$. The Krull dimension of a ring $\mathcal{A}_1$ is defined to be the supremum of the lengths of chains of prime ideals in $\mathcal{A}_1$, where a chain of prime ideals of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_\ell$ is said to be of length ℓ.

2.3 Restriction of trajectories to 1-D subspaces

Restriction of trajectories plays an important role in this paper. In this section, we look at restriction of trajectories to 1-D subspaces of the domain.

Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero vector $v = [v_1\ v_2]^T \in \mathbb{R}^2$. That is,

$$\mathcal{V} := \left\{ x \in \mathbb{R}^2 \,\middle|\, x = vt, t \in \mathbb{R} \right\}. \tag{5}$$

Definition 2.3. For a 2-D system $\mathfrak{B}$ and a 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$ spanned by a non-zero vector $v \in \mathbb{R}^2$, the restriction of $\mathfrak{B}$ to $\mathcal{V}$ is defined as the following set of 1-D trajectories:

$$\mathfrak{B}|_{\mathcal{V}} := \{w(vt) \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid w \in \mathfrak{B}\}. \tag{6}$$

An immediate question would be whether this subspace is free with respect to the system. We first define free-ness in this context.

Definition 2.4. A subspace $\mathcal{V} \subseteq \mathbb{R}^2$ is said to be *free* with respect to a given 2-D system $\mathfrak{B}$ if the restriction of $\mathfrak{B}$ to $\mathcal{V}$, as defined in equation (6), is equal to the space of 1-D real entire functions of exponential type. That is, $\mathfrak{B}|_{\mathcal{V}} = \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$.

Later in the paper, we use the notion of vectors being free with respect to a system. By this, we mean the subspace spanned by the vector to be free with respect to the system, according to Definition 2.4.

A necessary and sufficient condition for free-ness of a 1-D subspace with respect to a scalar system was provided in Pal and Pillai (2014). This characterization plays a crucial role in this paper. To state the result some constructions are required which we first note.

Denote by $\langle v, \partial\rangle$ the linear polynomial $v_1\partial_1 + v_2\partial_2$. Note that, the polynomial $\langle v, \partial\rangle$ is transcendental over $\mathbb{R}$. Therefore, the $\mathbb{R}$-algebra $\mathbb{R}[\langle v, \partial\rangle]$ is isomorphic to the polynomial ring in one indeterminate. Further note that $\mathbb{R}[\langle v, \partial\rangle]$ is a sub-algebra of $\mathbb{R}[\partial]$. Consider the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[\langle v, \partial\rangle] \to \mathcal{M}$ defined as follows:

$$\begin{array}{cccccc} \Phi : \mathbb{R}[\langle v, \partial\rangle] & \hookrightarrow & \mathbb{R}[\partial] & \twoheadrightarrow & \mathcal{M} \\ p & \mapsto & p & \mapsto & \overline{p}. \end{array} \tag{7}$$

Observe that $\ker\, \Phi = \mathfrak{a} \cap \mathbb{R}[\langle v, \partial\rangle]$. Since Φ is an $\mathbb{R}$-algebra homomorphism, $\ker\, \Phi$ is an ideal in $\mathbb{R}[\langle v, \partial\rangle]$. We call this the *intersection ideal* of $\mathfrak{a}$ and denote it by $\mathfrak{a}_{\mathcal{V}}$, that is $\mathfrak{a}_{\mathcal{V}} := \mathfrak{a} \cap \mathbb{R}[\langle v, \partial\rangle]$. Corresponding to the intersection ideal $\mathfrak{a}_{\mathcal{V}}$, we have the following 1-D behavior

$$\mathfrak{B}_{\mathcal{V}} := \left\{ \widetilde{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \,\middle|\, f(\frac{d}{dt})\widetilde{w} = 0 \;\forall\; f(\langle v, \partial\rangle) \in \mathfrak{a}_{\mathcal{V}} \right\}. \tag{8}$$

Proposition 2.5 relates the 1-D behavior, as defined in equation (8), to the behavior restricted to a 1-D subspace as defined in equation (6). The proof of the result can be found in Pal and Pillai (2014).

Proposition 2.5. Consider a scalar autonomous 2-D system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$. Let $\mathfrak{B}$ restricted to $\mathcal{V}$ be as defined in Definition 2.3. Consider the 1-D behavior as defined in equation (8). Then $\mathfrak{B}|_{\mathcal{V}} \subseteq \mathfrak{B}_{\mathcal{V}}$.

We now return to the question of free-ness of a subspace with respect to a given behavior. Proposition 2.6 gives an algebraic characterization of free-ness. The proof can be found in Pal and Pillai (2014).

Proposition 2.6. Consider a scalar autonomous 2-D system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$. Then the following are equivalent:

(1) $\mathcal{V}$ is free with respect to $\mathfrak{B}$.
(2) The intersection ideal $\mathfrak{a}_{\mathcal{V}} = \mathfrak{a} \cap \mathbb{R}[\langle v, \partial\rangle]$ is the zero ideal.
(3) The $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[\langle v, \partial\rangle] \to \mathcal{M}$, as defined in equation (7), is injective.

It also follows from the characterization in Proposition 2.6 that free subspaces are abundant in $\mathbb{R}^2$. In other words, a 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$ chosen at random will almost always be free with respect to a given 2-D behavior $\mathfrak{B}$. For details and proof please see Pal and Pillai (2014).

3. CHARACTERIZATION OF INITIAL DATA

In this section, we provide a constructive way of specifying initial data for a given system of PDEs using the notion of characteristic sets. Characteristic sets were initially defined for discrete 2-D systems (Valcher (2000)) and later extended to n-D system in Mukherjee and Pal (2016, 2017, 2019). Here, we define a characteristic set for the continuous case, that is for autonomous systems described by linear PDEs with real constant coefficients.

3.1 Characteristic sets

Definition 3.1. Given a system $\mathfrak{B}$, a subset $\mathcal{S}$ of the domain is called a *characteristic set* for $\mathfrak{B}$ if for every trajectory $w \in \mathfrak{B}$, the restriction of w to $\mathcal{S}$ allows to uniquely determine the remaining portion of w, that is $w|_{\mathbb{R}^2\setminus\mathcal{S}}$ can be uniquely determined if $w|_{\mathcal{S}}$ is known.

It is often infeasible to acertain if a given arbitrary subset of the domain is a characteristic set. In this paper, we provide a way of *constructing* a characteristic set for a given scalar 2-D autonomous system $\mathfrak{B}$.

Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ and corresponding quotient ring $\mathcal{M}$. The Krull dimension of $\mathcal{M}$ as an $\mathbb{R}[\partial]$-module can either be zero or one. For the case when the Krull dimension of $\mathcal{M}$ equals zero, $\mathcal{M}$ becomes a finite dimensional vector space over $\mathbb{R}$. In that case, $\mathfrak{B}$ is a *strongly autonomous* system and a characteristic set for $\mathfrak{B}$ is a set of finitely many points of the domain (see Fornasini et al. (1993)). The other possibility for an autonomous 2-D system is to have Krull dimension of $\mathcal{M}$ to be equal to one. It is not yet known what kind of subsets of $\mathbb{R}^2$ can then qualify as a characteristic set in this case. We address this in the following section.

3.2 Rectangular strips of finite width as characteristic sets

In this section, we show that rectangular strips of finite width containing a 1-D subspace, which is free with respect to the system, is a characteristic set for the system. This result is proved by first showing that for a system $\mathfrak{B}$ having Krull dimension equal to one and a 1-D subspace $\mathcal{V}$ which is free with respect to $\mathfrak{B}$, following Definition 2.4, the quotient ring $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\langle v, \partial\rangle]$. This result uses the idea of *integral ring extension.* We briefly discuss this here; for details please refer to (Atiyah and MacDonald, 1969, Chapter 5).

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rings such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ as a subring. Then an element $\alpha \in \mathcal{A}_2$ is said to be *integral* over $\mathcal{A}_1$ if

α satisfies a monic polynomial equation with coefficients from $\mathcal{A}_1$. When $\mathcal{A}_1 \subseteq \mathcal{A}_2$ as a subring, $\mathcal{A}_2$ is said to be an *integral extension* of $\mathcal{A}_1$ if every element of $\mathcal{A}_2$ is integral over $\mathcal{A}_1$. Proposition 3.2 summarizes the results on integral ring extension required for this paper. For proofs, please see (Atiyah and MacDonald, 1969, Chapter 5).

Proposition 3.2. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rings. Further, let $\mathcal{A}_2$ be a finitely generated algebra over $\mathcal{A}_1$. Then the following are equivalent.

(1) $\mathcal{A}_2$ is integral over $\mathcal{A}_1$.
(2) $\mathcal{A}_2$ is a finitely generated module over $\mathcal{A}_1$.

Theorem 3.3. Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ and corresponding quotient ring $\mathcal{M}$ having Krull dimension equal to one. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero vector $v \in \mathbb{R}^2$ such that $\mathcal{V}$ is free with respect to $\mathfrak{B}$. Recall the sub-algebra $\mathbb{R}[\langle v, \partial\rangle]$ of $\mathbb{R}[\partial]$. Then $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\langle v, \partial\rangle]$.

Proof: Recall the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[\langle v, \partial\rangle] \to \mathcal{M}$ as defined in equation (7). Since $\mathcal{V}$ is free with respect to $\mathfrak{B}$, Φ is injective (Statement 3 of Proposition 2.6). This shows that $\mathcal{M}$ is a faithful module over $\mathbb{R}[\langle v, \partial\rangle]$.

Note that, $\mathcal{M}$ is naturally a finitely generated algebra over $\mathbb{R}$. Since $\mathbb{R} \subseteq \mathbb{R}[\langle v, \partial\rangle]$, $\mathcal{M}$ is a finitely generated algebra over $\mathbb{R}[\langle v, \partial\rangle]$ as well. To show that $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\langle v, \partial\rangle]$, it then suffices to show that $\Phi : \mathbb{R}[\langle v, \partial\rangle] \to \mathcal{M}$ is integral (Proposition 3.2). We prove this by contradiction. Suppose Φ is not integral. Then there exists an element $\xi \in \mathcal{M}$ transcendental over $\mathbb{R}[\langle v, \partial\rangle]$ such that $\mathbb{R}[\langle v, \partial\rangle] \subsetneq \mathbb{R}[\langle v, \partial\rangle, \xi] \subseteq \mathcal{M}$. Now Krull dimension of $\mathbb{R}[\langle v, \partial\rangle, \xi]$ is one more than the Krull dimension of $\mathbb{R}[\langle v, \partial\rangle]$. Again $\mathbb{R}[\langle v, \partial\rangle, \xi] \subseteq \mathcal{M}$ implies Krull dimension of $\mathbb{R}[\langle v, \partial\rangle, \xi]$ is less than or equal to the Krull dimension of $\mathcal{M}$ which contradicts the assumption that the Krull dimension of $\mathcal{M}$ is equal to one. $\square$

For a given 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$ spanned by a non-zero vector $v \in \mathbb{R}^2$, let $\mathcal{U} \subseteq \mathbb{R}^2$ be a complementary subspace (not necessarily orthogonal) of $\mathcal{V}$ spanned by $u \in \mathbb{R}^2$ such that $\mathbb{R}^2 = \mathcal{V} \oplus \mathcal{U}$. Note that, $T := [v\ u]$ is non-singular. Define the linear polynomial $\langle u, \partial\rangle := u_1\partial_1 + u_2\partial_2$. As before, $\langle u, \partial\rangle$ is transcendental over $\mathbb{R}$.

Lemma 3.4. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be a complementary subspace of $\mathcal{V}$ spanned by $u \in \mathbb{R}^2$ such that $\mathbb{R}^2 = \mathcal{V} \oplus \mathcal{U}$. Then $\mathbb{R}[\partial] = \mathbb{R}[\langle v, \partial\rangle, \langle u, \partial\rangle]$ as $\mathbb{R}$-algebras.

Proof: Note that, $\mathbb{R}[\langle v, \partial\rangle, \langle u, \partial\rangle] \subseteq \mathbb{R}[\partial]$. To show $\mathbb{R}[\partial] \subseteq \mathbb{R}[\langle v, \partial\rangle, \langle u, \partial\rangle]$, we need to show that ∂_is can be expressed as $\mathbb{R}$-linear combinations of $\langle v, \partial\rangle$ and $\langle u, \partial\rangle$ for $i = 1, 2$. In other words, we need to show that there exists a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} = A \begin{bmatrix} \langle v, \partial\rangle \\ \langle u, \partial\rangle \end{bmatrix}.$$

Define $T := [v\ u] \in \mathbb{R}^{2\times 2}$. Note that, T is non-singular. It can be easily verified that $A = \left(T^T\right)^{-1}$, where T^T is the transpose of T. $\square$

We now state Corollary 3.5 which follows from Theorem 3.3 and Lemma 3.4. The proof follows from standard results in integral ring extension. For details please refer to (Atiyah and MacDonald, 1969, Chapter 5).

We use the shorthands $\partial_u := \langle u, \partial\rangle$ and $\partial_v := \langle v, \partial\rangle$ henceforth.

Corollary 3.5. Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ and corresponding quotient ring $\mathcal{M}$ having Krull dimension equal to one. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$ such that $\mathcal{V}$ is free with respect to $\mathfrak{B}$. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be a complementary subspace of $\mathcal{V}$ spanned by $u \in \mathbb{R}^2$. Then the following are true:

(1) $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\langle v, \partial\rangle]$.
(2) There exists $d \in \mathbb{N}$, such that
$$\partial_u^d + a_{d-1}(\partial_v)\partial_u^{d-1} + \ldots + a_1(\partial_v)\partial_u + a_0(\partial_v) \in \mathfrak{a}$$
where $a_i \in \mathbb{R}[\partial_v]$ for $i \in \{0, 1, \ldots, d-1\}$.

Using Corollary 3.5, we prove Lemma 3.6 which gives an explicit list of generators for $\mathcal{M}$ as a module over $\mathbb{R}[\langle v, \partial\rangle]$.

Lemma 3.6. Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ and corresponding quotient ring $\mathcal{M}$ having Krull dimension equal to one. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$ such that $\mathcal{V}$ is free with respect to $\mathfrak{B}$. Then there exist $u \in \mathbb{R}^2$ and $d \in \mathbb{N}$ such that

$$\left\{ \overline{\partial_u^j} \,\middle|\, 0 \leqslant j \leqslant d-1 \right\} \tag{9}$$

is a generating set for $\mathbb{R}[\partial]/\mathfrak{a}$ as a module over $\mathbb{R}[\langle v, \partial\rangle]$.

Proof: Since Krull dimension of $\mathcal{M}$ is equal to one and $\mathcal{V} \subseteq \mathbb{R}^2$ is a 1-D subspace it follows from Theorem 3.3 that $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\langle v, \partial\rangle]$. It follows from Lemma 3.4 that, there exists a complementary subspace $\mathcal{U} \subseteq \mathbb{R}^2$ of $\mathcal{V}$ spanned by $u \in \mathbb{R}^2$ such that $\mathbb{R}[\partial] = \mathbb{R}[\langle v, \partial\rangle, \langle u, \partial\rangle]$. Define $\partial_v := \langle v, \partial\rangle$ and $\partial_u := \langle u, \partial\rangle$. Then every polynomial in $\mathbb{R}[\partial]$ can be rewritten as a finite $\mathbb{R}$-linear combination of monomials of the form $\partial_v^a\partial_u^b$, where $a, b \in \mathbb{N}$. Since $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\langle v, \partial\rangle]$, according to Corollary 3.5, there exists $d \in \mathbb{N}$ such that

$$p(\partial_v, \partial_u) := \partial_u^d + a_{d-1}(\partial_v)\partial_u^{d-1} + \ldots + a_1(\partial_v)\partial_u + a_0(\partial_v) \in \mathfrak{a},$$

where $a_i \in \mathbb{R}[\partial_v]$ for $i \in \{0, 1, \ldots, d-1\}$. Now $p(\partial_v, \partial_u)$ being a monic polynomial in ∂_u, $\partial_v^a\partial_u^b$ can be divided by $p(\partial_v, \partial_u)$ using the Euclidean division algorithm. Thus we have

$$\partial_v^a\partial_u^b = p(\partial_v, \partial_u)q(\partial_v, \partial_u) + r(\partial_v, \partial_u),$$

where $p(\partial_v, \partial_u)q(\partial_v, \partial_u) \in \mathfrak{a}$ and the remainder $r(\partial_v, \partial_u)$ is an $\mathbb{R}[\partial_v]$-linear combination of monomials $\{1, \partial_u, \ldots, \partial_u^{d-1}\}$. Under the canonical surjection we have $\overline{\partial_v^a\partial_u^b} = \overline{r(\partial_v, \partial_u)}$. Thus every monomial $\overline{\partial_v^a\partial_u^b}$, where $a, b \in \mathbb{N}$, is equal to an $\mathbb{R}[\partial_v]$-linear combination of monomials $\{\overline{1}, \overline{\partial_u}, \ldots, \overline{\partial_u^{d-1}}\}$. $\square$

Theorem 3.7. Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ and corresponding quotient ring $\mathcal{M}$ having Krull dimension equal to one. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be a 1-D subspace spanned by a non-zero $v \in \mathbb{R}^2$ such that $\mathcal{V}$ is free with respect to $\mathfrak{B}$. Then a rectangular strip of finite width containing $\mathcal{V}$ is a characteristic set for $\mathfrak{B}$.

Proof: To show that a characteristic set for $\mathfrak{B}$ is given by a rectangular strip of finite width containing $\mathcal{V}$ we show that to evaluate a trajectory at an arbitrary point $x = (x_1, x_2) \in \mathbb{R}^2$, the information of the trajectory on this rectangular strip is sufficient. Recall, from Definition 2.1, that, $w(x)$ can be computed if w_ν is known for all $\nu \in \mathbb{N}^2$. Note that, $w_\nu = (\partial^\nu)\, w|_{x=0}$.

Applying Lemma 3.4, the solution trajectory in the transformed domain can be written as

$$w(x) = \sum_{\nu \in \mathbb{N}^2} \frac{\widetilde{w}_\nu}{\nu!} \xi^\nu,$$

where $x = T\xi$ with $T = [v\ u] \in \mathbb{R}^{2\times 2}$ and $\widetilde{w}_\nu = (\partial_v^{\nu_1}\partial_u^{\nu_2})\, w|_{x=0}$. Using Lemma 3.6, $\partial_v^{\nu_1}\partial_u^{\nu_2}$ is equivalent modulo the equation ideal to a finite $\mathbb{R}[\partial_v]$-linear combination of monomials $1, \partial_u, \ldots, \partial_u^{d-1}$ for some $d \in \mathbb{N}$. Thus to calculate $\widetilde{w}_\nu$ we must know the action of these monomials on a trajectory $w \in \mathfrak{B}$ evaluated at $x = 0$. In other words,

$$\left\{ (\partial_u^i \partial_v^j)\, w|_{x=0} \;\middle|\; 0 \leqslant i \leqslant d-1, j \in \mathbb{N} \right\} \tag{10}$$

lets us uniquely determine w at an arbitrary $x \in \mathbb{R}^2$.

Let $\mathcal{S} \subseteq \mathbb{R}^2$ be a strip containing the subspace $\mathcal{V}$ spanned by a non-zero $v \in \mathbb{R}^2$. That is,

$$\begin{aligned} \mathcal{S} := \big\{ x \in \mathbb{R}^2 \;\big|\; \exists\, \xi_1, \xi_2 \in \mathbb{R} \text{ such that } x = \xi_1 v + \xi_2 u, \\ \text{where } -\epsilon \leqslant \xi_2 \leqslant \epsilon \big\}. \end{aligned} \tag{11}$$

Then for all $x \in \mathcal{S}$, the action of $\partial_u^i \partial_v^j$ on w evaluated at x is given by

$$(\partial_u^i \partial_v^j)\, w|_x = \left((\partial_u^i \partial_v^j)\, w \right)(\xi_1 v + \xi_2 u) = \frac{\partial^i}{\partial \xi_2^i}\left(\frac{\partial^j w_T}{\partial \xi_1^j} \right) \tag{12}$$

where $w_T := w \circ T$. Thus w_T must be known for all $\xi_1 \in \mathbb{R}$ and $-\epsilon \leqslant \xi_2 \leqslant \epsilon$. Therefore the initial data, as specified in equation (10), can be calculated from $w|_{\mathcal{S}}$ which in turn helps in uniquely calculating w. $\square$

Remark 3.8. Note that, a characteristic set only refers to a subset of the domain having some special property. The property of uniquely extending the trajectory to the whole domain requires an *algorithm* for computing trajectories at various points in the domain. One such algorithm is the Oberst-Riquier algorithm as stated in (Pal and Pillai, 2014, Algorithm 22).

Remark 3.9. The Oberst-Riquier algorithm crucially uses Gröbner basis for computing a standard monomial set. A characteristic set is analogous to the standard monomial set having some nice structure and useful system theoretic properties.

4. EXISTENCE OF A RECTANGULAR STRIP OF FINITE WIDTH AS A CHARACTERISTIC SET

In Section 3.2 we have shown that for a given scalar autonomous 2-D system $\mathfrak{B}$ and a 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$, which is free with respect to $\mathfrak{B}$, a characteristic set for $\mathfrak{B}$ is given by a rectangular strip of finite width containing this 1-D subspace. In this section, we provide a systematic way of finding a 1-D subspace, with the desired specification of being free with respect to the given 2-D system, such that a characteristic set can be constructed using it. This follows from the application of a well-known result in commutative algebra – the Noether's Normalization Lemma (see Atiyah and MacDonald (1969)).

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear map represented by a non-singular matrix

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Let x and y denote the co-ordinate functions of the domain and co-domain, respectively. Then $y = Tx$. Define the T-induced linear map between the tangent spaces, $T^* : \mathcal{T}_x\mathbb{R}^2 \to \mathcal{T}_y\mathbb{R}^2$, in the following way. Let $y \mapsto w(y)$ be in $\mathfrak{Exp}(\mathbb{R}^2, \mathbb{R})$. Then for $i = 1, 2$,

$$\left(T^* \frac{\partial}{\partial x_i} \right)(w(y)) := \frac{\partial}{\partial x_i} w(Tx). \tag{13}$$

Applying the definition of T^* (equation (13)) to the co-ordinate function y_js, we have

$$\left(T^* \frac{\partial}{\partial x_i} \right) y_j = \frac{\partial}{\partial x_i} \sum_{k=1}^{2} t_{jk} x_k = t_{ji}.$$

By varying j, we have $\left(T^* \frac{\partial}{\partial x_i} \right) = \sum_{k=1}^2 t_{ji} \frac{\partial}{\partial y_j}$. Thus for $w \in \mathfrak{Exp}(\mathbb{R}^2, \mathbb{R})$, we have

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} w(Tx) = T^* \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} w(y) = T^T \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{bmatrix} w(y), \tag{14}$$

where T^T denotes the transpose of T. Define $\partial_x := \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ and $\partial_y := \{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$. Define the T-induced $\mathbb{R}$-algebra homomorphism $\psi : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$ defined in the following way

$$\begin{aligned} \psi : \mathbb{R}[\partial_x] &\to \mathbb{R}[\partial_y] \\ \partial_x &\mapsto T^T \partial_y. \end{aligned} \tag{15}$$

Since T is non-singular, T^T is also non-singular and thus ψ is an isomorphism of $\mathbb{R}$-algebras. Lemma 4.1 below relates the behaviors corresponding to equations written using the partial differential operators ∂_x and ∂_y owing to the co-ordinate transformation of the domain. The proof can be found in Pal and Pillai (2014).

Lemma 4.1. Let $T \in \mathbb{R}^{2\times 2}$ be a linear co-ordinate transformation on the domain $\mathbb{R}^2$ defined by $x \mapsto Tx =: y$. Consider the $\mathbb{R}$-algebra isomorphism $\psi : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$, induced by T, as defined in equation (15). Let $\mathfrak{a} \subseteq \mathbb{R}[\partial_x]$ be an ideal. Then $\psi(\mathfrak{a}) \subseteq \mathbb{R}[\partial_y]$ is also an ideal. Consider the behaviors

$$\mathfrak{B}_x := \left\{ w(x) \in \mathfrak{Exp}(\mathbb{R}^2, \mathbb{R}) \;\middle|\; m(\partial_x) w = 0 \ \forall\, m \in \mathfrak{a} \right\},$$
$$\mathfrak{B}_y := \left\{ w(y) \in \mathfrak{Exp}(\mathbb{R}^2, \mathbb{R}) \;\middle|\; m(\partial_y) w = 0 \ \forall\, m \in \psi(\mathfrak{a}) \right\}.$$

Let $v_x, v_y \in \mathbb{R}^2$ be such that $v_y = Tv_x$. Then there is a bijective map between $\mathfrak{B}_y|_{v_y}$ and $\mathfrak{B}_x|_{v_x}$ in the set-theoretic sense.

We now state a version of the Noether's normalization lemma suited for our purpose in Proposition 4.2 below. For details please refer to (Kreuzer and Robbiano, 2000, Tutorial 78, item ℓ).

Proposition 4.2. Let $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ be a non-zero ideal. Then there exists a non-singular matrix $T \in \mathbb{R}^{2\times 2}$ defined by $x \mapsto Tx =: y$ and the corresponding T-induced map $\psi : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$, such that $\mathbb{R}[\partial_y]/\psi(\mathfrak{a})$ is a finitely generated faithful module over $\mathbb{R}[\partial_{y_1}]$, where $\partial_{y_1} := \frac{\partial}{\partial y_1}$.

We now prove Theorem 4.3.

Theorem 4.3. Consider a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$ having Krull dimension equal to one. Then there exists a 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$ such that $\mathcal{V}$ is free with respect to $\mathfrak{B}$.

Proof: Let $T \in \mathbb{R}^{2\times 2}$ be the non-singular matrix representing the co-ordinate transformation on the domain $\mathbb{R}^2$ as defined in Proposition 4.2. It further follows from Proposition 4.2 that the $\mathbb{R}$-linear map $\mathbb{R}[\partial_{y_1}] \to \mathbb{R}[\partial_y]/\psi(\mathfrak{a})$ is injective and integral. Injectivity of the map implies that $e_1 = [1\ 0]^T$ is free with respect to $\mathfrak{B}_y$. Then $v := T^{-1}e_1$ is free with respect to $\mathfrak{B}$. Thus the 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$ that is free with respect to $\mathfrak{B}$ is given by the span of $v \in \mathbb{R}^2$. $\square$

Therefore, given a 2-D autonomous system $\mathfrak{B}$ with equation ideal $\mathfrak{a} \subseteq \mathbb{R}[\partial]$, there exists a non-singular matrix $T \in \mathbb{R}^{2\times 2}$ defined by $x \mapsto Tx =: y$ and the corresponding T-induced map $\psi : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$, such that $\mathbb{R}[\partial_y]/\psi(\mathfrak{a})$ is a finitely generated faithful module over $\mathbb{R}[\partial_{y_1}]$. Using Theorem 3.7, a characteristic set for $\mathfrak{B}_y$ is given by a rectangular strip of finite width containing e_1 in the transformed domain. Applying the inverse transformation we obtain a characteristic set for $\mathfrak{B}$. Thus, every scalar 2-D autonomous system, described by a set of linear PDEs having exponential solution, admits a characteristic set given by a rectangular strip of finite width containing a free subspace.

Remark 4.4. The theory of Gröbner bases can be applied for implementation and verification of the results presented in this paper.

5. CONCLUDING REMARKS

In this paper, we provided a method of constructing a region (in the form of subspaces and strips of finite width around such subspaces) in the domain such that trajectories restricted to these regions serve as initial/boundary data for the given scalar 2-D autonomous system of PDEs. Note that, we have considered only exponential trajectories; on bigger function spaces – like infinitely often differentiable functions or distributions – the main results of this paper often fail to hold. Formulating characteristic sets for these function spaces would be a matter of future research.

REFERENCES

Atiyah, M. and MacDonald, I. (1969). *Introduction to Commutative Algebra.* Addison-Wesley Publishing Company, Britain.

Cox, D., Little, J., and O'Shea, D. (2007). *Ideals, Varieties and Algorithms.* Springer, NY.

Fornasini, E. and Marchesini, G. (1976). State-space realization theory of 2-Dimensional filters. *IEEE Transactions on Automatic Control*, AC-21(4), 484–492.

Fornasini, E., Rocha, P., and Zampieri, S. (1993). State space realizations of 2-D finite-dimensional behaviours. *SIAM Journal on Control and Optimization*, 31(6), 1502–1517.

Kreuzer, M. and Robbiano, L. (2000). *Computational Commutative Algebra 2.* Springer.

Krstic, M. and Smyshlyaev, A. (2008). *Boundary Control of PDEs: A course on backstepping designs.* SIAM, Philadelphia.

Mukherjee, M. and Pal, D. (2016). On characteristic cones of scalar autonomous nD systems, with general n. *In the Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, 839–845.

Mukherjee, M. and Pal, D. (2017). Algorithms for verification of characteristic sets of discrete autonomous nD systems with $n \geqslant 2$. *In the Proceedings of the 20th IFAC World Congress*, 50(1), 1840–1846.

Mukherjee, M. and Pal, D. (2019). On characteristic cones of discrete nD autonomous systems: theory and an algorithm. *Multidimensional Systems and Signal Processing*, 30(2), 611–640.

Oberst, U. (1990). Multidimensional constant linear systems. *Acta Applicandae Mathematicae*, 20, 1–175.

Oberst, U. (2006). Stability and stabilization of multidimensional input/output systems. *SIAM Journal on Control and Optimization*, 45(4), 1467–1507.

Pal, D. (2017). Every discrete 2D autonomous system admits a finite union of parallel lines as a characteristic set. *Multidimensional Systems and Signal Processing*, 28(1), 49–73.

Pal, D. and Pillai, H.K. (2014). On restrictions of n-D systems to 1-d subspaces. *Multidimensional Systems and Signal Processing*, 25, 115–144.

Pillai, H.K. and Shankar, S. (1998). A behavioral approach to control of distributed systems. *SIAM Journal on Control and Optimization*, 37(2), 388–408.

Pommaret, J.F. and Quadrat, A. (1999). Algebraic analysis of linear multidimensional control systems. *IMA Journal of Mathematical Control and Information*, 16, 275–297.

Rocha, P. and Willems, J.C. (1989). State for 2-D systems. *Linear Algebra and its Applications*, 122-124, 1003–1038.

Shankar, S. (2000). Can one control the vibrations of a drum? *Multidimensional Systems and Signal Processing*, 11, 67–81.

Valcher, M.E. (2000). Characteristic cones and stability properties of two-dimensional autonomous behaviors. *IEEE Transactions on Circuits and Systems – Part I: Fundamental Theory and Applications*, 47(3), 290–302.

Willems, J.C. (1991). Paradigms and puzzles in theory of dynamical systems. *IEEE Transactions on Automatic Control*, 36(6), 259–294.

Zerz, E. (2000). *Topics in Multidimensional Linear Systems Theory.* Springer, London.