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Real-Time Optimization of Periodic Systems: A Modifier-Adaptation Approach

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Abstract: Modifier-Adaptation methodologies have been widely used to overcome plant-model mismatch and control a system to its steady-state optimal setpoint. They use gradient information of the real plant to design modifiers that correct the model, so that the first order necessary conditions for optimality of the model-based problem converge to those of the optimal one. In this paper, we get rid of the hypothesis that the plant optimum needs to be an equilibrium point. Instead, we only require it to be a periodic trajectory. We show the behaviour of the proposed approach by means of a motivating example that highlights the necessity of this formulation in cases where the system changes periodically through time.

Keywords: Model predictive control (MPC). Dynamic real-time optimization (DRTO). Economic design. Modifier-adaptation. Uncertainty. Nonlinear systems. Periodic references.

1. INTRODUCTION

Economic control of industrial systems has always been of great relevance in both literature and industry. The two-layer control scheme has become widely used for its simplicity and effectiveness. In this scheme, the upper layer, also known as real time optimizer (RTO), computes the state/trajectory which optimizes the economic cost of the plant, which in turn is provided to the lower controller as a reference to be tracked. Although there is a variety of solutions to the two-layer control, most of them need a model of the system, which can be obtained e.g. from historical data. These models differ from the real plant behaviour, since it is in general impossible to capture every minor dynamic or perturbance, even in small systems. In the lower layer, model predictive controllers (MPC) can deal with the plant-model mismatch by means of disturbance estimators, which modify the model to make its dynamic response converge to the plant's one locally. Modifier-adaptation methodologies have been widely studied [Chachuat et al. (2009), Marchetti et al. (2010), Marchetti et al. (2016), Rodríguez-Blanco et al. (2018)] and arise to modify the upper layer of control so that the necessary conditions for optimality (NCO) of the model-based problem match with those of the original one. By combining both, modifier-adaptation and disturbance estimators, an offset-free optimization of the steady-state economic cost can be achieved as shown in [Vaccari and Pannocchia (2017, 2018)].

In this paper we generalize these results, allowing the steady behaviour to be a periodic trajectory instead of a single equilibrium point. For this purpose, we use a dynamic RTO to compute the optimal economic trajectory of the (modified) model-based problem and an offset-free MPC to control the system to the trajectory found by the dynamic RTO. We show the modifiers required to make the model match the NCO of the original problem. The computation of the plant's gradients (needed for the computation of the modifiers) is beyond the scope of this paper. To show the performance of the proposed approach, we test it on the numerical example of the quadruple-tank process. Unlike state of the art approaches, ours is able to compute the periodic optimal trajectory of the plant and control the system to it with the knowledge of the gradients of the real plant.

The structure of the paper is the following: In Section 2 we study the formulation of the optimal problem and the its simplifications to make it tractable. Later, in Section 3 we analyse how to modify the model-based problem to make it match the first order NCO of the optimal one. Section 4 shows how to design the MPC so that it follows the periodic trajectory given by the RTO. In Section 5 we study in depth the two-layer control scheme and give details about its layers. Section 6 shows the algorithm proposed, which is used in Section 7 to solve a motivational example. Finally, Section 8 discusses about the relevance of the results and future works.

2. PROBLEM FORMULATION

Let $\mathbf{x}_T = [x_1,\ x_2,\ \cdots,\ x_T] \in \mathcal{R}^{Tn_x}$, $\mathbf{y}_T = [y_0,\ y_1,\ \cdots,\ y_{T-1}] \in \mathcal{R}^{Tn_y}$ and $\mathbf{u}_T = [u_0,\ u_1,\ \cdots,\ u_{T-1}] \in \mathcal{R}^{Tn_u}$ be the sequence of states, outputs and inputs of the system. Consider that the real system is defined by the following discrete-time T-steps ahead equations:

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$$\begin{aligned} &\mathbf{x}_T = F_X(x_0, \mathbf{u}_T) \\ &\mathbf{y}_T = F_Y(x_0, \mathbf{u}_T) \\ &H(\mathbf{x}_T, \mathbf{u}_T) = 0 \\ &G(\mathbf{x}_T, \mathbf{u}_T) \leq 0 \end{aligned} \tag{1}$$

The optimal economic control of the system consists of finding the sequence of inputs that reduce a given economic cost $\phi(y_k, u_k)$ over the time. That is, the optimal economic control of this system given the initial state x_0 can be expressed as the following optimization problem:

$$\begin{aligned} \min_{\mathbf{u}_\infty} \quad & \sum_{k=0}^{\infty} \phi(y_k, u_k) = \Phi(\mathbf{y}_\infty, \mathbf{u}_\infty) \\ \text{s.t.} \quad & \mathbf{x}_\infty = F_X(x_0, \mathbf{u}_\infty) \\ & \mathbf{y}_\infty = F_Y(x_0, \mathbf{u}_\infty) \\ & H(\mathbf{x}_\infty, \mathbf{u}_\infty) = 0 \\ & G(\mathbf{x}_\infty, \mathbf{u}_\infty) \leq 0. \end{aligned} \tag{2}$$

This ideal problem can not be implemented in general, since the number of decision variables is infinite and the functions F_X, F_Y, G and H that define the real system are usually unknown. To simplify the infinite horizon problem, we could take a practical approach such as an equilibrium state strategy, in which we look for the equilibrium state of the system that minimizes the economic cost and control the system to it. Instead, in this paper we follow a periodic strategy, that is, we find the best periodic trajectory and repeat it through time. Note that this always results in same or better performances than the equilibrium state strategy, since it is a generalized version of it, and, in the case of periodic systems with period T, is equivalent to the optimal problem (2) as proven in [Limon et al. (2014)]. Following a two-layer control scheme for this periodic approach, the optimal control problem (2) transforms into both the trajectory planner:

$$\min_{x_0, \mathbf{u}_T} \quad \Phi(\mathbf{y}_T, \mathbf{u}_T) \tag{3a}$$
$$\text{s.t.} \quad \mathbf{x}_T = F_X(x_0, \mathbf{u}_T) \tag{3b}$$
$$\mathbf{y}_T = F_Y(x_0, \mathbf{u}_T) \tag{3c}$$
$$H(\mathbf{x}_T, \mathbf{u}_T) = 0 \tag{3d}$$
$$G(\mathbf{x}_T, \mathbf{u}_T) \leq 0 \tag{3e}$$
$$x_0 = x_T, \tag{3f}$$

and a lower layer of control that would take the current state of the system to the path found by the trajectory planner. Note that the expression (3f) corresponds to the periodic constraint.

Although the previous formulation solves the optimal problem for the case of periodic systems, it requires the knowledge of the real system equations (1), which as commented before are unknown in practice. To deal with that, we use a model of the system and follow a two-layer control scheme. Consider that the discrete-time T-steps ahead model of system (1) is defined by:

$$\begin{aligned} &\hat{\mathbf{x}}_T = \hat{F}_X(x_0, \mathbf{u}_T) \\ &\hat{\mathbf{y}}_T = \hat{F}_Y(x_0, \mathbf{u}_T) \\ &\hat{H}(\hat{\mathbf{x}}_T, \mathbf{u}_T) = 0 \\ &\hat{G}(\hat{\mathbf{x}}_T, \mathbf{u}_T) \leq 0. \end{aligned} \tag{4}$$

Following a periodic strategy such as the one followed in (3), we define the dynamic RTO of the periodic model-based economic problem (P-MEP) as:

$$\begin{aligned} \min_{x_0, \mathbf{u}_T} \quad & \Phi(\hat{\mathbf{y}}_T, \mathbf{u}_T) \\ \text{s.t.} \quad & \hat{\mathbf{x}}_T = \hat{F}_X(x_0, \mathbf{u}_T) \\ & \hat{\mathbf{y}}_T = \hat{F}_Y(x_0, \mathbf{u}_T) \\ & \hat{H}(\hat{\mathbf{x}}_T, \mathbf{u}_T) = 0 \\ & \hat{G}(\hat{\mathbf{x}}_T, \mathbf{u}_T) \leq 0 \\ & \hat{x}_T = x_0. \end{aligned} \tag{5}$$

Given the initial state x_0 and the reference trajectory $\{\mathbf{x}_N^r, \mathbf{u}_N^r\}$ from (5), the MPC of the P-MEP is defined as:

$$\begin{aligned} \min_{\mathbf{u}_N} \quad & \ell(\hat{\mathbf{x}}_N, \mathbf{u}_N, \hat{\mathbf{x}}_N^r, \mathbf{u}_N^r) \\ \text{s.t.} \quad & \hat{\mathbf{x}}_N = \hat{F}_X(x_0, \mathbf{u}_N) \\ & \hat{h}_c(\hat{\mathbf{x}}_N, \mathbf{u}_N) = 0 \\ & \hat{g}_c(\hat{\mathbf{x}}_N, \mathbf{u}_N) \leq 0, \end{aligned} \tag{6}$$

where N is the control horizon considered and ℓ is a cost function that penalizes the distance to the trajectory $\{\mathbf{x}_N^r, \mathbf{u}_N^r\}$.

Despite the fact that the model-based economic problem (5, 6) is convenient for implementation, the model used makes the real plant converge to a different solution from the optimal economic problem (2), even when the real system is periodic. For this reason, we take a modifier-adaptation (MA) approach that modifies the model, so that the trajectory found by the P-MEP is the same that the one found by the optimal of problem (5) is the same that the optimal periodic one (3).

In the next section, we study how to modify the model to accomplish this.

3. KKT MATCHING

In this section we analyze the first order necessary conditions of optimality, also known as KKT conditions, for the two problems discussed above. For the sake of simplicity and comparison, we define $\theta = \begin{bmatrix} x_0 \\ \mathbf{u}_T \end{bmatrix}$ and compare problem (5) to problem (3), which, as commented before, results in an optimal steady-state solution for periodic systems with real period T. First, we verify that the solutions given by the RTO for both the optimal and the model-based problem are the same. Then, in section 4 we will design the MPC to follow this reference.

We formulate the dynamic RTO for the simplified version of the periodic optimal economic problem (3) as follows:

$$\min_{\theta} \quad \Phi^\theta(F_Y^\theta(\theta), \theta) \tag{7a}$$
$$\text{s.t.} \quad M_1 F_X^\theta(\theta) + M_2 \theta = 0 \tag{7b}$$
$$H^\theta(F_X^\theta(\theta), \theta) = 0 \tag{7c}$$
$$G^\theta(F_X^\theta(\theta), \theta) \leq 0, \tag{7d}$$

where $\Phi^\theta, F_X^\theta, F_Y^\theta, H^\theta$ and G^θ are functions derived from rewritting the ones in (3) in terms of θ and (7b) corresponds to the periodic constraint (3e).

We also define a modified version of the dynamic RTO for the P-MEP (5).

$$\begin{aligned} \min_{\theta} \quad & \Phi^\theta((\hat{F}_Y^\theta(\theta) + \mu_Y^\theta \theta), \theta) \\ \text{s.t.} \quad & M_1(\hat{F}_X^\theta(\theta) + \mu_X^\theta \theta) + M_2\theta = 0 \\ & \hat{H}^\theta((\hat{F}_X^\theta(\theta) + \mu_X^\theta \theta), \theta) + \mu_H^\theta \theta = 0 \\ & \hat{G}^\theta((\hat{F}_X^\theta(\theta) + \mu_X^\theta \theta), \theta) + \mu_G^\theta \theta \leq 0, \end{aligned} \tag{8}$$

where μ refers to the modifiers that will be used to match the KKT conditions of both problems.

The gradient of the Lagrangian function associated to the problem (7) is the following:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial \Phi^\theta}{\partial F_Y^\theta}(\frac{\partial F_Y^\theta}{\partial \theta}) + \frac{\partial \Phi^\theta}{\partial \theta} + \pi_1^T\Big(M_1(\frac{\partial F_X^\theta}{\partial \theta}) + M_2\Big) + \\ + \pi_2^T\Big[\frac{\partial H^\theta}{\partial F_X^\theta}(\frac{\partial F_X^\theta}{\partial \theta}) + (\frac{\partial H^\theta}{\partial \theta})\Big] + \pi_3^T\Big[\frac{\partial G^\theta}{\partial F_X^\theta}(\frac{\partial F_X^\theta}{\partial \theta}) + (\frac{\partial G^\theta}{\partial \theta})\Big].$$

Analogously, the gradient of the Lagrangian function associated to problem (8) is the following:

$$\frac{\partial \hat{\mathcal{L}}}{\partial \theta} = \frac{\partial \Phi^\theta}{\partial \hat{F}_Y^\theta}(\frac{\partial \hat{F}_Y^\theta}{\partial \theta} + \mu_Y^\theta) + \frac{\partial \Phi^\theta}{\partial \theta} + \pi_1^T\Big(M_1(\frac{\partial \hat{F}_X^\theta}{\partial \theta} + \mu_X^\theta) + \\ + M_2\Big) + \pi_2^T\Big[\frac{\partial \hat{H}^\theta}{\partial \hat{F}_X^\theta}(\frac{\partial \hat{F}_X^\theta}{\partial \theta} + \mu_X^\theta) + (\frac{\partial \hat{H}^\theta}{\partial \theta} + \mu_H^\theta)\Big] + \\ + \pi_3^T\Big[\frac{\partial \hat{G}^\theta}{\partial \hat{F}_X^\theta}(\frac{\partial \hat{F}_X^\theta}{\partial \theta} + \mu_X^\theta) + (\frac{\partial \hat{G}^\theta}{\partial \theta} + \mu_G^\theta)\Big].$$

Analogously to the arguments in Vaccari and Pannocchia (2017), to match the KKT conditions for both problems, we need to set the modifiers $\mu_X^\theta, \mu_Y^\theta, \mu_H^\theta$ and μ_G^θ so that:

$$\begin{aligned} \frac{\partial F_X^\theta}{\partial \theta} &= \frac{\partial \hat{F}_X^\theta}{\partial \theta} + \mu_X^\theta \\ \frac{\partial F_Y^\theta}{\partial \theta} &= \frac{\partial \hat{F}_Y^\theta}{\partial \theta} + \mu_Y^\theta \\ \frac{\partial H^\theta}{\partial \theta} &= \frac{\partial \hat{H}^\theta}{\partial \theta} + \mu_H^\theta \\ \frac{\partial G^\theta}{\partial \theta} &= \frac{\partial \hat{G}^\theta}{\partial \theta} + \mu_G^\theta. \end{aligned}$$

Thus, the modifiers $\{\mu_X^\theta, \mu_Y^\theta, \mu_H^\theta, \mu_G^\theta\}$ need to take the values:

$$\begin{aligned} \mu_X^\theta &= \frac{\partial F_X^\theta}{\partial \theta} - \frac{\partial \hat{F}_X^\theta}{\partial \theta} \\ \mu_Y^\theta &= \frac{\partial F_Y^\theta}{\partial \theta} - \frac{\partial \hat{F}_Y^\theta}{\partial \theta} \\ \mu_H^\theta &= \frac{\partial H^\theta}{\partial \theta} - \frac{\partial \hat{H}^\theta}{\partial \theta} \\ \mu_G^\theta &= \frac{\partial G^\theta}{\partial \theta} - \frac{\partial \hat{G}^\theta}{\partial \theta}. \end{aligned} \tag{9}$$

All in all, we have proven that, if we modify the initial model (4) with modifiers $\mu_X^\theta, \mu_Y^\theta, \mu_H^\theta$ and μ_G^θ, the solution of the dynamic RTO of the P-MEP will converge to that of the optimal problem. For this purpose, we only need to compute the gradients of the model and the real plant. The gradients of the model can be easily computed, for example, by numeric differentiation or by exploiting the knowledge of the model, whereas the gradients of the real system are more cumbersome and out of the scope of this paper.

In the next section, we will study how to design the MPC layer so that the system converges to the trajectory planned by the dynamic RTO.

4. OFFSET-FREE MPC

In order to make the model converge to the real plant dynamics in the lower layer of control, we propose an offset-free MPC for periodic systems based on the ones studied in Huang et al. (2011), Rawlings and Mayne (2009) and Limon et al. (2015). Offset-free control algorithms have been widely used and are usually based on augmented models which incorporate a measure of the difference between the model and the real system response, called disturbance [Maeder et al. (2009),Pannocchia and Rawlings (2003), Morari and Maeder (2012)]. State of the art formulations of offset-free MPCs use a single model with its associated disturbance to correct the model. The general form of these formulations is:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{f}_X(x_k, u_k, d_k) \\ \hat{y}_k &= \hat{f}_Y(x_k, d_k), \end{aligned}$$

where $\hat{f}_X$ and $\hat{f}_Y$ correspond to the model dynamics and d_k is the disturbance associated to the model at time step k, which computation depends on the disturbance model chosen.

In this paper, we use the idea of augmented models and design an offset-free MPC that converges to the periodic reference given by the dynamic RTO. We propose to set an independent disturbance model for each point in the trajectory. That way, we can transform a stationary model into a periodic one, which can ideally converge to the real system. For the sake of simplicity, we consider that we can measure the real states at any given time, otherwise, we would need a state estimator and the notation would get more complex. We formulate a simple disturbance model for a system with period T as:

$$d_k = d_{k-T} + (x_{k-T} - \hat{x}_{k-T}), \tag{10}$$

where x_{k-T} and $\hat{x}_{k-T}$ correspond to the real and the estimated state at time step $k - T$. Note that, upon convergence of the disturbance ($d_k = d_{k-T}$), the estimated state of the system is equal to the real value of the state. This disturbance model only requires knowledge of the state and predicted state at time step $k-T$ to converge to the real periodic system.

5. CONTROL SCHEME

In this section we give a proper formulation for the RTO and the MPC layers of the P-MEP. The model used is a sum of the original model plus the modifier-terms proposed in sections 3 and 4. From now on, we use a more explicit notation for the modifiers, where μ^θ splits into μ^x and μ^u to refer to the x_0 and $\mathbf{u}_T$ terms respectively. We also use the abuse of notation $\boldsymbol{\mu} x_0$ to refer to the composition of $[\mu_k x_0,\ \mu_{k+1} x_0,\ \cdots,\ \mu_{k+T-1} x_0]^T$. Finally, we use the disturbance model described in equation (10) and the contraction $\mathbf{d}_T = [d_0,\ d_1,\ \cdots,\ d_{T-1}]^T$.

5.1 Trajectory Planner Design

The trajectory planner also known as dynamic real-time optimizer (dynamic RTO) finds the trajectory (given by its initial state x_0 and the control sequence $\mathbf{u}_T$) that minimizes a given economic function Φ subject to the dynamics of the model, the periodic constraint and additional constraints of the problem. The mathematical formulation of the dynamic RTO is the following:

$$\begin{aligned} \min_{x_0,\mathbf{u}_T} \quad & \Phi(\hat{\mathbf{y}}_T, \mathbf{u}_T) \\ \text{s.t.} \quad & \hat{\mathbf{x}}_T = \hat{F}_X(x_0, \mathbf{u}_T) + \boldsymbol{\mu}_X^x x_0 + \boldsymbol{\mu}_X^u \mathbf{u}_T + \mathbf{d}_T \\ & \hat{\mathbf{y}}_T = \hat{F}_Y(x_0, \mathbf{u}_T) + \boldsymbol{\mu}_Y^x x_0 + \boldsymbol{\mu}_Y^u \mathbf{u}_T + \mathbf{d}_T \\ & M\hat{\mathbf{x}}_T + x_0 = 0 \\ & \hat{H}(\hat{\mathbf{x}}_T, \mathbf{u}_T) + \boldsymbol{\mu}_H^x x_0 + \boldsymbol{\mu}_H^u \mathbf{u}_T = 0 \\ & \hat{G}(\hat{\mathbf{x}}_T, \mathbf{u}_T) + \boldsymbol{\mu}_G^x x_0 + \boldsymbol{\mu}_G^u \mathbf{u}_T \le 0. \end{aligned} \tag{11}$$

To run the dynamic RTO, we need to set the value of T. This can be done either by previous knowledge of the system, or we could estimate it with different online algorithms such as the one presented in [Tsao and Qian (1993)]. In this paper, we will suppose that period T is always known beforehand.

5.2 Offset-free MPC Design

The offset-free MPC calculates the control sequence necessary to follow the trajectory given by the dynamic RTO, minimizing a cost function ℓ subject to constraints in the states and in the inputs of the system. The problem can be formulated as:

$$\begin{aligned} \min_{\mathbf{u}_N} \quad & \ell(\hat{\mathbf{x}}_N, \mathbf{u}_N, \hat{\mathbf{x}}_N^r, \mathbf{u}_N^r) \\ \text{s.t.} \quad & \hat{\mathbf{x}}_N = \hat{F}_X(x_0, \mathbf{u}_N) + \boldsymbol{\mu}_X^x x_0 + \boldsymbol{\mu}_X^u \mathbf{u}_N + \mathbf{d}_N \\ & \hat{g}_c(\hat{\mathbf{x}}_N, \mathbf{u}_N) \le 0 \\ & \hat{h}_c(\hat{\mathbf{x}}_N, \mathbf{u}_N) = 0, \end{aligned} \tag{12}$$

where x_0 is the current state of the system and $\{\hat{\mathbf{x}}_N^r, \mathbf{u}_N^r\}$ are the reference trajectory for states and inputs respectively given by (11).

In the next section, we detail the full algorithm used to achieve the optimal economic performance in periodic systems.

6. PERIODIC MODIFIER-ADAPTATION ALGORITHM

Given the real system defined in (1) and the model of it defined in (4), we propose the following algorithm to solve the P-MEP and obtain an optimal steady-state performance. Constraint functions H and G are ommited to simplify the algorithm.

(i) Initialize $k = 0$ and the predicted sequences of modifiers $\hat{\boldsymbol{\mu}}_X^x, \hat{\boldsymbol{\mu}}_X^u, \hat{\boldsymbol{\mu}}_Y^x, \hat{\boldsymbol{\mu}}_Y^u, \hat{\mathbf{d}}_T$ to zero.
(ii) Given the predicted sequences for the modifiers, compute the optimal trajectory with the dynamic RTO defined in (11).
(iii) Synchronize the trajectory given by the dynamic RTO with the current state, so that the starting point in the trajectory is the one closest to the current state (i.e. that minimizes a given norm).
(iv) Given the predicted sequences for the modifiers, the reference trajectory calculated by the dynamic RTO $\{\hat{\mathbf{x}}_N^r, \mathbf{u}_N^r\}$, and the current state of the system x_k, use the MPC defined in (12) to compute the input u_k.
(v) Apply input u_k to the system and get the following state x_{k+1}.
(vi) Estimate the gradients of the model and the real system and update the modifiers $\hat{\mu}_k$ with them:

$$\begin{aligned} \hat{\mu}_{X,k}^x &= \frac{\partial F_X}{\partial x_0}\Big|_{x_{k-T}} - \frac{\partial \hat{F}_X}{\partial x_0}\Big|_{x_{k-T}} \\ \hat{\mu}_{X,k}^u &= \frac{\partial F_X}{\partial \mathbf{u}_N}\Big|_{\mathbf{u}_{N,k-T}^r} - \frac{\partial \hat{F}_X}{\partial \mathbf{u}_N}\Big|_{\mathbf{u}_{N,k-T}^r} \\ \hat{\mu}_{Y,k}^x &= \frac{\partial F_Y}{\partial x_0}\Big|_{x_{k-T}} - \frac{\partial \hat{F}_Y}{\partial x_0}\Big|_{x_{k-T}} \\ \hat{\mu}_{Y,k}^u &= \frac{\partial F_Y}{\partial \mathbf{u}_N}\Big|_{\mathbf{u}_{N,k-T}^r} - \frac{\partial \hat{F}_Y}{\partial \mathbf{u}_N}\Big|_{\mathbf{u}_{N,k-T}^r} \end{aligned}$$

(vii) Estimate the perturbance d_k modifier with the disturbance model defined in (10):

$$d_k = \hat{d}_k + \Big(x_{k+1} - \big(\hat{F}_X(x_k, u_k) + \mu_{X,k}^x x_k + \mu_{X,k}^u u_k + \hat{d}_k\big)\Big),$$

(viii) Predict next period modifiers:

$$\begin{aligned} \hat{\mu}_{X,k+T}^x &= \hat{\mu}_{X,k}^x \\ \hat{\mu}_{X,k+T}^u &= \hat{\mu}_{X,k}^u \\ \hat{\mu}_{Y,k+T}^x &= \hat{\mu}_{Y,k}^x \\ \hat{\mu}_{Y,k+T}^u &= \hat{\mu}_{Y,k}^u \\ \hat{d}_{k+T} &= d_k, \end{aligned}$$

(ix) Update $k = k + 1$ and return to step (ii).

Remark 1. After step (vii), one can filter the modifiers to improve convergence at the expense of a slower update of the modifiers:

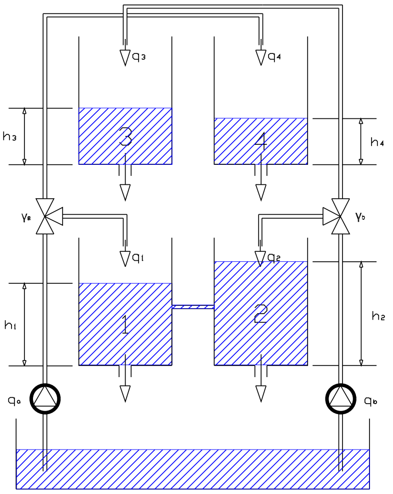

Fig. 1. Quadruple-tank system diagram, reproduced from Alvarado et al. (2011)

$$\begin{aligned}
\hat{\mu}^x_{X,k} &= (\mathrm{I} - K^x_X)\hat{\mu}^x_{X,k-T} + K^x_X \hat{\mu}^x_{X,k} \\
\hat{\mu}^u_{X,k} &= (\mathrm{I} - K^u_X)\hat{\mu}^u_{X,k-T} + K^u_X \hat{\mu}^u_{X,k} \\
\hat{\mu}^x_{Y,k} &= (\mathrm{I} - K^x_Y)\hat{\mu}^x_{Y,k-T} + K^x_Y \hat{\mu}^x_{Y,k} \\
\hat{\mu}^u_{Y,k} &= (\mathrm{I} - K^u_Y)\hat{\mu}^u_{Y,k-T} + K^u_Y \hat{\mu}^u_{Y,k} \\
d_k &= (\mathrm{I} - K_d)\hat{d}_k + K_d d_k,
\end{aligned}$$

where K corresponds to the filter matrices.

In the next section, we will use this algorithm to solve a benchmark problem.

7. MOTIVATIONAL EXAMPLE

To illustrate the algorithm proposed in the previous section, we use it in the economic control of a periodic version of the quadruple-tank process proposed in [Johansson (2000)] and used in [Alvarado et al. (2011), Shneiderman and Palmor (2010)] among others. We set parameter γ to change periodically through time so that it let us study the performance of the algorithm in a periodical system. The objective of this motivational example is to control the non-linear quadruple-tank system to its optimal steady-state trajectory using a linear time-invariant model of the plant.

The quadruple-tank system (Figure 1) consists of four tanks interconnected so that they share water according to the following equations:

$$\begin{aligned}
S\frac{dh_1}{dt} &= -a_1\sqrt{2gh_1} + a_3\sqrt{2gh_3} + \frac{\gamma_a q_a}{3600} \\
S\frac{dh_2}{dt} &= -a_2\sqrt{2gh_2} + a_4\sqrt{2gh_4} + \frac{\gamma_b q_b}{3600} \\
S\frac{dh_3}{dt} &= -a_3\sqrt{2gh_3} + (1-\gamma_b)\frac{q_b}{3600} \\
S\frac{dh_4}{dt} &= -a_4\sqrt{2gh_4} + (1-\gamma_a)\frac{q_a}{3600}.
\end{aligned}$$

We use a compact notation to define the parameters of the plant:

Table 1. Parameters of the plant

	Value	Unit	Description
S	0.03	m^2	Cross-section of the tanks
$\mathbf{a}$	$\begin{bmatrix} 1.31 \\ 1.51 \\ 0.927 \\ 0.882 \end{bmatrix} e^{-4}$	m^2	Discharge constants
$\mathbf{h}_{\max}$	$\begin{bmatrix} 1.36 \\ 1.36 \\ 1.30 \\ 1.30 \end{bmatrix}^T$	m	Maximum water level
$\mathbf{h}_{\min}$	$\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}^T$	m	Minimum water level
$\mathbf{q}_{\max}$	$\begin{bmatrix} 3.6 \\ 4.0 \end{bmatrix}^T$	m^3/h	Maximum water flow
$\mathbf{q}_{\min}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}^T$	m^3/h	Minimum water flow
g	9.81	m/s^2	Gravity acceleration

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} q_a \\ q_b \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} \gamma_a \\ \gamma_b \end{bmatrix}.$$

From a state-space point of view, the water levels $\mathbf{x}$ appoint the states of the system and $\mathbf{y}$ corresponds to the outputs of it. All these levels are measured in meters (m). Besides, $\mathbf{u}$ refers to the inputs of the system and are measured in m^3/h. Finally, $\boldsymbol{\gamma}$ refers to the parameters of the three-way valves, is adimensional and changes periodically through time in our example. Information about the rest of the parameters is given in table 1.

To model the system, we linearize it at the point

$$x_0 = \begin{bmatrix} 0.7293 \\ 0.8102 \\ 0.6594 \\ 0.9408 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1.948 \\ 2.00 \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}$$

and discretize it, with discretization time set to five seconds. This results in the following linear model:

$$\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0.945 & 0 & 0.040 & 0 \\ 0 & 0.940 & 0 & 0.032 \\ 0 & 0 & 0.959 & 0 \\ 0 & 0 & 0 & 0.967 \end{bmatrix} x_k + \begin{bmatrix} 0.0135 & 0.0006 \\ 0.0005 & 0.0180 \\ 0 & 0.0272 \\ 0.0319 & 0 \end{bmatrix} u_k \\
y_k &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u_k,
\end{aligned} \tag{13}$$

Given economic parameters $c = 1$ and $p = 20$, the economic cost of the plant at each time step is given by the following expression:

$$\phi(y_k, u_k) = (q_a^2 + c q_b^2) + p\frac{V_{\min}}{A(h_1 + h_2)}. \tag{14}$$

In this example, we consider a periodic nature of parameter $\boldsymbol{\gamma}$ given by (15), where the period is assumed to be $T = 10$. The non-linearity of the plant plus the periodicity of it, makes this periodic version of the quadruple-tank system an excellent example to test our algorithm.

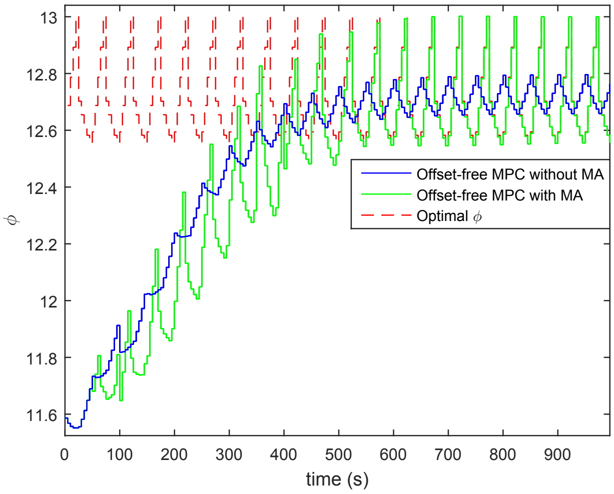

Fig. 2. Evolution of the economic cost

$$\gamma = \begin{bmatrix} 0.3 & 0.375 & 0.45 & 0.525 & 0.6 & 0.4 & 0.35 & 0.3 & 0.25 & 0.2 \\ 0.6 & 0.525 & 0.45 & 0.3750 & 0.3 & 0.4 & 0.425 & 0.45 & 0.475 & 0.5 \end{bmatrix}. \tag{15}$$

We use the linearization point x_0 as the initial state of the system, take control horizon $N = 10$ and period equal to that of γ, that is, $T = 10$ for the MPC and dynamic RTO respectively. We also use the filtering step mentioned in remark 1 with $K = 0.6$*I. These two optimization problems are solved using the CasADi optimization tool in Matlab [Andersson et al. (In Press, 2018)], which suits for non-linear optimization problems.

The results in Figure 2 show how, with the state of the art approach, the trajectory of the economic cost is not able to converge to the optimal cost trajectory. Whereas, with the periodic MA algorithm the economic cost follows the optimal cost trajectory when the system reaches its steady behaviour. This means that the modifiers have converged so that the model dynamics are equal to those of the real system locally, therefore the KKT optimal conditions hold for the periodic model-based economic problem.

8. CONCLUSIONS

In this paper we have presented the periodic modifier-adaptation algorithm, which uses a modifier-adaptation methodology to find the optimal trajectory of a real dynamic system given a model of it. We have proven that the proposed approach results in offset-free steady-state control when the optimal trajectory is periodic, as in the case of periodic systems, even when the model differs much from the real system. We have tested the proposed algorithm in the benchmark case of the quadruple-tank system, where it uses a linear stationary model to control the non-linear periodic system. As a result, the system converges to its steady-state optimal trajectory without offset, which is an improvement over the state of the art approach. Future works could study how to take advantage of the continuous stimulation of the system to compute the real plant gradients.

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