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Interval estimation for linear discrete-time delay systems

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Abstract: This paper deals with the problem of interval estimation for linear discrete-time systems with a constant time delay. First, an interval observer is designed based on cooperativity and Lyapunov-Krasovskii stability analysis. Second, a zonotope-based interval estimation, which is independent of cooperativity constraint, is proposed. It integrates robust observer design, based on multiple feedbacks, with reachability analysis via zonotopes. In order to enhance the accuracy of interval estimation, an H_∞ technique is introduced into observer design to reduce the effects of disturbances and noises. Finally, simulation results are given to illustrate the efficiency of the proposed method.

Keywords: Interval estimation, Time-delay systems, H_∞ approach, Zonotopes

1. INTRODUCTION

During the last decades, interval estimation methods, have been widely investigated and applied to several applications such as bioreactors (Moisan et al., 2007), nonlinear systems control (Raïssi et al., 2012), LPV systems (Efimov et al., 2012a) and fault diagnosis (Wang et al., 2018b). In the literature, two categories of interval estimation methods can be distinguished: the first is known as interval observer design which is based on the monotony systems theory (Gouzé et al., 2000). The second method is based on set-membership approach and aims to construct compact sets enclosing all the possible state values by using predefined geometrical sets such as ellipsoid (Liu et al., 2016), paralleltopes (Chisci et al., 1996) and zonotopes (Combastel, 2003). Among these sets, a zonotope-based approach can make a good trade-off between estimation accuracy and computation complexity.

Unlike this approach, interval observers have received considerable attention in recent years as an interesting alternative to deal with uncertainties (Sehli et al., 2019; Wang et al., 2018a). Under a general assumption that the uncertainties are bounded, interval observers can provide the upper and lower bounds of the state variables using the available data by two point observers such that their estimation error dynamics are both cooperative and stable. However, it is not a trivial to concept a cooperative and stable error system. Generally, the cooperativity constraint can be relaxed by a coordinate transformation but it can lead to some conservatism and limit the estimation accuracy (Chambon et al., 2016).
On the other hand, state estimation of time-delay systems has attracted much attention during the past three decades due to their frequent presence in engineering applications as in chemical and biological processes, hydraulic systems, and manufacturing processes. For instance, Sipahi et al. (2011) shows that the emergence of delays in dynamical systems may increase the complexity of observer design, degrades their performance and negatively affects their stability and robustness using functional differential equations (Richard, 2003). The literature shows that the interest has grown significantly in the past decade in regard to interval observer design for such systems. In Efimov et al. (2013) and Efimov et al. (2015b), the existing solutions are based on the delay-independent stability approach. Efimov et al. (2015a) and Efimov et al. (2016) used the delay-dependent positivity conditions to design interval observers for linear systems with delayed measurements with time-varying delays. However, all these methods are considered based on cooperativity constraint or coordinate transformation.

To overcome the aforementioned drawbacks, this paper deals with zonotope-based interval estimation for linear discrete-time delay systems with a constant time-delay subject to unknown but bounded disturbances and measurement noises. This approach, namely ”two-step method”, integrates observer design with reachability analysis technique via zonotopes (Tang et al., 2019). The main contribution of this work is to address the interval estimation problem for linear discrete-time delay systems using a zonotope-based method. Compared to interval observer theory, the proposed method is less restrictive since it overcomes the cooperativity constraints and avoids the additional conservatism caused by coordinate transformation. Then, by introducing H_∞ technique, the proposed method is effective in attenuating the effects of uncertainties and improving the accuracy of interval estimation.
The remainder of the paper is structured as follows. Some notations and preliminaries are briefly introduced in Section 2. The problem formulation is presented in Section 3. Section

4 presents an interval observer design for linear discrete-time delay systems. The proposed interval estimation method is presented in Section 5. Section 6 gives simulation results on two numerical examples. The last section is devoted to conclusions.

2. NOTATIONS & PRELIMINARIES

The n and $m\times n$ dimensional Euclidean spaces are denoted by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ respectively. $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. The symbol I_n is the identity matrix with dimensions $n\times n$ and E_n denotes $(n\times 1)$ vector whose elements are equal to 1. The relation $Q \succ 0$ ($Q \prec 0$) indicates that Q is positive (negative) definite. Lower and upper bounds $\bar{x}$ and $\underline{x}$ of x satisfy $\underline{x} \leq x \leq \bar{x}$, where the comparison operator $\leq$ should be understood elementwise for vectors and matrices. The operators $\oplus$ and $\odot$ represent the Minkowski sum and the linear image operators, respectively. The asterisk $*$ denotes the symmetric term in a symmetric block matrix. For a signal $x_k \in \mathbb{R}^n$, its $\mathcal{L}_2$-norm is defined as $||x||_2 = \sqrt{\sum_{k=0}^{\infty} x_k^T x_k}$.

2.1 Interval bounds

Given a matrix $M \in \mathbb{R}^{m\times n}$, define $M^+ = \max\{0, M\}$, $M^- = M^+ - M$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|M| = M^+ + M^-$. A matrix $M \in \mathbb{R}^{m\times n}$ is called Schur stable if all its eigenvalues have the norm less than one; it is called nonnegative if all its off-diagonal terms are nonnegative.

Lemma 1. (Efimov et al., 2012b) Let $z \in \mathbb{R}^n$ be a vector verifying $\underline{z} \leq z \leq \bar{z}$ and $B \in \mathbb{R}^{m\times n}$ is a constant matrix, then

$$B^+\underline{z} - B^-\bar{z} \leq Bz \leq B^+\bar{z} - B^-\underline{z}. \tag{1}$$

Lemma 2. (Haddad and Chellaboina, 2004) Consider a linear system with a constant delay

$$x(k+1) = A_0x(k) + A_1x(k-h) + w(k), \quad w : \mathbb{R}_+ \to \mathbb{R}^n{}_+ \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, h is a constant time delay and the matrices A_0 and A_1 have appropriate dimensions.

The system (2) is called cooperative or nonnegative for all $h \in \mathbb{R}_+$ if the matrix A_0 is Schur stable and nonnegative and A_1 is a positive matrix.

2.2 Zonotopic analysis

Definition 1. (Combastel, 2003) An s-zonotope $\mathcal{Z} \subset \mathbb{R}^n$ is the affine image of a hypercube $\mathbb{B}^s = [-1, 1]^s$ in $\mathbb{R}^n$ and can be expressed as follows:

$$\mathcal{Z} = \langle p, H\rangle = p + H\mathbb{B}^s = \{z \in \mathbb{R}^n : z = p + Hb\}, \tag{3}$$

where $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$ and $H \in \mathbb{R}^{n\times s}$ denotes the generator matrix of $\mathcal{Z}$.

Definition 2. (Tang et al., 2019) For a zonotope $\mathcal{Z} \subset \mathbb{R}^n$, its interval hull $Box(\mathcal{Z})$ is the smallest interval vector containing it, which is denoted by:

$$\mathcal{Z} \subseteq Box(\mathcal{Z}) = [\underline{z}, \bar{z}], \tag{4}$$

where $[\underline{z}, \bar{z}] = \{z \in \mathcal{Z}, \underline{z} \leq z \leq \bar{z}\}$ is an interval vector, $\underline{z}$ and $\bar{z}$ are the lower and upper bounds of z.

Property 1. (Combastel, 2015) The Minkowski sum of two zonotopes $\mathcal{Z}_1 = \langle p_1, H_1\rangle$ and $\mathcal{Z}_2 = \langle p_2, H_2\rangle$ is given by:

$$\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2 = \langle p_1 + p_2, [H_1 H_2]\rangle \tag{5}$$

Property 2. (Combastel, 2015) The image of a zonotope $\mathcal{Z} = \langle p, H\rangle$ by a linear mapping K can be computed by a standard matrix such as $K \odot \mathcal{Z} = \langle Kp, KH\rangle$.

Property 3. (Tang et al., 2019) For a zonotope $\mathcal{Z} = \langle p, H\rangle \subset \mathbb{R}^n$, its interval hull, $Box(\mathcal{Z}) = [\underline{z}, \bar{z}]$, can be obtained by:

$$\begin{cases} \underline{z}_i = p_i - \sum_{j=1}^{m} |H_{i,j}|, \; i = 1, ..., n \\ \bar{z}_i = p_i + \sum_{j=1}^{m} |H_{i,j}|, \; i = 1, ..., n \end{cases} \tag{6}$$

According to the Definitions 1 and 2, the interval hull of the zonotope $\mathcal{Z} = \langle p, H\rangle$ can also be denoted by $\mathcal{Z} \subseteq Box(\mathcal{Z}) = \langle p, \overline{H}\rangle$ where $\overline{H} \in \mathbb{R}^{n\times n}$ is a diagonal matrix given by:

$$\overline{H} = diag(\left[\sum_{j=1}^{m} |H_{1,j}| \;\cdots\; \sum_{j=1}^{m} |H_{n,j}\right].$$

Property 4. (Combastel, 2003) A high-dimensional zonotope can be bounded by a lower one via a reduction operator denoted by $\downarrow_q(.)$, defined by:

$$\mathcal{Z} = \langle p, H\rangle \subseteq \langle p, \downarrow_q(H)\rangle \subseteq Box(\mathcal{Z}), \quad n < q < m, \tag{7}$$

where q is the maximum number of columns of the generator matrix after reduction.

3. PROBLEM FORMULATION

Consider the following linear discrete-time delay system:

$$\begin{cases} x(k+1) = A_0x(k) + A_1x(k-1) + Bu(k) + Dw(k), \\ y(k) = Cx(k) + Fv(k), \end{cases} \tag{8}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ denote respectively the state, input and measurement output vectors. A_0, A_1, B, D, C and F are known constant matrices with the corresponding dimensions. $w \in \mathbb{R}^{n_w}$ and $v \in \mathbb{R}^{n_v}$ are the process disturbances and measurement noises.

The goal of this paper is to find an interval vector $[\underline{x}(k), \bar{x}(k)]$ that contains the real state $x(k)$ such that

$$\underline{x}(k) \leq x(k) \leq \bar{x}(k), \quad k \in \mathbb{Z}_+.$$

4. INTERVAL OBSERVER DESIGN FOR LINEAR DISCRETE-TIME DELAY SYSTEMS

This section introduces an interval observer for the linear discrete-time delay system (8), which can estimate respectively upper and lower bounds of the real state. The following assumption is considered.

Assumption 1. Let $x(0) \in [\underline{x}(0), \bar{x}(0)]$ for some known $\underline{x}(0), \bar{x}(0) \in \mathbb{R}^{n_x}$; let also two functions $\underline{w}$ and $\overline{w}$ and a constant scalar $K > 0$ be given such that

$$\underline{w} \leq w(k) \leq \overline{w}, \; -KE_{n_v} \leq v(k) \leq KE_{n_v}.$$

Then, the following interval observer structure for the system (8) is proposed:

$$\begin{cases} \underline{x}(k+1) = A_0\underline{x}(k) + A_1\underline{x}(k-1) + Bu(k) + L_0(y(k) - C\underline{x}(k)) \\ \qquad + L_1(y(k-1) - C\underline{x}(k-1)) + D^+\underline{w} - D^-\overline{w} \\ \qquad - (|L_0F| + |L_1F|)KE_{n_v}, \\ \bar{x}(k+1) = A_0\bar{x}(k) + A_1\bar{x}(k-1) + Bu(k) + L_0(y(k) - C\bar{x}(k)) \\ \qquad + L_1(y(k-1) - C\bar{x}(k-1)) + D^+\overline{w} - D^-\underline{w} \\ \qquad + (|L_0F| + |L_1F|)KE_{n_v}, \end{cases} \tag{9}$$

where $L_0, L_1 \in \mathbb{R}^{n_x\times n_y}$ are the observer gain matrices to be determined.

The dynamics of the lower and upper state estimation errors $\underline{e} = \underline{x} - x$ and $\overline{e} = \overline{x} - x$ are described by:

$$\begin{cases} \underline{e}(k+1) = (A_0 - L_0C)\underline{e}(k) + (A_1 - L_1C)\underline{e}(k-1) + \underline{\Psi} - \Psi(k), \\ \overline{e}(k+1) = (A_0 - L_0C)\overline{e}(k) + (A_1 - L_1C)\overline{e}(k-1) + \overline{\Psi} - \Psi(k), \end{cases} \quad (10)$$

where

$$\Psi = Dw(k) - L_0Fv(k) - L_1Fv(k-1),$$
$$\overline{\Psi} = D^+\overline{w} - D^-\underline{w} + (|L_0F| + |L_1F|)KE_{n_v},$$
$$\underline{\Psi} = D^+\underline{w} - D^-\overline{w} - (|L_0F| + |L_1F|)KE_{n_v}.$$

The observer design consists in finding two matrices L_0 and L_1 for ensuring the estimation error convergence. To limit the effect of system uncertainties, an H_∞ formalism is introduced to tune the observer gain matrices.
For brevity, define

$$\overline{d}(k) = \begin{bmatrix} \overline{\Psi} - Dw(k) \\ v(k) \\ v(k-1) \end{bmatrix}, \; \underline{d}(k) = \begin{bmatrix} \underline{\Psi} - Dw(k) \\ v(k) \\ v(k-1) \end{bmatrix}, \quad (11)$$

where $\underline{d}$ and $\overline{d}$ depend on observer gain matrices L_0 and L_1. Consequently, the error dynamics in (10) can be rewritten as:

$$\begin{cases} \underline{e}(k+1) = (A_0 - L_0C)\underline{e}(k) + (A_1 - L_1C)\underline{e}(k-1) + B_d\underline{d}(k), \\ \overline{e}(k+1) = (A_0 - L_0C)\overline{e}(k) + (A_1 - L_1C)\overline{e}(k-1) + B_d\overline{d}(k), \end{cases} \quad (12)$$

where $B_d = [I_{n_x} \; L_0F \; L_1F]$. The following proposed theorem provides sufficient conditions under LMI formulation to synthesis an interval observer in order to attenuate error estimation.

Theorem 1. Given system (8) and the observer structure (9). Let Assumption 1 be satisfied and the matrices $(A_0 - L_0C)$ and $(A_1 - L_1C)$ be nonnegative. Then the relation

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k), \quad (13)$$

is satisfied for all $k \geq 0$ provided $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.
In addition, for a given scalar $\gamma > 0$, if there exist a diagonal matrix P, a symmetric matrix Q and two matrices K_0 and K_1 such that the following matrix inequalities are verified:

$$\begin{bmatrix} -P+Q+I_{n_x} & * & * & * & * & * \\ 0 & -Q & * & * & * & * \\ 0 & 0 & -\gamma^2 I_{n_w} & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * \\ 0 & 0 & 0 & 0 & -\gamma^2 I_{n_v} & * \\ PA_0 - K_0C & PA_1 - K_1C & P & K_0F & K_1F & -P \end{bmatrix} \prec 0, \quad (14)$$

$$P \succ 0, \quad (15)$$
$$Q \succ 0, \quad (16)$$
$$PA_0 - K_0C \geq 0, \quad (17)$$
$$PA_1 - K_1C \geq 0, \quad (18)$$

then, (9) in an interval observer for the system (8) and satisfies $||\overline{e}||_2 < \gamma^2 ||\overline{d}||_2$ and $||\underline{e}||_2 < \gamma^2 ||\underline{d}||_2$. Moreover, the observer gains can be deduced from

$$\begin{cases} L_0 = P^{-1}K_0, \\ L_1 = P^{-1}K_1. \end{cases} \quad (19)$$

Proof. Using Lemma 1, the following relations hold

$$\begin{cases} \underline{\Psi} - \Psi \leq 0, \\ \overline{\Psi} - \Psi \geq 0. \end{cases} \quad (20)$$

In addition, $x(0) \in [\underline{x}(0), \overline{x}(0)]$ indicates that $\overline{e}(0) \geq 0$ and $\underline{e}(0) \leq 0$. Applying Lemma 2. to (10), the relation $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$ for all $k \in \mathbb{Z}_+$ holds if the matrices $(A_0 - L_0C)$ and $(A_1 - L_1C)$ are nonnegative, then the system (10) is cooperative.
Moreover, in order to calculate the matrices L_0 and L_1 for ensuring the estimation error convergence, consider a Lyapunov-Krasovskii function for the upper estimation error (similarly for the lower estimation error) defined as

$$V(\overline{e}(k)) = \overline{e}(k)^T P\overline{e}(k) + \overline{e}(k-1)^T Q\overline{e}(k-1), \quad P, Q \succ 0 \quad (21)$$

To satisfy the constraints $||\overline{e}||_2 < \gamma^2 ||\overline{d}||_2$ and $||\underline{e}||_2 < \gamma^2 ||\underline{d}||_2$, it is sufficient to find a Lyapunov candidate satisfying

$$\Delta V + \overline{e}(k)^T \overline{e}(k) - \gamma^2 \overline{d}(k)^T \overline{d}(k) \leq 0. \quad (22)$$

Then, the following matrix inequality holds

$$\begin{bmatrix} G_0^T PG_0 - P + Q + I_{n_x} & * & * \\ G_1^T PG_0 & G_1^T PG_1 - Q & * \\ B_d^T PG_0 & B_d^T PG_1 & B_d^T PB_d - \gamma^2 I_{n_x+2n_v} \end{bmatrix} \prec 0, \quad (23)$$

where $G_0 = A_0 - L_0C$, $G_1 = A_1 - L_1C$ and γ represents the attenuation level of the disturbances.
By applying the Schur complement lemma, the above inquality is equivalent to

$$\begin{bmatrix} -P+Q+I_{n_x} & * & * & * & * & * \\ 0 & -Q & * & * & * & * \\ 0 & 0 & -\gamma^2 I_{n_w} & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * \\ 0 & 0 & 0 & 0 & -\gamma^2 I_{n_v} & * \\ PA_0 - PL_0C & PA_1 - PL_1C & P & PL_0F & PL_1F & -P \end{bmatrix} \prec 0. \quad (24)$$

By letting $K_0 = PL_0$ and $K_1 = PL_1$, the inequality (24) becomes the LMI in (14). Moreover, the nonnegativeness of the matrices $A_0 - L_0C$ and $A_1 - L_1C$ is ensured if the inequalities (17) and (18) are verified.

Remark 1. The main limitation of interval observers synthesis consists in providing simultaneously the cooperativity and the stability of the interval estimation error dynamics. Then, a coordinate transformation can be introduced to relax the conditions of interval observers design but it may engender extra conservatism and reduce the estimation accuracy (Chambon et al., 2016).
To deal with this problem, a zonotope-based interval estimation method is proposed. Independent of the cooperativity constraint, this method combines a robust observer design with zonotopic analysis technique (Tang et al. (2019)).

5. ZONOTOPE-BASED INTERVAL ESTIMATION

This section proposes a zonotope-based interval estimation method for the system (8), that combines an observer design with zonotopic analysis to achieve guaranteed state estimation. The following hypotheses are considered.
Assumption 2. The initial system state vector $x(0)$, disturbances vector $w(k)$ and measurement noises vector $v(k)$ are assumed to be unknown but bounded by the following zonotopes:

$$x(0) \in \langle p_0, H_0 \rangle, \; w(k) \in \mathcal{W} = \langle 0, H_w \rangle, \; v(k) \in \mathcal{V} = \langle 0, H_v \rangle, \quad (25)$$

where $p_0 \in \mathbb{R}^{n_x}$, $H_0 \in \mathbb{R}^{n_x \times n_x}$, $H_w \in \mathbb{R}^{n_w \times n_w}$ and $H_v \in \mathbb{R}^{n_v \times n_v}$ are known vector and matrices.

5.1 Observer design based on H_∞ approach

Consider the following robust observer structure for the system (8):

$$\hat{x}(k+1) = A_0\hat{x}(k) + A_1\hat{x}(k-1) + Bu(k) + L_0(y(k) - C\hat{x}(k)) + L_1(y(k-1) - C\hat{x}(k-1)), \tag{26}$$

where $L_0, L_1 \in \mathbb{R}^{n_x \times n_y}$ are the observer gains to be computed. By defining the estimation error as

$$e(k) = x(k) - \hat{x}(k), \tag{27}$$

the error dynamics are given by:

$$e(k+1) = (A_0 - L_0C)e(k) + (A_1 - L_1C)e(k-1) + Ed(k), \tag{28}$$

where

$$E = \begin{bmatrix} D & -L_0F & -L_1F \end{bmatrix}, \; d = \begin{bmatrix} w(k) & v(k) & v(k-1) \end{bmatrix}^T.$$

Then, an H_∞ approach is introduced to tune the observer gain matrices L_0 and L_1 to obtain accurate interval state estimation ensuring uncertainties attenuation. This result is summerized in the following proposed theorem.

Theorem 2. Given a scalar $\gamma > 0$, (26) is called a robust observer for the system (8) and satisfies $||e||_2 < \gamma^2||d||_2$, if there exist two symmetric and positive definite matrices $P, Q \in \mathbb{R}^{n_x \times n_x}$ and two matrices R_0 and R_1 such that the following matrix inequality is verified

$$\begin{bmatrix} -P+Q+I_{n_x} & * & * & * & * & * \\ 0 & -Q & * & * & * & * \\ 0 & 0 & -\gamma^2 I_{n_w} & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * \\ 0 & 0 & 0 & 0 & -\gamma^2 I_{n_v} & * \\ (PA_0 - R_0C) & (PA_1 - R_1C) & PD & -R_0F & -R_1F & -P \end{bmatrix} \prec 0. \tag{29}$$

Then, the observer gain matrices L_0 and L_1 can be determined by:

$$\begin{cases} L_0 = P^{-1}R_0, \\ L_1 = P^{-1}R_1. \end{cases} \tag{30}$$

Proof. Let us consider the Lyapunov-Krasovskii candidate defined as

$$V(k) = V_1(k) + V_2(k) \tag{31}$$

where

$$V_1(k) = e(k)^T Pe(k), \quad P^T = P \succ 0 \tag{32}$$

$$V_2(k) = e(k-1)^T Qe(k-1) \quad Q^T = Q \succ 0 \tag{33}$$

Then, the time difference of $V(k)$ is given by

$$\Delta V = \begin{bmatrix} e(k) \\ e(k-1) \\ w(k) \\ v(k) \\ v(k-1) \end{bmatrix}^T \begin{bmatrix} \Omega_{11} & * & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & * \\ \Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} \end{bmatrix} \begin{bmatrix} e(k) \\ e(k-1) \\ w(k) \\ v(k) \\ v(k-1) \end{bmatrix}, \tag{34}$$

where

$$\begin{array}{ll} \Omega_{11} = G_0^T PG_0 - P + Q, & \\ \Omega_{21} = G_1^T PG_0, & \Omega_{22} = G_1^T PG_1 - Q, \\ \Omega_{31} = D^T PG_0, & \Omega_{32} = D^T PG_1, \\ \Omega_{33} = D^T PD, & \Omega_{41} = -(L_0F)^T PG_0, \\ \Omega_{42} = -(L_0F)^T PG_1, & \Omega_{43} = -(L_0F)^T PD, \\ \Omega_{44} = (L_0F)^T PL_0F, & \Omega_{51} = -(L_1F)^T PG_0, \\ \Omega_{52} = -(L_1F)^T PG_1, & \Omega_{53} = -(L_1F)^T PD, \\ \Omega_{54} = (L_1F)^T PL_0F, & \Omega_{55} = (L_1F)^T PL_1F, \\ G_0 = A_0 - L_0C, & G_1 = A_1 - L_1C. \end{array} \tag{35}$$

To satisfy the constraint $||e|_2| < \gamma^2||d||_2$, it is sufficient to find such a Lyapunov function under the condition

$$\Delta V + e(k)^T e(k) - \gamma^2 w(k)^T w(k) - \gamma^2 v(k)^T v(k) - \gamma^2 v(k-1)^T v(k-1) \leq 0, \tag{36}$$

that holds if

$$\begin{bmatrix} \Omega_{11} + I_{n_x} & * & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} - \gamma^2 I_{n_w} & * & * \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} - \gamma^2 I_{n_v} & * \\ \Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} - \gamma^2 I_{n_v} \end{bmatrix} \prec 0. \tag{37}$$

It is clair that (37) is not a standard LMI. By applying the Schur complement lemma, the above matrix inequality is satisfied if

$$\begin{bmatrix} -P+Q+I_{n_x} & * & * & * & * & * \\ 0 & -Q & * & * & * & * \\ 0 & 0 & -\gamma^2 I_{n_w} & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * \\ 0 & 0 & 0 & 0 & -\gamma^2 I_{n_v} & * \\ PG_0 & PG_1 & PD & -PL_0F & -PL_1F & -P \end{bmatrix} \prec 0. \tag{38}$$

By replacing G_0 and G_1 by their expressions and letting $R_0 = PL_0$ and $R_1 = PL_1$, the inequality (38) becomes the LMI in (29).

5.2 Interval state estimation

After designing the proposed observer (26) the interval estimation of the state can be realized based on the zonotopic analysis. From (27), we can deduce

$$x(k) = \hat{x}(k) + e(k). \tag{39}$$

Consequently, the interval state estimation is transformed as interval analysis of the error system $e(k)$ and can be obtained from:

$$\begin{cases} \underline{x}(k) = \hat{x}(k) + \underline{e}(k) \\ \overline{x}(k) = \hat{x}(k) + \overline{e}(k) \end{cases} \tag{40}$$

Based on the zonotopic technique, the interval estimation of the state can be obtained using the following proposed theorem.

Theorem 3. Consider the system (8) that (25) is satisfied, then the state $x(k)$ belongs into a zonotope $\hat{\mathcal{X}}_k = \langle \hat{x}(k), \hat{H}_k \rangle$ where $\hat{x}(k)$ is given in (26) with $\hat{x}(0) = p_0$ and the interval state estimation can be obtained as follows:

$$\begin{cases} \underline{x}(i,k) = \hat{x}(i,k) - \sum_{j=1}^{m} |\hat{H}_{i,j}|, & i = 1,...,n \\ \overline{x}(i,k) = \hat{x}(i,k) + \sum_{j=1}^{m} |\hat{H}_{i,j}|, & i = 1,...,n \end{cases} \tag{41}$$

where m denotes the column number of $\hat{H}_k$ and $\hat{H}_k$ has the following expression:

$$\begin{cases} \hat{H}_{k+1} = [(A_0 - L_0C) \downarrow_q (\hat{H}_k) \quad DH_w \quad -L_0FH_v]; & k = 0, \\ \hat{H}_{k+1} = [(A_0 - L_0C) \downarrow_q (\hat{H}_k) \quad (A_1 - L_1C)\hat{H}_{k-1} \\ \qquad DH_w \quad -L_0FH_v \quad -L_1FH_v]; & k \geq 1, \end{cases} \tag{42}$$

and $\hat{H}_0 = H_0$.

Proof. For brevity, denote the reachable set of $e(k)$ as Ω_k. From (28), the error dynamics $e(k)$ can be split into two subsystems as:

$$\begin{cases} e(k+1) = (A_0 - L_0C)e(k) + Dw(k) - L_0Fv(k); & k = 0, \\ e(k+1) = (A_0 - L_0C)e(k) + (A_1 - L_1C)e(k-1) + Dw(k) \\ \qquad -L_0Fv(k) - L_1Fv(k-1); & k \geq 1. \end{cases} \tag{43}$$

From (25) and (43), Ω_k can be obtained by

$$\begin{cases} \Omega_{k+1} = (A_0 - L_0 C) \odot \Omega_k \oplus D \odot \mathcal{W} \oplus (-L_0) F \odot \mathcal{V}; \quad k = 0, \\ \Omega_{k+1} = (A_0 - L_0 C) \odot \Omega_k \oplus (A_1 - L_1 C) \odot \Omega_{k-1} \oplus D \odot \mathcal{W} \\ \quad \oplus (-L_0 F) \odot \mathcal{V} \oplus -(L_1 F) \odot \mathcal{V}; \quad k \geq 1. \end{cases} \quad (44)$$

Since $x(0) \in \langle p_0, H_0 \rangle$ and $\hat{x}(0) = p_0$, we have

$$e(0) = x(0) - \hat{x}(0) \in \langle 0, H_0 \rangle. \quad (45)$$

Then,

$$\Omega_0 = \langle 0, H_0 \rangle, \quad (46)$$

and we obtain

$$e(k) \in \Omega_k = \langle 0, \hat{H}_k \rangle, \quad (47)$$

where $\hat{H}_k$ is given by (42).
According to (39), we have

$$x(k) \in \langle \hat{x}(k), 0 \rangle \oplus \Omega_k = \langle \hat{x}(k), \hat{H}_k \rangle. \quad (48)$$

Using the Property 3., the interval estimation of $x(k)$ is given in (41) which ends this proof.

Remark 2. It is clair that compared with interval observer theory, the zonotope-based interval estimation method is intuitive and independent of cooperativity and coordinate transformation. Therefore, the proposed method provides high computational efficiency and can enhance the estimation accuracy by integrating robust observer design and zonotopic techniques.

6. SIMULATIONS

In this section, two numerical examples are provided to illustrate the effectiveness of the proposed method.

6.1 *Example 1: Case of cooperative estimation error*

Consider a numerical time-delay linear system in the form of (8) with:

$$A_0 = \begin{bmatrix} 0.5 & 0.3 \\ -0.8 & 0.1 \end{bmatrix}, A_1 = \begin{bmatrix} -0.11 & 0.03 \\ 0.17 & 0.11 \end{bmatrix}, F = 0.1$$

$$B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, D = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, C = [\, 1 \;\; 0 \,].$$

In the simulation study, the known input is chosen as $u(k) = 0.1$ and the disturbance and measurement noise are bounded as

$$w(k) \in \langle 0, H_w \rangle, \quad v(k) \in \langle 0, H_v \rangle$$

where $H_w = 0.1$ and $H_v = 0.01$.
The initial state is bounded by the zonotope $\mathcal{X} = \langle 0, H_0 \rangle$ where

$$H_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

In this case, the cooperativity condition is satisfied and the interval observer is designed by solving the LMIs given in Theorem 1., with $\gamma = 3.8$ and

$$L_0 = \begin{bmatrix} 0.36 \\ -0.91 \end{bmatrix}, L_1 = \begin{bmatrix} -0.25 \\ 0.04 \end{bmatrix},$$

$$A_0 - L_0 C = \begin{bmatrix} 0.13 & 0.3 \\ 0.11 & 0.10 \end{bmatrix}, A_1 - L_1 C = \begin{bmatrix} 0.14 & 0.03 \\ 0.12 & 0.11 \end{bmatrix}.$$

Moreover, by solving the optimization problem in (29), we obtain the H_∞ index $\gamma = 1.94$ and the following matrices:

$$L_0 = \begin{bmatrix} 0.5 \\ -0.79 \end{bmatrix}, L_1 = \begin{bmatrix} -0.10 \\ 0.16 \end{bmatrix}.$$

To illustrate the efficiency of the zonotope-based interval estimation method, a comparison is made with the interval observer design method. The simulation results are presented in Fig.1 and Fig.2 where the pink and blue dotted lines correspond respectively to upper and lower bounds of the estimate obtained by the interval observer. However the red and green dashed lines correspond respectively to upper and lower bounds of the state obtained by the proposed method. These figures show that the proposed method gives more accurate interval estimation results than the interval observer design method.

In the following, in order to better illustrate the feasibility

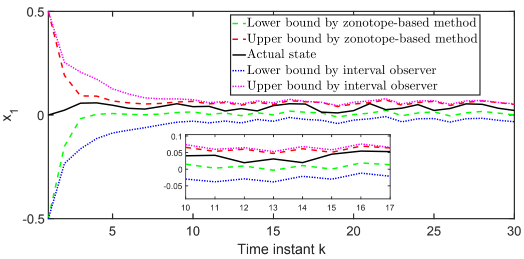

Fig. 1. The interval estimation of $x_1(k)$

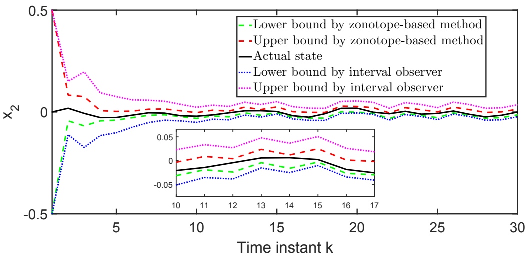

Fig. 2. The interval estimation of $x_2(k)$

and effectiveness of the proposed method, a numerical example from (Lam et al., 2015) is used to compare the proposed method with interval observer design method.

6.2 *Example 2: Case of non cooperative estimation error*

Consider a numerical time-delay linear system in the form of (8) with:

$$A_0 = \begin{bmatrix} 0.5 & -0.3 \\ -0.8 & 0.1 \end{bmatrix}, A_1 = \begin{bmatrix} -0.11 & 0.03 \\ 0.17 & -0.11 \end{bmatrix}, F = 0.1$$

$$B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, D = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, C = [\, 1 \;\; 0 \,].$$

The known input is chosen as $u(k) = 0.1$ and the disturbance and measurement noise are bounded as

$$w(k) \in \langle 0, H_w \rangle, \quad v(k) \in \langle 0, H_v \rangle$$

where $H_w = 0.1$ and $H_v = 0.01$.
The initial state is assumed to be $x(0) \in \langle 0, H_0 \rangle$ where

$$H_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

In this case, the interval observer can not be designed since the LMIs given in Thorem 1. are not solvable. However, independent of the cooperativity constraint, an interval estimation, based on the zonotope-based method, can be implemented and by solving the LMIs in (29), we obtain $\gamma = 1.94$. and :

$$L_0 = \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix}, L_1 = \begin{bmatrix} -0.11 \\ 0.17 \end{bmatrix}.$$

The simulation results are givinig Fig 3.and Fig 4. On these figures, the state coordinates are shown with the corresponding bounding variables from the zonotope-based interval estimation method. Compared with the interval observer design, the zonotope-based interval estimation method is independent from the cooperativity constraint and gives more accurate estimation results.

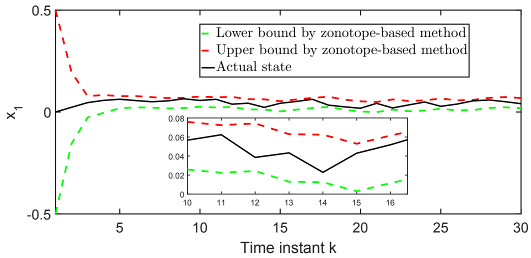

Fig. 3. The interval estimation of $x_1(k)$

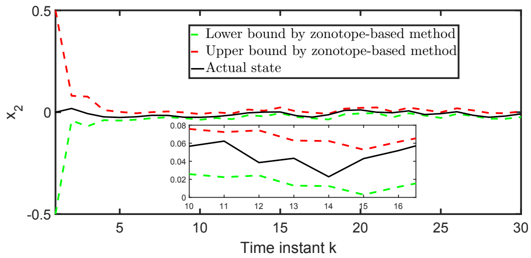

Fig. 4. The interval estimation of $x_2(k)$

7. CONCLUSIONS

This paper proposes an interval observer design and zonotope-based interval estimation methods for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises. An interval observer is designed based on cooperativity conditions of error dynamics. However, the zonotope-based interval estimation method is proposed by via a robust observer design based on H_∞ technique and zonotopic analysis. Compared with interval observers, the proposed method is independent of cooperativity constraint and coordinate transformation and gets more accurate estimation results. In further works, the proposed method will be extended to delay-dependent stability approach and robust diagnosis for discrete-time systems with time-delay will be investigated.

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