**Preprints of the 21st IFAC World Congress (Virtual)**
**Berlin, Germany, July 12-17, 2020**

# Robust Regulator Design of General Linear Systems with Sampled Measurements

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**Abstract:** This paper studies the robust output regulation problem of general linear continuous-time systems with periodically sampled measurements, consisting of both the regulation errors and the extra measurements. With some standard conditions, we propose a novel robust implementable regulator design paradigm, that is comprised of a generalized zero-order hold device, a discrete-time compensator, a discrete-time washout filter and a discrete-time stabilizer.

Keywords: Robust output regulation, internal model, washout filter, sampled measurements

## 1. INTRODUCTION

The problem of output regulation is to design a controller so as to achieve asymptotic trajectory tracking and/or disturbance compensation. Taking robustness into consideration, the internal model principle has been regarded as the most effective design and analysis tool since the seminal work Francis and Wonham [1976] for linear continuous-time systems. Internal model-based methods have been well developed for continuous-time nonlinear systems with continuous measurements(e.g., Isidori and Byrnes [1990], Huang [2004], Wang and Kellett [2019]), hybrid systems (e.g., Marconi and Teel [2013], Forte et al. [2017]) and networked systems (e.g., Wang et al. [2019]).

In general, the internal model-based regulator consists of two main components: the internal model (compensating for the steady state input) and the stabilizer (stabilizing the closed loop for regulation purposes). According to the location of the stabilizer in the regulator structure, there are two kinds of control architectures: post-processing and pre-processing schemes (see e.g. Isidori and Marconi [2012]). Regarding the latter, the stabilizer is directly cascaded with the controlled plant, processing the regulation errors, while in the former the internal model is cascaded with the plant and hence processes the regulation errors. For single-input single-error (SISE) systems, both schemes are fundamentally equivalent. In spite of this, recently the pre-processing scheme has been shown to be more feasible in some cases, such as for nonlinear systems with non-vanishing extra measurements (e.g. Wang et al. [2020], Toledo et al. [2006]), and multi-rate systems (e.g. Antunes et al. [2014]).

On the other hand, measurement information for feedback design is frequently obtained from periodically sampling sensors, rather than accessible continuously, and the regulator is practically implemented by digital devices or in combination with some simple analog devices, for example generalized zero-order hold devices. This naturally motivates the design of an implementable regulator driven by sampled measurements. In Castillo et al. [1997] for linear continuous-time systems, a state-feedback solution is studied locally by proposing a fully discrete-time regulator, which fulfills the internal model principle solely at the sampling time. To fulfill the continuous-time internal model property, in Lawrence and Medina [2001] a hybrid internal model is proposed. It is shown that the continuous-time steady state input can be compensated and there always exists a discrete-time stabilizer achieving the desired regulation purpose, though in the absence of robustness analysis. Motivated by this work, Marconi and Teel [2013] further develops a robust solution for SISE linear systems. In all the aforementioned results, the controlled continuous-time systems are required to be detectable by the regulation errors, which might not be fulfilled in practice, for example, as with the inverted pendulum on the cart considered in Section 4 below.

Motivated by the previous analysis, this paper studies the robust sampled-data regulation problem of *general* linear continuous-time systems, for which the detectability property is fulfilled by the whole measurements, consisting of both the regulation errors and the extra non-vanishing measurements. To compensate for the continuous-time steady state input, motivated by Lawrence and Medina [2001] we first design a generalized zero-order hold device. By cascading this device with the controlled plant, it is shown that the desired regulation objective can be achieved by designing a discrete-time output feedback stabilizer fulfilling two conditions, i.e., stabilizing the closed loop at the origin and compensating for the steady state input. Both conditions are shown to be sufficient and necessary for systems with the same number of inputs and regulation errors. To fulfill both conditions, we further propose a discrete-time compensator and a discrete-time washout filter, which in turn simplifies the problem to the design of a discrete-time output feedback stabilizer for a super-augmented discrete-time linear system that is stabilizable and detectable. This, in turn, is easily solved by using standard linear control theory. It is worth noting that the zero-order hold device and the discrete-time compensator are copies of the exosystem and its discretized form, respectively. By regarding both together as an internal model, the proposed robust implementable regulator naturally matches the pre-processing internal model-based structure proposed in Wang et al. [2020].

This paper is organized as follows. In Section 2 the considered problem is explicitly formulated and some standing assumptions are presented. Section 3 presents the main results. To show the effectiveness of the proposed approach, the linearly approximated model of the inverted pendulum on the cart is studied in Section 4. Finally, a brief conclusion is made in Section 5.

## 2. PROBLEM STATEMENT

Consider the output feedback regulation problem for linear systems

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= A\,x + B\,u + P\,w \\ y &= C\,x + Q\,w \end{aligned} \tag{1}$$

with exogenous states $w \in \mathbb{R}^d$, states $x \in \mathbb{R}^n$, inputs $u \in \mathbb{R}^m$ and measurements $y \in \mathbb{R}^q$. We deal with a *general* class of linear systems in which the measurements $y$ consist of regulation errors, to be steered to zero asymptotically, and also extra measurements on which no specific regulation requirements are fixed, both of which are periodically sampled with the sample time $T > 0$. In other words, the measurements available for feedback are given by the sampled regulated error $e(t) := C_e x(t_k) + Q_e w(t_k) \in \mathbb{R}^{q_e}$ and the sampled extra measurement $y_m(t) := C_m x(t_k) + Q_m w(t_k) \in \mathbb{R}^{q_m}$ for $t \in [t_k, t_{k+1})$, with $q = q_e + q_m$, $t_k = kT$ and $k \in \mathbb{N}_+$. As customary in the field of output regulation, we assume that $S$ is neutrally stable and there exists an invariant compact set $\mathcal{W} \in \mathbb{R}^d$ such that $w(t) \in \mathcal{W}$ for all $t \geq 0$.

In this setting, the control objective is to design a *robust implementable* regulator driven by the sampled measurements $(e(t), y_m(t))$ such that the resulting closed-loop trajectories are bounded, and the continuous-time regulation error $\hat{e}(t) := C_e x(t) + Q_e w(t)$ asymptotically converges to zero. As in Francis and Wonham [1976], Wang et al. [2020], we are interested in a *robust* solution, i.e., the above control objective is still guaranteed even if all system matrices in (1) except $S$ vary in a (small) neighborhood of their nominal forms [1]. Additionally, this paper presents an *implementable* solution that can be directly implemented by a computer only, or together with some simple analog devices, such as generalized zero-order hold devices (see Lawrence and Medina [2001]).

We will require some standard assumptions, previously used for a robust continuous-time solution (see e.g. Francis and Wonham [1976]).

*Assumption 1.* (i) The matrix triplet $(A, B, C)$ is stabilizable and detectable;

(ii) The non-resonance condition

$$\operatorname{rank}\begin{pmatrix} A - \lambda I_n & B \\ C_e & 0 \end{pmatrix} = n + q_e\,, \quad \forall \lambda \in \sigma(S) \tag{2}$$

holds.

With Assumption 1, it immediately follows that for any pairs of matrices $(P, Q_e)$, there exist $\Pi_x \in \mathbb{R}^{n\times d}$ and $\Psi \in \mathbb{R}^{m\times d}$ such that the regulator equations

$$\begin{aligned} \Pi_x S &= A\Pi_x + B\Psi + P \\ 0 &= C_e\Pi_x + Q_e\,. \end{aligned} \tag{3}$$

are satisfied.

On the other hand, as in Lawrence and Medina [2001], Castillo et al. [1997], in order to preserve the stabilizability and detectability of system (1) after discretization, the following assumption is made.

*Assumption 2.* Suppose that the sampling period $T$ is not pathological from the pair $(A, S)$. That is, for any distinct $\lambda_i, \lambda_j \in \sigma(A) \bigcup \sigma(S)$, $\lambda_i - \lambda_j \neq \frac{2k\pi}{T}$ for any $k \in \mathbb{N}$.

Note that due to the presence of the sampled measurements, the resulting system is fundamentally hybrid. In this paper, we will follow the notation from Goebel et al. [2012] to represent a hybrid system as a combination of a flow dynamics and a jump dynamics, which are described by a differential equation (e.g., $\dot{x}$) and a difference equation (e.g., $x^+$) , respectively. More explicitly, as in Marconi and Teel [2013], the jump is triggered by a clock

$$\begin{aligned} \dot{\tau} &= 1\,, \quad \tau \in [0, T) \\ \tau^+ &= 0\,, \quad \tau = T \end{aligned}$$

with the sample interval $T > 0$. Namely, the jumps occur every $T$ time units. Thus, the action of sampling the measurements $e$ and $y_m$ leads to

$$\left\{\begin{aligned} \dot{e} &= 0 \\ \dot{y}_m &= 0 \end{aligned}\right. \qquad \left\{\begin{aligned} e^+ &= C_e x + Q_e w \\ y_m^+ &= C_m x + Q_m w\,. \end{aligned}\right.$$

## 3. MAIN RESULTS

### 3.1 Problem Transformation

It is well-known (see Francis and Wonham [1976]) that the steady state input forcing the desired regulation objective of system (1) is a continuous-time signal of the form $u_{ss}(t) = \Psi w(t)$ with $\Psi$ provided by the regulator equations (3). Note that this continuous-time steady state input cannot be perfectly re-constructed by a discrete-time compensator. This naturally motivates us to embed the regulator with a continuous-time signal reconstructor, which compensates for the steady-state effect of the control input during flows, and needs to be implementable in our setting. In view of this, motivated by Lawrence and Medina [2001], we deal with a cascade of the controlled plant (1) and a generalized zero-order hold device, the latter described by

$$\begin{aligned} \dot{\zeta} &= (\Phi \otimes I_{q_e})\zeta \\ \zeta^+ &= v_\zeta \end{aligned} \tag{4}$$

where state $\zeta \in \mathbb{R}^{dq_e}$, input $v_\zeta \in \mathbb{R}^{dq_e}$ is used to reset the value of $\zeta$ at the sampling time, and the matrix $\Phi \in \mathbb{R}^{d\times d}$

[1] Note that if there exist uncertainties on matrix $S$, then the idea of adaptive internal model (e.g. Serrani et al. [2001]) can be employed.

has the minimal polynomial coincident with that of $S$, denoted by $\mathcal{P}_S(\lambda) = \sum_{i=0}^{d-1} s_i\lambda^i + \lambda^d$. To ease subsequent analysis, there is no loss of generality to let

$$\Phi = \begin{pmatrix} \mathbf{0}_{d-1} & I_{d-1} \\ -s_0 & (-s_1 \cdots - s_{d-1}) \end{pmatrix},$$

with $\mathbf{0}_{d-1}$ being a zero vector of dimension $d-1$.

The feedback control law is designed as

$$u = L\zeta + v_u \tag{5}$$

where $v_u \in \mathbb{R}^m$ denotes the residual control input to be determined later, and $L \in \mathbb{R}^{m\times dq_e}$ is such that the pair $(\Phi \otimes I_{q_e}, L)$ is observable.

By augmenting (4) and (5) with (1), we obtain

$$\begin{cases} \dot{\tau} = 1 \\ \dot{w} = S\,w \\ \dot{x} = A\,x + B\,L\,\zeta + B\,v_u + P\,w \\ \dot{\zeta} = (\Phi \otimes I_{q_e})\,\zeta \\ \dot{e} = 0 \\ \dot{y}_m = 0 \end{cases} \tag{6}$$

during the flow, and

$$\begin{cases} \tau^+ = 0 \\ w^+ = w \\ x^+ = x \\ \zeta^+ = v_\zeta \\ e^+ = C_e\,x + Q_e\,w \\ y_m^+ = C_m\,x + Q_m\,w \end{cases} \tag{7}$$

during the jump.

Note that for the sake of practical implementation, in the following both inputs $(v_u, v_\zeta)$ will be provided by a discrete-time dynamics, i.e., $\dot{v}_u = 0$ and $\dot{v}_\zeta = 0$. With this in mind, defining $w_k = w(k, t_k)$, $x_k = x(k, t_k)$, $\zeta_k = \zeta(k, t_k)$, $v_{u,k} = v_u(k, t_k)$ and $v_{\zeta,k} = v_\zeta(k, t_k)$ yields a *discrete-time* system

$$\begin{aligned} w_{k+1} &= S_D\,w_k \\ x_{k+1} &= A_D\,x_k + L_D\,\zeta_k + B_D\,v_{u,k} + P_D w_k \\ \zeta_{k+1} &= v_{\zeta,k} \\ y_k :&= \begin{pmatrix} e_k \\ y_{m,k} \end{pmatrix} = \begin{pmatrix} C_e x_k + Q_e w_k \\ C_m x_k + Q_m w_k \end{pmatrix} \end{aligned} \tag{8}$$

with $S_D = e^{S\,T}$, $A_D = e^{A\,T}$, $B_D = \int_0^T e^{A\,r}\mathrm{d}r\,B$, $L_D = \int_0^T e^{A\,(T-r)}BLe^{(\Phi\otimes I_{q_e})r}\mathrm{d}r$, and $P_D = \int_0^T e^{A\,(T-r)}Pe^{S\,r}\mathrm{d}r$. For this *discrete-time* system (8), the following holds.

*Lemma 1.* Suppose Assumptions 1 and 2 hold. System (8) is stabilizable and detectable with respect to inputs $v_k := \mathrm{col}\ (v_{u,k}, v_{\zeta,k})$ and outputs $y_k$ when $w_k = 0$, and there exist $\Pi_x \in \mathbb{R}^{n\times d}, \Pi_\zeta \in \mathbb{R}^{dq_e\times d}$ such that

$$\begin{aligned} \Pi_x S_D &= A_D \Pi_x + L_D\,\Pi_\zeta + P_D \\ \Pi_\zeta S_D &= (\Phi_D \otimes I_{q_e})\Pi_\zeta \\ 0 &= C_e \Pi_x + Q_e \end{aligned} \tag{9}$$

with $\Phi_D = e^{\Phi\,T}$.

*Proof.* With Assumption 1.(i) and 2, according to Kimura [1990] it can be deduced that $(A_D, B_D, C)$ is stabilizable and detectable. Then using the PBH test, it can be easily verified that system (8) is stabilizable and detectable with respect to inputs $v_k$ and outputs $y_k$ when $w_k = 0$. As for the solution of (9), we observe that, for any $\Psi \in \mathbb{R}^{m\times d}$, since $(\Phi\otimes I_{q_e}, L)$ is observable, there always exists a unique solution $\Pi_\zeta \in \mathbb{R}^{dq_e\times d}$ such that

$$\begin{aligned} \Pi_\zeta S &= (\Phi \otimes I_{q_e})\Pi_\zeta \\ \Psi &= L\Pi_\zeta\,. \end{aligned} \tag{10}$$

In view of the fact that (9) is indeed the discretized form of equations (3) (derived by Assumption 1.(ii)) and (10), this indicates that such $(\Pi_x, \Pi_\zeta)$ is also the solution of (9). The proof is thus done. □

The desired robust regulator can be completed by designing an output feedback stabilizer for the discrete-time system (8), having the form

$$\begin{aligned} z_{k+1} &= A_z\,z_k + B_z\,y_k\,, \quad z_k \in \mathbb{R}^{n_z} \\ v_k &= K_z\,z_k + L_z\,y_k\,. \end{aligned} \tag{11}$$

*Theorem 1.* Suppose Assumptions 1 and 2 hold. The robust sampled output regulation problem of system (1) is solved by the regulator (4), (5), and (11) *if*

(a) the origin of the closed-loop discrete-time system (8), (11) with $w_k = 0$ is globally exponentially stable, and
(b) for any $Y_m \in \mathbb{R}^{q_m\times d}, \Pi_\zeta \in \mathbb{R}^{dq_e\times d}$, there exists a solution $\Pi_z \in \mathbb{R}^{n_z\times d}$ for the linear equations

$$\begin{aligned} \Pi_z S_D &= A_z\,\Pi_z + B_z\,(\,\mathbf{0}_{d\times q_e} \quad Y_m^\top\,)^\top \\ \begin{pmatrix} \mathbf{0}_{m\times d} \\ \Pi_\zeta S_D \end{pmatrix} &= K_z\,\Pi_z + L_z\,(\,\mathbf{0}_{d\times q_d} \quad Y_m^\top\,)^\top\,. \end{aligned} \tag{12}$$

*Proof.* By setting $\chi_k := \mathrm{col}\ (x_k, \zeta_k, z_k)$, the resulting closed-loop (8), (11) can be compactly described by

$$\begin{aligned} w_{k+1} &= S_D\,w_k \\ \chi_{k+1} &= A_{cl}\chi_k + P_{cl} w_k \end{aligned} \tag{13}$$

for some appropriately defined matrices $A_{cl}, P_{cl}$. With the requirement (a), it immediately follows that $|\sigma(A_{cl})| < 1$, i.e., all eigenvalues of $A_{cl}$ lie within the unit circle. Thus there exists a *unique* $\Pi_{cl} \in \mathbb{R}^{(n+dq_e+n_z)\times d}$ such that

$$\Pi_{cl} S_D = A_{cl}\Pi_{cl} + P_{cl}\,. \tag{14}$$

On the other hand, let $\Pi_x, \Pi_\zeta$ be a solution of (3), (10), which fulfill (9) by the proof of Lemma 1. Furthermore, by letting $Y_m = C_m\Pi_x + Q_m$, and with condition (b), there exists a solution $\Pi_z$ for the equations (12). With the derived triplet $(\Pi_x, \Pi_\zeta, \Pi_z)$ being the case, it can be easily concluded that $\Pi_{cl} = \mathrm{col}\ (\Pi_x, \Pi_\zeta, \Pi_z)$ is a solution of (14), and thus the unique one.

Let $\rho := \mathrm{col}\ (x, \zeta, e, y_m, z)$, which compactly expresses the hybrid system (6), (7), and (11) as the form

$$\begin{aligned} \dot{w} &= Sw\,, \quad w^+ = w \\ \dot{\rho} &= F_{cl}\rho + P_F w \\ \rho^+ &= J_{cl}\rho + P_J w \end{aligned} \tag{15}$$

with some appropriately defined matrices $F_{cl}, J_{cl}, P_F, P_J$, with $|\sigma(J_{cl}e^{F_{cl}T})| < 1$ by the requirement (a). Thus system (15) is exponentially stable at the invariant set $\mathcal{M} = \{(\tau, w, \rho) : \rho = \widehat{\Pi}_{cl}(\tau)w\}$ with $\widehat{\Pi}_{cl}(\tau) : [0, T) \to \mathbb{R}^{(n+dq_e+n_z)\times d}$ being the *unique* solution of the equations

$$\begin{aligned} \frac{d\widehat{\Pi}_{cl}(\tau)}{d\tau} + \widehat{\Pi}_{cl}(\tau)S &= F_{cl}\widehat{\Pi}_{cl}(\tau) + P_F \\ \widehat{\Pi}_{cl}(0) &= J_{cl}\widehat{\Pi}_{cl}(T) + P_J \,. \end{aligned} \tag{16}$$

With (3), (10), and (12), simple calculations show that

$$\widehat{\Pi}_{cl}(\tau) := \mathrm{col}\,(\Pi_x, \Pi_\zeta, 0, Y_m\, e^{-S\tau}, \Pi_z\, e^{-S\tau})$$

is a solution of (16), and thus is the unique one. Since $C_e\Pi_x + Q_e = 0$ in (3), it indicates that $\hat{e}(t) := C_e x(t) + Q_e w(t)$ vanishes in the set $\mathcal{M}$. The proof is thus completed. $\square$

*Corollary 1.* Suppose Assumptions 1 and 2 hold. The sampled robust output regulation problem of system (1) with $m = q_e$ is solved by the regulator (4), (5), and (11) *if and only if* the requirements (a) and (b) in Theorem 1 are fulfilled.

*Proof.* The "if" part has been proved in Theorem 1. As for the proof of "only if" part, we can see that the requirement (a) is clear. Thus, we now focus on the proof of the requirement (b). Using the notations in the proof of Theorem 1, we use (15) to denote the resulting hybrid closed-loop system (1), (4), (5), and (11). Simple calculations show that (16) has the unique solution $\hat{\Pi}_{cl}(\tau)$, which can be partitioned as

$$\widehat{\Pi}_{cl}(\tau) := \mathrm{col}\,(\Pi_x(\tau), \Pi_\zeta(\tau), 0, Y_m\, e^{-S\tau}, \Pi_z\, e^{-S\tau})$$

and

$$0 = C_e\Pi_x(\tau) + Q_e\,. \tag{17}$$

To be explicit, we can equivalently rewrite (16) as

$$\begin{aligned} &\begin{cases} \dot{\Pi}_x(\tau) = -\Pi_x(\tau)S + A\,\Pi_x(\tau) + B\Psi(\tau) + P \\ \Pi_x(0) = \Pi_x(T) \end{cases} \\ &\left\{ \dot{\Pi}_\zeta(\tau) = -\Pi_\zeta(\tau)S + (\Phi \otimes I_{q_e})\Pi_\zeta(\tau) \right. \\ &\left\{ \Pi_z S_D = A_z\,\Pi_z + B_z\,\begin{pmatrix} \mathbf{0}_{d\times q_e} & Y_m^\top \end{pmatrix}^\top \right. \end{aligned} \tag{18}$$

where $Y_m = C_m\Pi_x(0) + Q_m$, and $\Psi(\tau) = \Psi_\zeta(\tau) + \Psi_{v_u} e^{-S\tau}$ with $\Psi_\zeta(\tau) = L\Pi_\zeta(\tau)$ and

$$\begin{pmatrix} \Psi_{v_u} \\ \Pi_\zeta(0)S_D \end{pmatrix} = K_z\,\Pi_z + L_z\,\begin{pmatrix} \mathbf{0}_{d\times q_d} & Y_m^\top \end{pmatrix}^\top .$$

Putting (17) and the first part of (18) together, due to the nonresonance condition (2) and $m = q_e$, we observe that they reduce to the continuous-time regulator equations (3) and have the unique constant solution $\Pi_x(\tau), \Psi(\tau)$, i.e., independent of $\tau$. On the other hand, taking the second of (18) into consideration, we can deduce that $\Pi_\zeta$ is also independent of $\tau$ since it allows for a solution $\Pi_\zeta(\tau)$ if and only if $\dot{\Pi}_\zeta(\tau) = 0$. Furthermore, we observe that $\Psi_{v_u}$ is constant, and satisfies $\Psi_{v_u} = [\Psi - \Psi_\zeta]e^{S\tau}$. Since $\Psi$ and $\Psi_\zeta$ are constant and $e^{S\tau}$ is nonsingular for all $\tau > 0$, it is clear that $\Psi = \Psi_\zeta$, leading to $\Psi_{v_u} = 0$. Therefore, in view of the previous observations, the requirement (b) can be easily concluded by using the fact that $P, Q$ are arbitrary matrices. $\square$

### 3.2 Design Output Feedback Stabilizer (11)

In the previous subsection, with the cascade of a generalized zero-order hold device, we have shown that the original output regulation problem with sampled measurements can be transformed into the design of a discrete-time output feedback stabilizer (11) such that the requirements (a) and (b) in Theorem 1 are fulfilled.

Observe that the second equation of (12) can be partitioned into two parts, consistent with the left side of $\mathbf{0}_{m\times d}$ and $\Pi_\zeta S_D$. Then it can be seen that the requirement of Theorem 1.(b) characterizes two extra features of the stabilizer (11), in addition to the stabilizing purpose. One is to block the steady state of the extra measurements $y_m$, denoted by $Y_m w_k$, while the second is to compensate for the steady state of input $v_{\zeta,k}$, denoted by $\Pi_\zeta S_D w_k$.

Regarding the first, motivated by Wang et al. [2020], we propose a washout filter to $y_{m,k}$ so as to block its zeros synchronized with the discretized exosystem of the form

$$\begin{aligned} \xi_{k+1} &= F_{\mathrm{f}}\,\xi_k + G_{\mathrm{f}}\,y_{m,k} \\ y_{\mathrm{f,k}} &= y_{m,k} - \Gamma_{\mathrm{f}}\,\xi_k \end{aligned} \tag{19}$$

where $y_{\mathrm{f,k}}$ is the filter output, $(F_{\mathrm{f}}, \Gamma_{\mathrm{f}}) \in \mathbb{R}^{dq_m\times dq_m} \times \mathbb{R}^{q_m\times dq_m}$ is an observable pair with $|\sigma(F_{\mathrm{f}})| < 1$, and $G_{\mathrm{f}} \in \mathbb{R}^{dq_m\times q_m}$ is such that matrix $\Phi_D \otimes I_{q_m} = F_{\mathrm{f}} + G_{\mathrm{f}}\Gamma_{\mathrm{f}}$.

This in turn indicates that, given any $Y_m \in \mathbb{R}^{q_m\times d}$ there exists a unique solution $\Pi_{\mathrm{f}} \in \mathbb{R}^{q_m d\times d}$ for the linear matrix equation

$$\begin{aligned} \Pi_{\mathrm{f}}\,S_D &= F_{\mathrm{f}}\,\Pi_{\mathrm{f}} + G_{\mathrm{f}}\,Y_m \\ 0 &= Y_m - \Gamma_{\mathrm{f}}\,\Pi_{\mathrm{f}}\,. \end{aligned} \tag{20}$$

To compensate for the steady state of the input $v_{\zeta,k}$, we propose a compensator as

$$\begin{aligned} \eta_{k+1} &= (\Phi_D \otimes I_{q_e})\eta_k + N\,\mu_k \\ v_{\zeta,k} &= \eta_k + \mu_k \end{aligned} \tag{21}$$

where $\eta_k \in \mathbb{R}^{dq_e}$, inputs $\mu_k \in \mathbb{R}^{dq_e}$ will be determined later, and $N \in \mathbb{R}^{dq_e\times dq_e}$ is such that $\Phi_D \otimes I_{q_e} - N$ is a matrix with all eigenvalues strictly smaller than 1. This in turn indicates that, by letting $\Pi_\eta = \Psi_{v_\zeta} := \Pi_\zeta S_D$, the linear matrix equations

$$\Pi_\eta\,S_D = (\Phi_D \otimes I_{q_e})\,\Pi_\eta \tag{22}$$

hold by recalling the second equation of (9).

*Lemma 2.* Suppose Assumptions 1 and 2 hold. The super-augmented discrete-time system (8), (19), and (21) is stabilizable and detectable with respect to inputs $(v_{u,k}, \mu_k)$ and outputs $\mathrm{y}_k := \begin{pmatrix} e_k \\ y_{\mathrm{f,k}} \end{pmatrix}$, when $w_k = 0$.

With this lemma, we then can design a discrete-time output feedback stabilizer for the super-augmented discrete-time system (8), (19), and (21) when $w_k = 0$, which takes the form

$$\begin{aligned} \vartheta_{k+1} &= A_\vartheta\,\vartheta_k + B_\vartheta \mathrm{y}_k\,, \quad \vartheta \in \mathbb{R}^{n_z - dq} \\ \begin{pmatrix} v_{u,k} \\ \mu_k \end{pmatrix} &= C_\vartheta\,\vartheta_k + D_\vartheta\,\mathrm{y}_k\,. \end{aligned} \tag{23}$$

It turns out that the cascade of the systems (19), (21), and (23), which is a system with inputs $(e_k, y_{m,k})$ and outputs $(v_{u,k}, v_{\zeta,k})$, fulfills the requirements of Theorem 1 as formalized in the following. The proof is straightforward with $\Pi_z = \begin{pmatrix} \Pi_{\mathrm{f}}^\top & \Pi_\eta^\top & \mathbf{0}_{d\times(n_z-dq)} \end{pmatrix}^\top$ and is thus omitted.

*Theorem 2.* Let (23) be a stabilizer for the super-cascaded system (8), (19), and (21). Then with system (11) defined by the cascade of (19), (21), and (23), the requirements (a) and (b) in Theorem 1 are fulfilled.

Before the close of this section, we observe that the generalized zero-order hold device (4) contains $q_e$ copies of the exosystem and the compensator (19) contains $q_e$ copies of the discretized exosystem. By regarding both components as a unit, it naturally fulfills the requirements for the internal model in Lawrence and Medina [2001], Marconi and Teel [2013]. In other words, the generalized zero-order hold device (4) and the compensator (19) together can be regarded as the internal model. By doing so, as shown in Fig. 1, we further observe that the proposed regulator structure matches the pre-processing internal model-based scheme proposed in Wang et al. [2020].

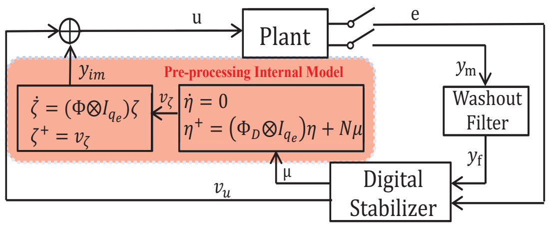


Fig. 1. Structure of the proposed regulator.

## 4. AN EXAMPLE

Consider the output regulation problem of an inverted pendulum on a cart, whose linearly approximated model is described by

$$\begin{aligned} m_0\ddot{q} &= -mg\theta - \mu_f\dot{q} + u + P_1 w \\ m_0\ell\ddot{\theta} &= (m_0+m)g\theta + \mu_f\dot{q} - u + P_2 w \end{aligned} \tag{24}$$

where $q$ is the distance of the cart from the zero reference, $\theta$ is the angle of the pendulum with respect to the vertical axis, input $u$ is the horizontal force applied to the cart and $w$ is the state of the exosystem having the form

$$\dot{w} = Sw\,, \quad S = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}.$$

All other parameters are as in Wang et al. [2019]. Suppose both $q$ and $\theta$ are measured periodically by some sensors, with the sample period $T = 0.1$. In this setting, the problem in question is to design an implementable regulator taking advantage of the sampled measurements such that all signals in (24) are bounded and the regulation output $\theta(t)$ asymptotically converges to zero.

By setting $x := \mathrm{col}\,(q+\ell\theta, \dot{q}+\ell\dot{\theta}, \theta, \dot{\theta})$, we can rewrite (24) in the form (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\mu_f}{m_0\ell} & \frac{(m_0+m)g}{m_0\ell} & -\frac{\mu_f}{m_0} \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ \frac{P_1+P_2}{m_0} \\ 0 \\ \frac{P_2}{m_0\ell} \end{pmatrix},$$
$$B^\top = \begin{pmatrix} 0 & 0 & 0 & \frac{-1}{m_o\ell} \end{pmatrix}, \qquad C_e = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix},$$
$$Q_e = Q_m = 0\,.$$

It is clear that the pair $(A, C_e)$ is not detectable. Namely, in order to solve the robust output regulation problem at hand, we need an extra measurement. In the following, to simplify the subsequent analysis, there is no loss of generality to choose the extra measurement $y_m := x_1 = q+\ell\theta$, i.e., $C_m = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. Straightforward calculations show that Assumption 1.(i) is fulfilled.

Following the design paradigm proposed in Section 3, we design the generalized zero-order hold device (4), the feedback law (5) with $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$, the washout filter (19) with

$$F_{\mathrm{f}} = \begin{pmatrix} 0.5657 & 0.0998 \\ -0.3325 & 0.9950 \end{pmatrix}, G_{\mathrm{f}} = \begin{pmatrix} 0.4293 \\ 0.2327 \end{pmatrix}, \Gamma_{\mathrm{f}}^\top = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the compensator (21).

With the above design, the remaining problem is to consider the output feedback stabilization of the corresponding discrete-time system (8), (19), and (21), for which the desired stabilizer can be easily derived by solving two discrete-time Riccati equations.

The simulation is performed with $m_0 = 0.5, m = 2, \mu_f = 0.2, g = 9.8$ and $\ell = 0.3$. Assumption 1.(ii) and Assumption 2 can be easily verified to be true. As seen from Fig. 2, it can be seen that the regulation error $e(t)$ asymptotically converges to zero, while $y_m(t)$ is bounded.

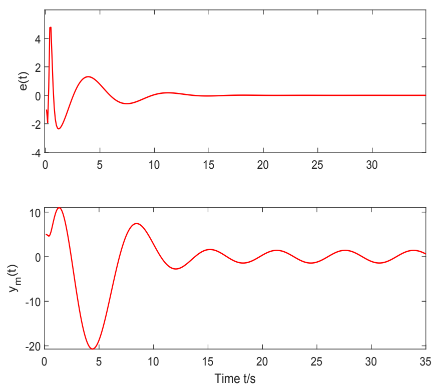


Fig. 2. Trajectories of $e(t)$ and $y_m(t)$.

## 5. CONCLUSION

In this paper, the robust output regulation problem is investigated for general linear continuous-time systems with periodically sampled measurements, consisting of both the regulation errors and extra measurements that are generally non-vanishing in steady state. By designing a generalized zero-order hold device, it is shown that the original problem can be transformed into designing an output feedback stabilizer fulfilling two conditions for a discrete-time system. Furthermore, by designing a discrete-time compensator and a discrete-time washout filter, it is shown there always exists a discrete-time output feedback stabilizer for the resulting super-augmented system, which together with the previously designed compensator and filter completes the design such that such conditions are fulfilled. Note that the proposed regulator structure naturally matches with the pre-processing scheme proposed in Wang et al. [2020] by regarding the generalized zero-order hold device and the discrete-time compensator as an internal model.

## Appendix A. PROOF OF LEMMA 2

Instrumental to the subsequent analysis is the following lemma, whose proof can be easily derived using Assumptions 1.(i) and 2 and arguments in Kimura [1990].

*Lemma 3.* Suppose Assumptions 1 and 2 hold. Let $\mathcal{T}_1(\lambda) = \mathrm{col}\,(1, \lambda, \dots, \lambda^{d-1}) \otimes I_{q_e}$. Then

(a) the matrix triplet $(A_D, B_D, C)$ is stabilizable and detectable, and
(b) *the discrete-time non-resonance condition*
$$\mathrm{rank}\begin{pmatrix} A_D - \lambda I_n & L_D \mathcal{T}_1(\lambda) \\ C_e & 0 \end{pmatrix} = n + q_e \qquad \text{(A.1)}$$
holds for all $\lambda \in \sigma(S_D)$.

With the above lemma, we now proceed to use the PBH test and Lemma 3 to verify the stabilizability and detectability of (8). Regarding the stabilizability, it holds if and only if all rows of the following matrix
$$\begin{pmatrix} F_\mathrm{f} - \lambda I & G_\mathrm{f} C_m & 0 & 0 & 0 & 0 \\ 0 & A_D - \lambda I & L_D & 0 & B_D & 0 \\ 0 & 0 & -\lambda I & I & 0 & N \\ 0 & 0 & 0 & (\Phi_D - \lambda I) \otimes I_{q_e} & 0 & I \end{pmatrix}$$
are independent for all $\lambda \in \{\lambda \in \mathbb{C} | \, |\lambda| \geq 1\}$. Recalling that $|\sigma(F_\mathrm{f}) < 1$ and that $(A_D, B_D)$ is stabilizable, the above verification problem can be simplified to verifying
$$\mathrm{rank}\begin{pmatrix} \lambda I & I & N \\ 0 & (\Phi_D - \lambda I) \otimes I_{q_e} & I \end{pmatrix} = \sharp\mathrm{rows} \qquad \text{(A.2)}$$
for all $\lambda \in \{\lambda \in \mathbb{C} | \, |\lambda| \geq 1\}$, which is clearly true.

To further explore the detectability, it is true if and only if for all $\lambda \in \{\lambda \in \mathbb{C} | \, |\lambda| \geq 1\}$, the matrix
$$\begin{pmatrix} F_\mathrm{f} - \lambda I & G_\mathrm{f} C_m & 0 & 0 \\ 0 & A_D - \lambda I & L_D & 0 \\ 0 & 0 & -\lambda I & I \\ 0 & 0 & 0 & (\Phi_D - \lambda I) \otimes I_{q_e} \\ 0 & C_e & 0 & 0 \\ -\Gamma_\mathrm{f} & C_m & 0 & 0 \end{pmatrix}$$
is full-column-rank. Since $F_\mathrm{f} + G_\mathrm{f}\Gamma_\mathrm{f} = \Phi_D \otimes I_{q_m}$ by construction, the above verification reduces to show the matrix
$$\begin{pmatrix} (\Phi_D - \lambda I) \otimes I_{q_m} & 0 & 0 & 0 \\ 0 & A_D - \lambda I & L_D & 0 \\ 0 & 0 & -\lambda I & I \\ 0 & 0 & 0 & (\Phi_D - \lambda I) \otimes I_{q_e} \\ 0 & C_e & 0 & 0 \\ -\Gamma_\mathrm{f} & C_m & 0 & 0 \end{pmatrix}$$
is full-column-rank for all $\lambda \in \{\lambda \in \mathbb{C} | \, |\lambda| \geq 1\}$. For all $\lambda \in \{\lambda \in \mathbb{C} | \, |\lambda| \geq 1, \lambda \notin \sigma(\Phi_D)\}$, the above matrix is full-column-rank if and only if $(A_D, C)$ is detectable, which has been shown to be true.

With this being the case, we turn to investigate the case that $\lambda \in \sigma(\Phi_D)$. Since both $(\Phi_D \otimes I_{q_m}, \Gamma_\mathrm{f})$ is observable by construction, by taking appropriate column transformation, the previous verification reduces to show
$$\mathrm{rank}\begin{pmatrix} A_D - \lambda I & L_D \mathcal{T}_1(\lambda) \\ C_e & 0 \end{pmatrix} = n + q_e \quad \forall \lambda \in \sigma(\Phi_D)\,,$$
which clearly is true by recalling (A.1) and the fact that $\sigma(\Phi_D) = \sigma(S_D)$. The proof is thus completed.

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