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Tube-Based Internal Model Control of Minimum-Phase Input-Affine MIMO Systems under Input Constraints

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Abstract: In this paper an optimal control approach based on a combination of inversion-based control and internal model control (IMC) is designed to keep the controlled states of a minimum-phase input-affine MIMO system within predefined tubes while respecting input constraints. This contribution extends recently presented results for the SISO case to nonlinear input-affine MIMO systems. The developed approach uses ideas developed in the design of inversion-based IMC controllers for setpoint tracking and extends them to 'tube tracking'. It shows the interesting result that for input-affine systems the control task of maintaining each controlled state within a tube while minimizing energy consumption and respecting input constraints can be expressed as a convex quadratic optimization problem. This concept allows to handle dynamic systems where the number of control inputs differs from the number of controlled outputs are not equal. The control approach is illustrated by three simulation examples.

Keywords: internal model control, feedforward control, model-based control, optimal control, differentially flat systems, input-affine systems, building technology, tube control, band control

1. INTRODUCTION

In many control applications where high performance trajectory tracking is not required, it is desired to keep the controlled state within some predefined interval. A classical example for 'tube control' or 'band control' arises from the building sector, where the comfort requirements are usually given in terms of an upper and lower bound of the comfort range (Brelih, 2013). An aerospace application of 'tube control' is given by satellites' path control where the desired satellite's path is delimited by two orbits. The control task of remaining within some predefined band can be achieved by the perfect tracking of any trajectory lying in the band. However, to address the growing energy saving demand the question arises about developing an optimal control approach to achieve a robust tube control with minimal energy consumption.

In comparison to the broad spectrum of controllers for trajectory tracking, band control has been exclusively treated by model predictive control (MPC). By transforming the control task into a sequence of dynamic optimization problems, band control can be easily addressed by incorporating state constraints for the controlled variable. However, the major drawback of MPC lies in the related computational load. An application of an MPC approach in bulidings' automation can be a challenging task due to the limited computational power offered by the standard processing hardware. Therefore, the computational feasibility is considered as an important aspect during the design of control strategies for systems with low processing power.

Aside from the predictive aspect, IMC and MPC have in some sort similar properties. Both control concepts rely upon a mathematical model of the process to calculate the suitable input leading to the desired output. While MPC uses the model for dynamic optimization to predict the suitable control input, IMC employs an inverse system model to produce a stabilizing control law. Due to its straightforward design, strong robustness properties and low computational cost, IMC is considered as an attractive model-based control strategy in many industrial applications. However, the conventional IMC structure is exclusively designed for setpoint tracking and lacks the ability to incorporate a control interval despite the knowledge of the system dynamics disposed in the IMC approach through the model.

This paper proposes an extension of the conventional design of inversion-based IMC controllers for setpoint tracking into tube tracking. It shows that for input-affine systems the control task of keeping the control outputs within predefined intervals while respecting input constraints and minimizing energy consumption can be achieved by solving a convex optimization problem. The basic idea of the approach consists in considering the references for the control outputs as constrained optimization variables in the calculation of the optimal control inputs. The tubes in which the controlled variables should be kept are expressed in terms of box constraints for the variable references. The proposed approach shows the interesting ability to handle systems with unequal numbers of control inputs and control outputs. Thus, systems showing a high degree freedom with more control inputs than control outputs as well as systems with conflicted control outputs where the number of control inputs is reduced can be handled by this approach.

This paper is structured as follows: In Section 2, previously developed ideas in the design of inversion-based IMC controllers for setpoint tracking are presented. Section 3 presents the main contribution of this paper: an extension of the inversion-based IMC approach for setpoint tracking to cover tube tracking. An illustrative example of the developed approach is given in Section 4, while Section 5 concludes the paper.

2. INVERSION-BASED INTERNAL MODEL CONTROL FOR SETPOINT TRACKING

This section gives a survey on the IMC concept and previously developed ideas in the design of inversion-based IMC controllers for setpoint tracking.

2.1 IMC structure

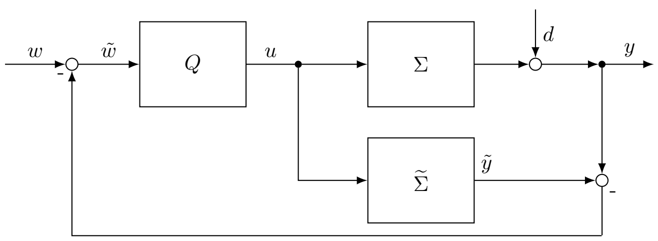

Fig. 1. IMC structure for setpoint tracking

Fig. 1 illustrates the IMC structure, where Q is the IMC controller, Σ is the plant and $\widetilde{\Sigma}$ is the plant model (Morari and Zafiriou, 1989). Two major differences distinguish the IMC structure from the standard control loop. First, the plant model constitutes an explicit component of the control scheme in the IMC structure. Second, the information routed back within the feedback signal in the two control schemes are not identical. In the standard control loop, the feedback represents the output of the plant. In the IMC structure the difference between the plant output y and the model output $\widetilde{y}$ is routed back and the feedback expresses therefore the mismatch between the plant and the model. The IMC strategy can be explained as follows: assuming a perfect match between plant and model and that no disturbances occur, plant and model have the same output and the feedback signal vanishes. In this ideal case, the IMC controller acts as a feedforward controller. In the presence of process-model mismatch or disturbances, the IMC controller tries to attenuate the occuring uncertainties. Thus, for systems with small model uncertainties the IMC controller is usually designed as a feedforward controller.

2.2 Definition of an input-affine system

Consider the nonlinear input-affine plant model

$$\widetilde{\Sigma}: \quad \dot{x} = f(x) + \sum_{j=1}^{q} g_j(x)u_j,\ x(0) = x_0 \tag{1a}$$
$$\widetilde{y}_1 = h_1(x) \tag{1b}$$
$$\vdots \tag{1c}$$
$$\widetilde{y}_p = h_p(x) \tag{1d}$$

with time $t \in \mathbb{R}$, state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^q$ and p outputs $\widetilde{y}_i \in \mathbb{R},\ i = 1, ..., p$. The vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_j : \mathbb{R}^n \to \mathbb{R}$ and the functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be sufficiently smooth. The components of the input u are constrained by

$$\mathcal{U} = \{u \in \mathbb{R}^q | u_j \in [u_j^-, u_j^+], j = 1, ..., q\}. \tag{2}$$

These conditions account for the fact that in many applications certain input constraints have to be taken into account, e.g. limited cooling/heating power for air condioning systems or maximal speed of a fan in a ventilation system. The model is assumed to be stable and not to contain a time-delay.

2.3 Input-output normal form

The inversion-based control design used in this work is based on the input-output normal form of the considered system (Isidori, 1995). The prerequisite for the input-output representation of the MIMO system (1) is the definition of the vector relative degree $\{r_1, ..., r_p\}$.

Definition 1: The nonlinear MIMO system (1) has a well-defined vector relative degree $\{r_1, ..., r_p\}$ at $x = \overline{x}$ if for all x in a neighbourhood of $\overline{x}$ and all admissible inputs $u = (u_1, ..., u_q)$ the following two conditions are fulfilled:

- $L_{g_i} L_f^k h_j(x) = 0, \forall i \in \{1, ..q\}, \forall j \in \{1, ..p\}, \forall k \in \{1, ..., r_j - 2\}$
- $L_{g_i} L_f^{r_j-1} h_j(x) \neq 0$ for at least one $i \in \{1, ..q\}, \forall j \in \{1, ..p\}$

The operators L_f and L_{g_i} represent the Lie derivatives along the vector fields f and g_i respectively. Literally, the element r_k of the vector relative degree $\{r_1, ..., r_p\}$ denotes the number of times the k-th output $\widetilde{y}_k$ has to be differentiated until at least one component $u_i, k = 1, ..., q$ of the input vector u appears explicitly. In the following, it is assumed that the relative degree $\{r_1, ..., r_p\}$ is well-defined at least locally in the neighborhood of $\overline{x}$.
The time derivatives $\widetilde{y}_i^{(r_i)}, i = 1, ..., p$ are given by

$$\begin{bmatrix} \widetilde{y}_1^{(r_1)} \\ \vdots \\ \widetilde{y}_{p-1}^{(r_{p-1})} \\ \widetilde{y}_p^{(r_p)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_{p-1}} h_{p-1}(x) \\ L_f^{r_p} h_p(x) \end{bmatrix}}_{b(x)} + D(x) \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_{q-1} \\ u_q \end{bmatrix}}_{u} \tag{3}$$

with the $(p \times q)$-decoupling matrix

$$D(x)=\begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1 & L_{g_2}L_f^{r_1-1}h_1 & \dots & L_{g_q}L_f^{r_1-1}h_1 \\ L_{g_1}L_f^{r_2-1}h_2 & L_{g_2}L_f^{r_2-1}h_2 & \dots & L_{g_q}L_f^{r_2-1}h_2 \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_p-1}h_p & L_{g_2}L_f^{r_p-1}h_p & \dots & L_{g_q}L_f^{r_p-1}h_p \end{bmatrix} \tag{4}$$

With the diffeomorphism

$$z=\begin{bmatrix} z_1\\ \vdots \\ z_n\end{bmatrix}=\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\begin{bmatrix}\xi_{1,1}\\ \xi_{1,2}\\ \vdots\\ \xi_{1,r_1}\\ \vdots\\ \xi_{p,1}\\ \vdots\\ \xi_{p,r_p}\\ \phi_{r+1}(x)\\ \vdots\\ \phi_n(x)\end{bmatrix}=\Phi(x)=\begin{bmatrix}h_1(x)\\ L_fh_1(x)\\ \vdots\\ L_f^{r_1-1}h_1(x)\\ \vdots\\ h_p(x)\\ \vdots\\ L_f^{r_p-1}h_p(x)\\ \phi_{r+1}(x)\\ \vdots\\ \phi_n(x)\end{bmatrix} \tag{5}$$

the system (1) is transformed into the following Byrnes-Isidori normal form

$$\widetilde{\Sigma}_1=\begin{cases} \dot{\xi}_{1,1} &= \xi_{1,2}\\ \dot{\xi}_{1,2} &= \xi_{1,3}\\ &\vdots\\ \dot{\xi}_{1,r_1} &= \widetilde{b}_1(\xi,\eta)+\sum_{j=1}^{q}\widetilde{D}_{1,j}(\xi,\eta)u_j\\ &\vdots\\ \dot{\xi}_{p,1} &= \xi_{p,2}\\ \dot{\xi}_{p,2} &= \xi_{p,3}\\ &\vdots\\ \dot{\xi}_{p,r_p} &= \widetilde{b}_p(\xi,\eta)+\sum_{j=1}^{q}\widetilde{D}_{p,j}(\xi,\eta)u_j \end{cases}.$$

$$\widetilde{\Sigma}_2=\begin{cases} \dot{\eta}_1 &= q_1(\xi,\eta)+\sum_{j=1}^{q}P_{1,j}(\xi,\eta)u_j\\ &\vdots\\ \dot{\eta}_{n-r} &= q_{n-r}(\xi,\eta)+\sum_{j=1}^{q}P_{n-r,j}(\xi,\eta)u_j \end{cases}.$$

where

$$\widetilde{b}_j(\xi,\eta)=b_j(\Phi^{-1}(\xi,\eta))=L_f^{r_j}h_j(\Phi^{-1}(\xi,\eta)), j=1,...,p\ , \tag{6a}$$

$$\widetilde{D}_{l,j}(\xi,\eta)=D_{l,j}(\Phi^{-1}(\xi,\eta))=L_{g_j}L_f^{r_l-1}h_l(\Phi^{-1}(\xi,\eta)),\quad j=1,...,q, l=1,...,p \tag{6b}$$

$$q_i(\xi,\eta)=L_f\phi_{r+i}(\Phi^{-1}(\xi,\eta)), i=1,...,n-r\ , \tag{6c}$$

$$P_{i,j}(\xi,\eta)=L_{g_j}\phi_{r+i}(\Phi^{-1}(\xi,\eta)), i=1,...,n-r, j=1,...,q, \tag{6d}$$

$$r=\sum_{i=1}^{p}r_i \tag{6e}$$

In this representation, the system $\widetilde{\Sigma}_1$ describes the input-output dynamics and the system $\widetilde{\Sigma}_2$ represents the internal dynamics. In this work, we assume that the internal dynamics $\widetilde{\Sigma}_2$ is stable, thus only minimum phase systems are covered.

For the purpose of the developed control approach, we define the function γ_i describing the r_i-th derivative of the i-th output. The function γ_i, $i=1,...,p$, is given by

$$\gamma_i(u,\xi,\eta)=\widetilde{y}_i^{(r_i)} \tag{7a}$$

$$=\widetilde{b}_i(\xi,\eta)+\sum_{j=1}^{q}\widetilde{D}_{i,j}(\xi,\eta)u_j \tag{7b}$$

2.4 Trajectory generation

In this section, the generation of the desired outputs $\widetilde{y}_{\mathrm{d},i}, i=1,...,p$, is discussed. This step is performed through a filtering process (Schwarzmann et al., 2006). For this purpose, a linear IMC filter F_i is designed for each output $\widetilde{y}_i$ that filters the reference signal $\widetilde{w}_i$ such that the filter output $\widetilde{y}_{\mathrm{d},i}$ can be realized by the model output $\widetilde{y}_i$. Additionaly, this filter will give the r_i derivatives of $\widetilde{y}_{\mathrm{d},i}$, namely $\widetilde{y}_{\mathrm{d},i}^{(k)}, k=0,...,r_i$ (Nitsche et al., 2007). The filter F_i is designed with the following transfer function

$$F_i(s)=\frac{\widetilde{Y}_{\mathrm{d},i}(s)}{\widetilde{W}_i(s)}=\frac{1}{k_{i,r_i}s^{r_i}+k_{i,r_{i-1}}s^{r_i-1}+\cdots+1} \tag{8}$$

with $\widetilde{w}_i=w_i-(y_i-\widetilde{y}_i)$. Defining the vectors k_i and ξ_i^{d} as

$$k_i=[1/k_{r_i}\ k_{i,1}/k_{i,r_i}\ \cdots\ k_{i,r_i-1}/k_{i,r_i}]\,, \tag{9}$$

$$\xi_i^{\mathrm{d}}=\left[\widetilde{y}_{\mathrm{d},i}\ \dot{\widetilde{y}}_{\mathrm{d},i}\ \cdots\ \widetilde{y}_{\mathrm{d},i}^{(r_i-1)}\right]^T, \tag{10}$$

leads to the following setup function γ_i^{d} for the r_i-th derivative of the desired trajectory

$$\widetilde{y}_{\mathrm{d},i}^{(r_i)}=\gamma_i^{\mathrm{d}}(w_i,y_i,\widetilde{y}_i,\xi_i^{\mathrm{d}}) \tag{11a}$$

$$=\frac{1}{k_{i,r_i}}(w_i-(y_i-\widetilde{y}_i))-k_i\xi_i^{\mathrm{d}}. \tag{11b}$$

The desired trajectories generated by the p IMC filters are concatenated together to produce the overall desired trajectory vector ξ^{d} given by

$$\xi^{\mathrm{d}}=\left[\xi_1^{\mathrm{d},T}\ \xi_2^{\mathrm{d},T}\ \cdots\ \xi_p^{\mathrm{d},T}\right]^T. \tag{12}$$

2.5 Inversion-based IMC as optimization problem

Since the control input u acts firstly on $\widetilde{y}_i^{(r_i)}$, achieving the desired trajectory $\widetilde{y}_{\mathrm{d},i}$ begins with achieving its r_i-th derivative $\widetilde{y}_{\mathrm{d},i}^{(r_i)}$. Thus, the distance between $\widetilde{y}_{\mathrm{d},i}^{(r_i)}$ and $\widetilde{y}_i^{(r_i)}$ ought to be minimized while respecting the input constraints (Kotman, 2018). In mathematical terms, the task at hand can be stated as the constrained optimization problem

$$\min_{u\in\mathcal{U}} d(\widetilde{y}_{\mathrm{d},i}^{(r_i)},\widetilde{y}_i^{(r_i)}) \tag{13}$$

The distance measure $d(x,y)$ has to fulfill the conditions of a semi-metric, i.e $d(x,y)\geq 0 with d(x,y)=0$ if and only if $x=y$, and $d(x,y)=d(y,x)$. These conditions are fulfilled by the square of the L_2-norm-induced metric, i.e. $d(x,y)=(x-y)^2$. Substituing $\widetilde{y}_{\mathrm{d},i}^{(r_i)}$ and $\widetilde{y}_i^{(r_i)}$ in (14) by their analytical descriptions γ_i^{d} and γ_i given in (7) and (11) respectively , leads to the following constrained optimization problem

$$\min_{u\in\mathcal{U}}(\gamma_i^{\mathrm{d}}(w_i, y_i, \widetilde{y}_i, \xi_i^{\mathrm{d}}) - \gamma_i(u, \xi^{\mathrm{d}}, \eta))^2 \tag{14}$$

Note that the vector ξ in the function γ_i has been substituted by the desired trajectory vector ξ^{d}. The vector η of the internal dynamics needed by the function γ_i can be calculated by an online simulation of the internal dynamics (Fig.3) or by using a state observer along with the diffeomorphism $\Phi(x)$ (Fig.4).

Solving the optimization problem (14) enables to achieve the desired trajectory $\widetilde{y}_{\mathrm{d},i}$ for the i-th output while respecting the input constraints. In order to take all control states into account, problem (14) ought to be expanded to cover all the control outputs. For this purpose, the following optimization problem is proposed

$$\min_{u\in\mathcal{U}} \sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i, y_i, \widetilde{y}_i, \xi_i^{\mathrm{d}}) - \gamma_i(u, \xi^{\mathrm{d}}, \eta))^2 \tag{15}$$

where $\alpha_i > 0, i = 1, ..., p$ represents the scale factor related to the output $\widetilde{y}_i$. The structure and the solution of the optimization problem (15) is discussed in detail later on. The inversion-based IMC control scheme for setpoint tracking described as an optimization problem is shown as a block diagram in Fig.2.

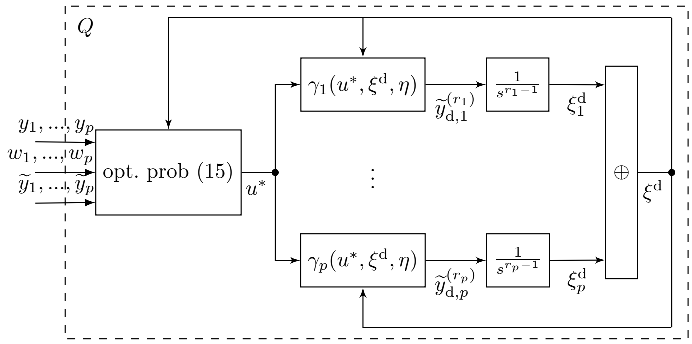

Fig. 2. Optimization-based IMC controller for setpoint tracking

As it can be seen in Fig.2, solving the optimization problem (15) enables to calculate the control input u^* allowing to achieve the r_i-th derivatives of the desired trajectories $\widetilde{y}_{\mathrm{d},i}$. The vector ξ^{d} is then calculated through simple chains of integrators.

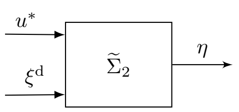

Fig. 3. Online simulation of the internal dynamics

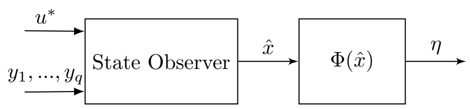

Fig. 4. State observer

3. OPTIMAL INVERSIONS-BASED INTERNAL MODEL CONTROL FOR TUBE TRACKING

This section introduces the main idea of this contribution, namely using the inversion-based feedforward IMC controller to push the controlled variables to remain within predefined intervals or 'tubes' with the minimal energy consumption.As it turns out, that this task can be stated as a convex optimization problem that can be solved easily with an analytical approach.

3.1 Reference as Optimization Variable

The optimization problem (15) is a multivariable optimization problem, where the optimization variables are the control inputs $u_i, i = 1, ..., q$. Solving this optimization problem enables to calculate the control input allowing to achieve well-defined references w_i for each output $\widetilde{y}_i, i = 1, ..., p$. In the following steps, the references w_i are considered as additional optimization variables of the problem (15) and are subject to the following box constraints

$$\mathcal{W} = \{w \in \mathbb{R}^p | w_i \in [w_i^-, w_i^+], i = 1, ..., p\}. \tag{16}$$

The values w_i^- and w_i^+ define the lower and upper bound of the tube in which the controlled variable $\widetilde{y}_i$ should remain. Moreover, an energetic term expressed by $\sum_{i=1}^{q} \epsilon_i u_i^2$ is added to the cost function (15) in order to establish a trade-off between tracking performance and energy consumption. The resulting optimization problem is given by

$$\min_{w\in\mathcal{W},u\in\mathcal{U}} \sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i, y_i, \widetilde{y}_i, \xi_i^{\mathrm{d}}) - \gamma_i(u, \xi^{\mathrm{d}}, \eta))^2 + \sum_{i=1}^{q} \epsilon_i u_i^2 \tag{17}$$

where γ_{d} and γ are defined according to equations (11) and (7) respectively. The vectorization of the above mentioned optimization problem leads to the following matrix representation

$$\min_{w\in\mathcal{W},u\in\mathcal{U}} [w\ u]\, A \begin{bmatrix} w \\ u \end{bmatrix} + B^T \begin{bmatrix} w \\ u \end{bmatrix} + c^2 \tag{18}$$

with $A = \begin{pmatrix} X^T\alpha X & X^T\alpha Z \\ (X^T\alpha Z)^T & Z^T\alpha Z + \epsilon \end{pmatrix}$, $B^T = [2\widetilde{c}\alpha X\ 2\widetilde{c}\alpha Z]$, $c = \sqrt{\widetilde{c}^T\alpha c}$

where $X = \begin{pmatrix} \frac{1}{k_{1,r_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{k_{p,r_p}} \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_p \end{pmatrix}$,

$Z = -\widetilde{D}(\xi^{\mathrm{d}}, \eta)$, $\widetilde{c} = [\widetilde{c}_1 \cdots \widetilde{c}_q]^T$,
and $\widetilde{c}_i = -\big(\frac{1}{k_{i,r_i}}(y_i - \widetilde{y}_i) - k_i\xi_i^{\mathrm{d}} - \widetilde{b}_i(\xi^{\mathrm{d}})\big)$.

Problem (18) is a convex quadratic optimization problem with box constraints. The convexity of problem (18) is due to the positive definiteness of the matrix A. The optimality conditions of convex quadratic optimization problems with box constraints are discussed in detail by Deangelis and Toraldo (2009).

In the following we discuss the uniqueness of the solution of problem (18). If the matrix A is positive definite, the cost function of problem (18) ist strictly convex and the problem admits a unique solution. The positive definiteness of the matrix A can be investigated using Schur's theorem as stated below:

Proposition 1: For any symmetric matrix M of the form $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$, with M_{11} having full rank, the following properties hold:

- $M > 0$ iff $M_{11} > 0$ and $M_{22} - M_{12}M_{11}^{-1}M_{12}^T > 0$.
- If $M_{11} > 0$, then $M \geq 0$ iff $M_{22} - M_{12}M_{11}^{-1}M_{12}^T \geq 0$.

According to Proposition 1 , since the left upper block $X^T\alpha X$ of A is invertible and positive definite , the matrix A is positive definite, i.e. $A \in S_+^n$, if and only if $Z^T\alpha Z + \epsilon - (X^T\alpha Z)(X^T\alpha X)^{-1}(X^T\alpha Z)^T > 0$, which is equivalent to $\epsilon > 0$. The matrix A is positive definite, if and only if $\epsilon_i > 0, \forall i = 1, ..., q$. In the case that at least one ϵ_i is equal to zero, the matrix A is positive semidefinite and the cost function is not strictly convex. In this case, the uniqueness of the solution of problem (18) ist not guaranteed.

3.2 Suboptimal tube control

In this section, the case where $\epsilon_i > 0, \forall i = 1, ..., q$, is discussed. Due to the existence of the energetic term $\sum_{i=1}^{q} \epsilon_i u_i^2$ in the cost function, the desired trajectories planned within $\mathcal{W}$ are not tracked perfectly. However, the balance between energy consumption and tube tracking performance can be fine-tuned with the parameters ϵ_i. A large weighting factor ϵ_i emphasizes the importance of energy reduction regarding the control input u_i and leads therefore to a reduced tube tracking performance. On the other hand, very small positives values for ϵ_i allow for a more accurate tracking of the desired trajectory.

When the weights $\epsilon_i, i = 1, ..., q$, are strictly positive, the cost function of problem (18) is strictly convex. In this case, problem (18) admits a unique solution. A geometric interpretation of the problem for the case $p = q = 1$ is given by (Ben Jemaa et al., 2019). The Karush-Kuhn-Tucker (KKT) conditions are therefore not only necessary but also sufficient to determine the optimal solution (w^*, u^*) (Boyd and Vandenberghe, 2009). The KKT conditions of problem (18) are given by

$$2A \begin{bmatrix} w^* \\ u^* \end{bmatrix} + B + \begin{bmatrix} \lambda_{u^+} - \lambda_{u^-} \\ \lambda_{w^+} - \lambda_{w^-} \end{bmatrix} = 0 \tag{19a}$$
$$\lambda_{u^+}(u^* - u^+) = 0 \tag{19b}$$
$$\lambda_{u^-}(-u^* + u^-) = 0 \tag{19c}$$
$$\lambda_{w^+}(w^* - w^+) = 0 \tag{19d}$$
$$\lambda_{w^-}(-w^* + w^-) = 0 \tag{19e}$$
$$u^* - u^+ \leq 0 \tag{19f}$$
$$-u^* + u^- \leq 0 \tag{19g}$$
$$w^* - w^+ \leq 0 \tag{19h}$$
$$-w^* + w^- \leq 0 \tag{19i}$$
$$\lambda_{u^+} \geq 0,\ \lambda_{u^-} \geq 0 \tag{19j}$$
$$\lambda_{w^+} \geq 0,\ \lambda_{w^-} \geq 0 \tag{19k}$$

where λ_{u^+}, λ_{u^-}, λ_{w^+} and λ_{w^-} are the Lagrange multipliers associated with the box constraints where $u^+ = [u_1^+, ..., u_q^+]$, $u^- = [u_1^-, ..., u_q^-]$, $w^+ = [w_1^+, ..., w_p^+]$ and $w^- = [w_1^-, ..., w_p^-]$ respectively.
The optimal solution can be found through iterating the cases in which the box constraints are active or inactive. Since the box constraints are given by q inequality conditions for the inputs and q inequality conditions for the outputs and each inequality condition can be either active or inactive, 2^{p+q} cases can be distinguished. For SISO systems ($p = q = 1$) following cases can be distinguished:

- Case 1: $u^* \notin [u^-, u^+]$, $w^* \notin [w^-, w^+]$
- Case 2: $u^* \in [u^-, u^+]$, $w^* \in [w^-, w^+]$
- Case 3: $u^* \notin [u^-, u^+]$, $w^* \in [w^-, w^+]$
- Case 4: $u^* \in [u^-, u^+]$, $w^* \notin [w^-, w^+]$

For instance, the optimal solution (w^*, u^*) in Case 1 is equal to $-BA^{-1}$ if the box constraints are maintained. Similarly, w^*, u^* and the Lagrange multipliers are calculated in the remaining cases and the box constraints and the positive definiteness of the Lagrange multipliers are checked afterwards to ensure the feasibility of the solution. For MIMO systems, an analytical solution based on the KKT optimality conditions can have as much computational load as a pure numerical solution due to the high number of cases (2^{p+q}) that should be distinguished.

Whether problem (18) is solved analytically or numerically, the inversion-based IMC control scheme for tube tracking can be described wih the block diagram in Fig.5.

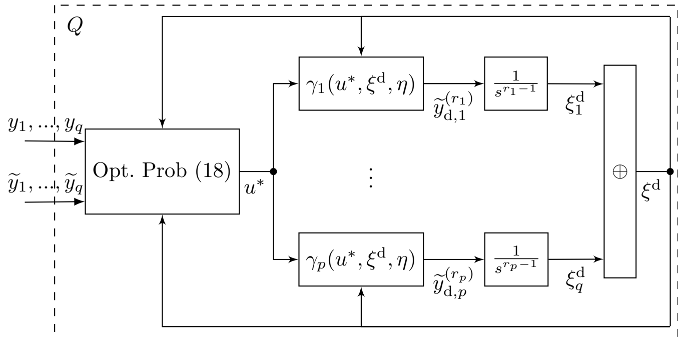

Fig. 5. Optimization-based IMC controller for tube tracking

As can be seen in Fig. 5, the calculation of the optimal setpoints w^* and the optimal control input u^* is done within the same optimization step. Therefore, there is no need to use the optimal setpoints w^* as input for the conventional IMC controller for setpoint tracking.

3.3 Optimal tube control

Now we discuss another formulation of the optimization problem leading to a better tube tracking performance while minimizing the energy consumption. A better tube tracking performance can be achieved by setting the energetic term in the cost function of problem (18) to zero. This can be achieved by setting all the parameters $\epsilon_i, i = 1, ..., q$ to zero. In this case the energetic term vanishes in the cost function and only tube tracking is taken into account through the minimization of the term $\sum_{i=1}^{p} \alpha_i(\gamma_i^d(w_i, y_i, \widetilde{y}_i, \xi_i^d) - \gamma_i(u, \xi^d, \eta))^2$.

However, when at least one $\epsilon_i = 0$, the cost function of problem (18) is not strictly convex, thus uniqueness of the solution of problem (18) is not guaranteed. A geometric interpretation of the SISO case with $\epsilon = 0$ is given in (Ben Jemaa et al., 2019). In order to ensure uniqueness of the solution and energetic optimality, the following optimization problem is proposed

$$\min_{u} \sum_{i=1}^{q} \epsilon_i u_i^2 \tag{20a}$$

$$\text{s.t.} \sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i) - \gamma_i(u))^2 = \arg \min_{w' \in \mathcal{W}, u' \in \mathcal{U}} \sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i') - \gamma_i(u'))^2 \tag{20b}$$

where $\epsilon_i > 0, \forall i \in 1, ..., q$. The above mentioned optimization problem (20) can be interpreted as follows. Tube tracking is addressed by the equality constraint (20b). With this strict formulation, it is ensured that the difference between the desired trajectories and the fulfilled trajectories is minimal. The second optimization level in this problem consists in minimizing the energy consumption through the minimization of the cost function $\sum_{i=1}^{q} \epsilon_i u^2$. In mathematical terms, the feasible set of problem (20) is given by

$$\mathcal{F} = \{(u, w) | u \in \mathcal{U}, w \in \mathcal{W},$$
$$\sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i') - \gamma_i(u))^2 = \arg \min_{w' \in \mathcal{W}, u' \in \mathcal{U}} \sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i') - \gamma_i(u'))^2\} \tag{21}$$

Due to the convexity of the function $\sum_{i=1}^{p} \alpha_i(\gamma_i^{\mathrm{d}}(w_i) - \gamma_i(u))^2$ the feasible set $\mathcal{F}$ is a convex set. Problem (20) consists therefore in minimizing a strictly convex cost function over a convex set. Therefore problem (20) is also a convex optimization problem and admits a unique solution.

4. EXAMPLES

In this section three different simulation examples are shown. In the first example, the case where the number of control inputs and the number of control outputs are equal i.e. $p = q$ is dicussed. In the second example the case of conflicted control outputs where $q < p$ is investigated. In the last example the case where more control inputs than control outputs are available i.e. $q > p$ is discussed.

4.1 Example 1: Control of temperature and CO_2*-concentration* $(p = q = 2)$

Consider the plant model

$$\frac{\mathrm{d}T_Z}{\mathrm{d}t} = \frac{1}{C_Z}(\dot{m}_{IN} c_p (T_{IN} - T_Z) + \dot{Q}_{dist}) \tag{22a}$$

$$\frac{\mathrm{d}X_Z}{\mathrm{d}t} = \frac{1}{m_Z}(\dot{m}_{IN}(X_{IN} - X_Z) + \dot{m}_{Z,CO_2}) \tag{22b}$$

where T_Z is the temperature in a room Z, C_Z is the heact capacity of the room, $\dot{m}_{IN}$ is the incoming air mass flow through a ventilation system, c_p is the specific heat capacity of the air, T_{IN} is the temperature of the incoming air and $\dot{Q}_{dist}$ is the disturbing heat signal resulting from occupany, solar radiation and heat exchange with the walls. X_Z represents the CO_2-concentration in the room, m_Z is the total mass of the air in the room, X_{IN} is the CO_2-concentration of the incoming fresh air and $\dot{m}_{Z,CO_2}$ is the CO_2 mass flow exhaled by the occupants in the the room. The control task consists in maintaining the temperature and the CO_2-concentration in a predefined comfort zone determined by an upper and lower bound for each controlled state. Defining the control inputs as $u_1 = \dot{m}_{IN} c_p T_{IN}$ and $u_2 = \dot{m}_{IN}$ leads to the following system formulation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{1}{a_1}(u_1 - b_1 u_2 x_1 + d_1) \tag{23a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{a_2}(u_2(b_2 - x_2) + d_2) \tag{23b}$$

$$\widetilde{y}_1 = x_1 \tag{23c}$$

$$\widetilde{y}_2 = x_2 \tag{23d}$$

The vector relative degree is [1,1]. Fig. 6 shows the response of the controlled system for $a_1 = a_2 = b_1 = b_2 = 1$, $u_1 \in [1, 2]$, $u_2 \in [1, 2]$, $w_1 \in \mathcal{W}_1 = [1.5, 2.5]$, $w_2 \in \mathcal{W}_2 = [1, 1.5]$,

$$d_1(t) = 0.5\sin\big((2\pi/30)t\big) + 1.5$$

and

$$d_2(t) = 0.5\sin\big((2\pi/30)t + \pi\big) + 0.5.$$

The trade-off between tube tracking performance and energy consumption can be seen in the case of the suboptimal control for the different values of ϵ_i. As the parameters ϵ_1 and ϵ_2 increase, the violation of the tubes $\mathcal{W}_1$ and $\mathcal{W}_2$ increases. In the case of optimal tube control no violation of the comfort zone occurs.

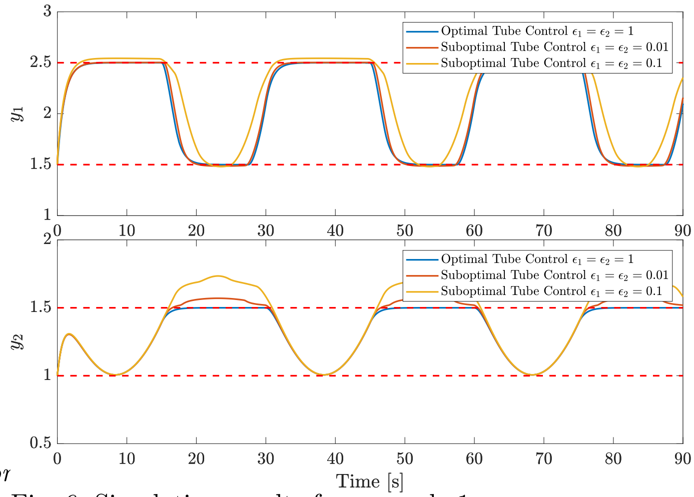

Fig. 6. Simulation results for example 1

4.2 Example 2: Control of humidity and CO_2*-concentration* $(p = 2, q = 1)$

Consider the plant model

$$\frac{\mathrm{d}X_Z}{\mathrm{d}t} = \frac{1}{m_Z}(\dot{m}_{IN}(X_{IN} - X_Z) + \dot{m}_{Z,CO_2}) \tag{24a}$$

$$\frac{\mathrm{d}H_Z}{\mathrm{d}t} = \frac{1}{m_Z}(\dot{m}_{IN}(H_{IN} - H_Z) + \dot{m}_{Z,H_2O}) \tag{24b}$$

$$\tag{24c}$$

where H_Z is the absolute humidity of the air in the room Z, H_{IN} is the absolute humidity of the incoming fresh air and $\dot{m}_{Z,H_2O}$ is the water amount stemming from the occupancy. In this example we assume that $\dot{m}_{IN}$ is the unique control input. The control of CO_2-concentration and the control of humidity could be conflicted control task. For example, if the outside fresh air is too humid, the ventilation rate should be minimized which can affect the

CO_2-concentration in the room. The system formulation takes the following form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{1}{a_1}(u(b_1 - x_1) + d_1) \tag{25a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{a_2}(u(b_2 - x_2) + d_2) \tag{25b}$$

$$\tilde{y}_1 = x_1 \tag{25c}$$

$$\tilde{y}_2 = x_2 \tag{25d}$$

The vector relative degree is {1,1}. Fig. 7 shows the response of the controlled system for $a_1 = a_2 = b_1 = b_2 = 1$, $u \in [1, 2]$, $w_1 \in \mathcal{W}_1 = [1.5, 2.5]$, $w_2 \in \mathcal{W}_2 = [1, 1.5]$,

$$d_1(t) = 0.5\sin\big((2\pi/30)t\big) + 1.5$$

and

$$d_2(t) = 0.5\sin\big((2\pi/30)t + \pi\big) + 0.5.$$

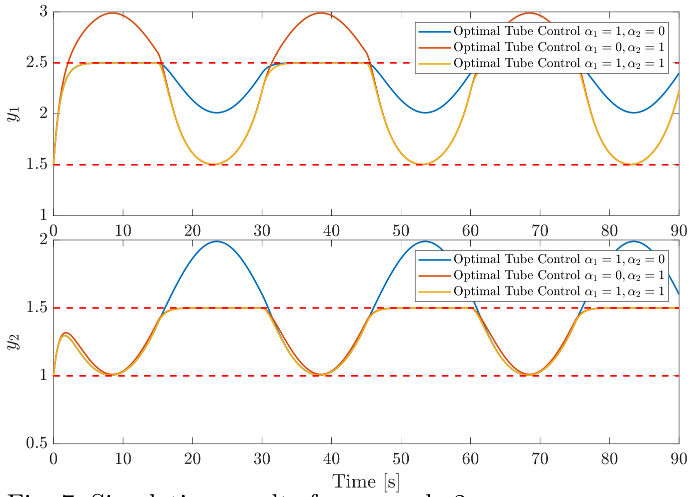

Fig. 7. Simulation results for example 2

The simulation results show the ability of the control strategy to prioritize a control output over an other. For $\alpha_1 = 1$ and $\alpha_2 = 0$, only the CO_2-concentration is regulated while $\alpha_1 = 0$ and $\alpha_2 = 1$ correspond to humidity-only control. Choosing $\alpha_1 = 1$ and $\alpha_2 = 1$ enables to handle both control outputs simultaneously.

4.3 Example 3: Water-based heating vs. ventilation-based heating ($p = 1, q = 2$)

Consider the plant model

$$\frac{\mathrm{d}T_Z}{\mathrm{d}t} = \frac{1}{C_Z}(\dot{m}_{IN} c_p (T_{IN} - T_Z) + \dot{Q}_{Rad} + \dot{Q}_{dist}) \tag{26}$$

where $\dot{Q}_{Rad}$ denotes the heat flow of the radiators. We assume that the system has two control inputs: $u_1 = \dot{m}_{IN}$ and $u_2 = \dot{Q}_{Rad}$. Thus, the zone can be heated through ventilation or by using the radiators. The system formulation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{a_1}(u_1 b_1 (b_2 - x) + u_2 + d_1) \tag{27a}$$

$$\tilde{y} = x \tag{27b}$$

Fig. 8 shows the response of the controlled system for $a_1 = b_1 = 1$, $b_2 = 3$, $u_1 \in [1, 2]$, $u_2 \in [1, 2]$, $w \in \mathcal{W} = [1.5, 2.5]$,

$$d_1(t) = 0.5\sin\big((2\pi/30)t\big) + 1.5$$

The simulation results show the ability of the control approach to prioritize the less expensive control input in achieving the control task of remaining within the tube $\mathcal{W}$. For example, for $\epsilon_1 = 1$ and $\epsilon_2 = 10$ the control input u_2 is 10 times more expensive than u_1. Therefore, the input u_2 is kept at its minimum value over the whole time interval.

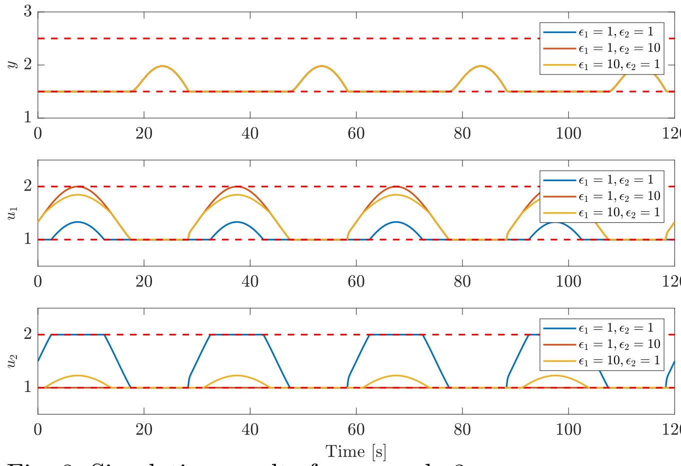

Fig. 8. Simulation results for example 3

5. CONCLUSION

An optimal control approach for keeping the outputs of a minimum-phase input-affine MIMO system in predefined tubes is presented in this paper. The proposed approach is an extension of the conventional inversion-based IMC controller for setpoint control to tube control. The paper shows that the control task of tube tracking under input constraints while minimizing the energy consumption can be stated as a convex quadratic optimization problem with box constraints. Within the control strategy, the calculation of the optimal setpoints and the optimal control inputs is performed simultaneously within the same optimization step. The proposed control strategy shows the interesting ability to handle systems with different number of control inputs and control outputs.

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