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A Bi-level Approach to MPC for Switching Nonlinear Systems ⋆

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Abstract: In this paper, a nonlinear model predictive control scheme for switching dynamical systems is presented. The controller comprises of two layers of optimization. The upper layer is based on the embedding transformation technique, hence it does not require prior knowledge of the switching sequence. In particular, it provides the optimal relaxed switching sequence together with the optimal regulating inputs and the corresponding trajectories of the states. Within the lower layer, the integrality constraints are restored and a switching solution is recomputed to minimize the error with respect to the trajectories given from the upper-level optimization. The scheme is presented and the bounds of the integer approximation errors are evaluated together with brief recursive feasibility analysis. Simulation results of a tracking and an economics optimizing nonlinear MPC for a supermarket refrigerator system show the applicability and efficiency of the proposed approach.

Keywords: Switching dynamical systems, MPC for switching systems, Supermarket refrigeration

1. INTRODUCTION

Switching dynamical systems arise as a result of the interaction between discrete and continuous dynamics which can be observed in many processes in diverse fields of engineering (Liberzon (2003)). Such processes attain several modes of operation, for instance as a result of the presence of discontinuous actuating devices, phase changes, switching on or off pieces of equipment, and changes in raw materials or product specifications. Such processes can be treated within the framework of switching systems, where the evolution of the states can be described by a set of vector fields that are valid in different sections of the state space (modes of operation).

Model predictive control (MPC) is an iterative model-based controller which solves an open-loop optimal control problem (OCP) to produce a sequence of control moves that are optimal with respect to a certain performance criterion. In order to realize feedback from the controlled process, only the first part of the calculated input sequence is sent to the process, then the evolution of the process is captured through measurements (estimations) and is used to reinitialize the next iteration in a moving horizon fashion (Rawlings and Mayne (2009)).

It is well known that MPC for switching nonlinear systems is a challenging task due to the discontinuity in the underlying process model, which adds to the nonlinearity and prohibits the direct usage of the standard gradient-based algorithms. The main source of difficulty is the determination of the optimal trajectories of the discrete degrees of freedom including the mode sequence. In consequence, a mixed integer optimal control problem (MIOCP) has to be solved in real time which is significantly more expensive compared to the purely continuous counterparts.

Many algorithms were proposed for MPC of switching systems which are mostly tailored to linear or piecewise affine systems (Camacho et al. (2010)). In the mixed logical dynamical (MLD) framework (Bemporad and Morari (1999)), the optimization problem is cast into a mixed integer linear (MILP) or quadratic (MIQP) problem by replacing the logical decisions by a set of linearly constrained binary indicators which are determined within the optimization problem. However long time horizons lead to a combinatorial increase in the size and complexity as reported in Till et al. (2004). Hence, the MLD approach is tailored to linear and mildly nonlinear systems, but it does not scale well to large problems and long horizons. In Baotić et al. (2006) explicit MPC for small scale linear switching systems was developed to minimize the online computation time by shifting the expensive computations of the control laws to be done offline.

For nonlinear systems and in order to avoid solving the underlying mixed integer nonlinear program (MINLP) online, many of the MPC algorithms assume that the switching sequence is either predefined or the switches can be detected almost as soon as they occur, which is a realistic assumption in some processes. In Mhaskar et al. (2005) a Lyapunov-based MPC was developed for switching systems with a predefined switching signal. A nonlinear MPC algorithm was presented in Müller et al. (2012) assuming that the switches are restricted by a system dependent average dwell time and can be detected as soon as they occur with a small tolerance. On the other hand, some works consider the case when the switching

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sequence is considered as a degree of freedom that has to be determined online. A numerically efficient approach for nonlinear MPC was proposed in (Kirches et al. (2013)) for truck control, which is based on a relaxation of the integer constraints within the embedding transformation formulation (Bengea and DeCarlo (2005); Sager (2009)). After solving a relaxed problem, the integer feasibility is restored via a rounding scheme with the aim of getting the closest integer feasible input trajectories to the relaxed optimized ones however, neither optimality nor feasibility, e.g., with respect to path constraints, can be guaranteed. In Rawlings and Risbeck (2017), the general nominal stability theory of sub-optimal MPC was proven to hold equally for exogenous switching systems, given a way to solve the underlying MINLP while respecting the real time restrictions, which is very difficult with the currently available solvers.

In this paper, the problem of reference tracking and economic nonlinear MPC for switching dynamical systems is addressed. Two layers of optimization are considered within each MPC iteration to produce a sub-optimal control input for the switching system which is guaranteed to satisfy the state and the input constraints. The upper layer solves a continuous optimal control problem over a reformulated process model to generate the reference input and state trajectories as well as a relaxed discrete switching sequence. The lower layer is devoted to driving the switching system in a bounded vicinity of the reference trajectories after restoring the integrality constraints. The lower layer is comprised of: 1) a simple MILP to generate switching input trajectories with a minimum integral difference to the relaxed optimal ones, and 2) a tracking auxiliary nonlinear MPC to find the optimal switching times of the rounded inputs in order to further improve the tracking of the reference state trajectories.

The innovative contribution of this work is in the combination of the embedding transformation algorithm and the switching time optimization in an MPC framework, which guarantees feasibility in terms of state and input constraints satisfaction for the switching system. Furthermore, the approach is computationally efficient and reduces the bound of the error between the optimal relaxed trajectories and the switching ones, which leads to less sub-optimal switching control. For validation purposes, the algorithm is applied to a nontrivial benchmark problem which contains endogenous (state dependent) as well as exogenous switches.

The rest of the paper is structured as follows: Section 2 introduces the problem statement together with necessary preliminaries, while the details of the proposed nonlinear MPC scheme are explained in section 3. Section 4 illustrates the application of the algorithm to a version of the supermarket refrigeration system. Finally, part 5 concludes the paper and points out open research issues.

2. PRELIMINARIES

Switching dynamical systems are characterized by additional relatively simple discrete dynamics compared to the continuous counterparts and by a switching function that determines the active mode and accordingly the model equations that are valid at each time instant. The switching function is either state dependent for endogenous type switches or time dependent for exogenous ones (Liberzon (2003)). Noting that the endogenous switches can be reformulated as exogenous, and vice versa (Bock et al. (2018)).

It is desired to design an MPC controller for the nonlinear switching dynamical system described by:

$$\begin{aligned} \dot{x}(t) &= f_{i(t,x(t))}(x(t), u(t), v(t)) \qquad t \in [t_0, t_f], \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where $f_{i(\cdot)} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the continuous states and input variables. $v \in \nu \triangleq \{v^1, \ldots, v^{n_v}\}, v^j \in \mathbb{R}^p$ are the discrete control variables. The function $i : [0, \infty) \times \mathbb{R}^n \to \mathbb{I} \triangleq \{1, 2, \ldots, \mu\}$ is a right continuous piecewise constant function which defines the index of the active mode at each time instant, while μ is the total number of modes. The switching signal is a set of $N \in [1, \infty)$ ordered pairs of modes $\sigma \triangleq \{i_s \in \mathbb{I}\}_{s=1}^N$, and the corresponding switching instants $\tau \triangleq \{\tau_s\}_{s=0}^N$, such that $t_0 \triangleq \tau_0 < \tau_1 < \cdots < \tau_N \triangleq t_f$ with t_0, t_f are the initial and final times.

2.1 Problem Formulation

We assume that all the subsystems evolve in $\mathbb{R}^n$ without state jumps, and the bounds of the state variables are enforced globally over all the subsystems. Consider the underlying MIOCP that has to solved online:

$$\min_{x(\cdot),u(\cdot),i(\cdot),v(\cdot)} \int_{t_0}^{t_f} L_{i(t,x(t))}(x(t), u(t), v(t))dt + \varphi(x(t_f)) \tag{2a}$$

$$\begin{aligned} \textbf{s.t.} \quad & \text{ODE equations (1)}, && (2b)\\ 0 \geq & h(x(t), u(t)), & t \in [t_0, t_f], \quad & (2c)\\ 0 \geq & g_{i(t,x(t))}(x(t), u(t), v(t)), & t \in [t_0, t_f], \quad & (2d)\\ v(t) \in & \nu \triangleq \{v^1, v^2, \ldots, v^{n_v}\}, & t \in [t_0, t_f], \quad & (2e)\\ x(t_f) \in & \mathcal{X}_f && (2f) \end{aligned}$$

where the mode dependent running cost $L_{i(\cdot)} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^+$ and the terminal cost $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ are assumed to be $\mathcal{C}^2$ functions. Constraints $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^h$ denote the state bounds and the mode independent constraints, while the combinatorial constraints $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^g$ are explicitly dependent on the discrete variables, e.g., prescribing maximum number of switches, or the logical constraints due to the transformation of the endogenous switches into exogenous ones. $\mathcal{X}_f \subseteq \bar{\mathcal{X}}$ denotes the terminal set, where $\bar{\mathcal{X}}$ is the feasible region of the target mode.

2.2 Embedding Transformation

In addition to the nonlinearity of the process model, the discrete decision variables, i.e., mode sequence together with the discrete inputs trajectories, result in a very demanding MIOCP. Embedding transformation (Bengea and DeCarlo (2005); Sager (2009)) presents a computationally efficient algorithm to solve such problems. Within the embedding transformation, a continuous optimal control problem (OCP) is solved over a convex hull relaxation of the switching model rather than an MIOCP over the original discontinuous model. Thus, the switching model is embedded into a family of systems, and its state trajectories are dense inside the set of trajectories of the embedding system. Consider the following reformulation of model (1):

$$\dot{x}(t) = \sum_{j=1}^{\mu \cdot n_v} \omega_j^b(t) \cdot f^j(x(t), u(t), v^j), \qquad x(t_0) = x_0. \quad (3)$$

The function $f^j(\cdot)$ denotes the dynamics of the switching system evaluated at every discrete assignment. The set of binary multipliers $\omega^b \in \{0,1\}^{\mu.n_v}$ enumerate the feasible discrete assignments in all modes of the switching system and are constrained by the special set order property of type one (SOS-1) constraint $\sum_{j=1}^{\mu.n_v} \omega_j^b(t) = 1$ to impose a single assignment at any time instant. It should be emphasized that the two formulations for the switching dynamics are equivalent due to the one-to-one transformation between the discrete assignments and the binary multipliers. Since the space of feasible assignments is sought rather than the trajectories of the discrete variables, the forbidden modes or assignments can be rejected by construction (Ebrahim et al. (2018)). Next, the switching system is embedded into a bundle of continuous systems by dropping the integrality constraint of the binary multipliers to $\omega_j \in [0,1]^{\mu.n_v}$. The reformulation results in the following OCP:

$$\min_{x(\cdot),u(\cdot),\omega(\cdot)} \int_{t_0}^{t_f} \sum_{j=1}^{\mu \cdot n_v} \omega_j^b(t) \cdot L^j(x(t), u(t), v^j)dt + \varphi(x(t_f)) \quad (4a)$$

$$\begin{aligned} \textbf{s.t.} \quad & \text{ODE equations (3)}, && (4b)\\ 0 \geq & h(x(t), u(t)), \quad t \in [t_0, t_f], && (4c)\\ 0 \geq & \omega_j(t) g^j(x(t), u(t), v^j), \quad t \in [t_0, t_f], && (4d)\\ x(t_f) \in & \mathcal{X}_f, && (4e)\\ \omega_j(t) \in & [0,1], \qquad \sum_{i=j}^{\mu.n_v} \omega_j(t) = 1, && (4f) \end{aligned}$$

where $L^j(\cdot)$ denotes the running cost evaluated at every discrete assignment. As a consequence of the relaxed formulation, the feasible region $\mathcal{X} \subset \mathbb{R}^n$ is not disjoint and $\mathcal{X}_f \subseteq \mathcal{X}$. The combinatorial constraints (4d) are evaluated at all feasible discrete assignments and are only enforced when the corresponding binary multiplier assumes a value other than zero. In order to avoid compensation effects, feasibility is enforced separately for each possible assignment (Kirches and Lenders (2016)). It should be noted that the discrete trajectories including the switching sequence are captured by the relaxed binary multipliers which are included as optimization variables. Moreover, the optimal solution of (4) represents a tight lower bound for any solution of switching type.

2.3 Integer Approximation Error Bound

Despite the fact that the integrality constraints were dropped in (4) and the optimal solution is not necessarily switching, it is computationally efficient to find an attainable lower bound relaxed solution and to then search for the closest integer solution, e.g., by a rounding scheme, in a subsequent separate step. The following theorem provides bounds of the integer approximation error in the differential states for nonlinear systems in the from of (3).

Theorem 1. (Sager et al. (2012)). Consider the following two model functions with $t \in [t_0, t_f]$:

$$\dot{x}(t) = f(x(t), \bar{u}(t)) \cdot \bar{\omega}(t), \qquad \bar{\omega} \in [0,1]^{\mu.n_v}, \quad (5)$$
$$\dot{z}(t) = f(z(t), \bar{u}(t)) \cdot \omega^b(t), \qquad \omega^b \in \{0,1\}^{\mu.n_v}, \quad (6)$$

where $\omega^b(t)$ is a switching input approximation of the relaxed optimal trajectory $\bar{\omega}(t)$ and $\bar{u}(t)$ denotes the optimal continuous input obtained from the OCP (4). Both state trajectories $x(\cdot), z(\cdot)$ attain the same initial conditions at t_0. It is assumed that the function f is globally Lipschitz continuous with respect to the state vector and the Lipschitz constant is denoted by $L > 0$. Furthermore f is uniformly bounded by $M_1 > 0$ and its time derivative is bounded by $M_2 > 0$ almost everywhere on $t \in [t_0, t_f]$:

$$\|f(x(t), \bar{u}(t))\| \leq M_1, \quad (7)$$
$$\left\| \frac{d}{dt} f(x(t), \bar{u}(t)) \right\| \leq M_2, \quad (8)$$
$$\|f(z(t), \bar{u}(t)) - f(x(t), \bar{u}(t))\| \leq L\|z(t) - x(t)\|. \quad (9)$$

If the integral error between the two input trajectories is bounded by $\epsilon \geq 0$,

$$\| \int_{t_0}^{t} \bar{\omega}(s) - \omega^b(s) ds \| \leq \epsilon, \quad (10)$$

then it holds that the error in the state trajectories is bounded for all $t \in [t_0, t_f]$ by,

$$\|z(t) - x(t)\| \leq (M_1 + M_2(t - t_0))\epsilon e^{L(t-t_0)}. \quad (11)$$

It can be observed that the solution of the OCP (4) presents a lower bound which can be attained by a switching solution only if $\epsilon = 0$. An MILP can be set up to seek the binary trajectories which minimizes the integral error with respect to the optimal relaxed input trajectories while respecting the maximum allowed number of switches (Sager et al. (2011)). For the sake of computational efficiency, a direct simultaneous method, e.g., direct collocation, is the method of choice to solve the OCP (4). The time horizon $t \in [t_0, t_f]$ is partitioned into intervals of length Δt, thus forming an equidistant time discretization grid and the control input is parameterized by a vector of piecewise constant functions and accordingly computed only at time instants $(t_0 < t_1 < \cdots < t_f)$. If an MILP is used to generate the binary multipliers from the relaxed trajectories, then the error bound is dependent on the granularity of the underlying time grid as follows (Kirches and Lenders (2016)),

$$\epsilon \leq \Delta t \cdot \frac{((\mu.n_v) - 1)^2}{(\mu.n_v)}, \quad (12)$$

where $\mu.n_v$ is the number of the binary multipliers, Δt is the interval length in case of a uniform discretization or the maximum of the interval lengths in case of a non-uniform discretization. Note that the given bound is only valid in case the maximum number of switches is only constrained by the underlying discretization grid.

3. PREDICTIVE CONTROLLER FOR SWITCHING SYSTEMS

In this section, the proposed nonlinear MPC algorithm is established and the formulation of the underlying optimal control problems is discussed. Moreover, it is shown how the state dependent constraints can be satisfied after restoring the integrality constraints.

Consider figure (1), in which the continuous and discrete control inputs (u_k, v_k) that are fed to the switching system are computed in each iteration by two successive layers of optimization. The trajectories $\bar{x}(\cdot)$, $\bar{u}(\cdot)$ and $\bar{\omega}(\cdot)$ are

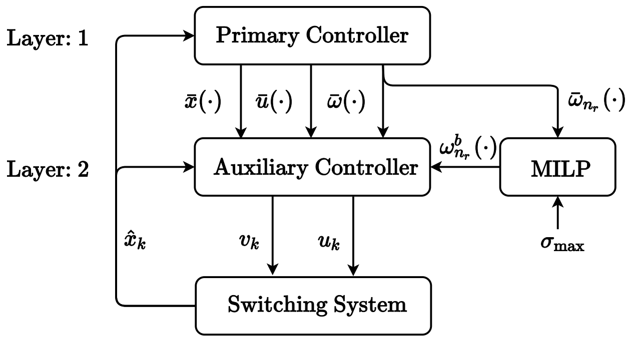

Fig. 1. MPC scheme for switching systems

the optimal state, continuous input and relaxed binary multipliers along the prediction horizon. $\bar{\omega}_{n_r}(\cdot)$ denotes the first $(n_r > 1)$ values of the relaxed multipliers trajectory, while $\omega^b_{n_r}(\cdot)$ is the corresponding binary feasible trajectory with less than σ_{max} switches. The feedback from the controlled system is realized by $\hat{x}_k$ which denotes the measured state vector.

3.1 Layer 1: Primary Controller

The primary controller is based on the embedding transformation formulation (4), with all the endogenous switches transformed into exogenous switches and integrated into the OCP by logically constrained binary multipliers. Thus the combinatorial constraints (4d) will contain logical constraints indicating the active mode at each time instant. This controller aims at generating a tight lower-bound reference relaxed solution for the second layer.

It can be inferred from (12) that in order to reduce the error bound (11) for the state trajectories, the sampling interval should be reduced such that, as $\Delta t \to 0$, the relaxed state trajectories can be reproduced by a switching solution. Since this significantly increases the computational effort and is usually not possible due to the real time restrictions, it is proposed to divide the first sampling interval into n_r equidistant intervals, which will be denoted as the rounding horizon in the sequel. Thus the time discretization grid for the primary controller will be non-equidistant as follows:

$$t_0 < \frac{\Delta t}{n_r} + t_0 < \frac{2\Delta t}{n_r} + t_0 \cdots < t_1 < \cdots < t_f. \tag{13}$$

Based on the previous discussion, the integer approximation error in the state trajectory at any time t is bounded by,

$$\|z(t) - x(t)\| \leq \tag{14}$$
$$(M_1 + M_2(t - t_0))\frac{((\mu.n_v) - 1)^2}{(\mu.n_v)}\frac{\Delta t}{n_r}e^{L(t-t_0)}.$$

Despite the reduction in the error bound introduced by n_r, the propagation of the error along the prediction horizon may lead to violation of states bounds or path constraints and accordingly simple tightening of the constraints might not be enough. Therefore a second controller is introduced to provide feedback in order to contain the evolution of the error between the actual switching system trajectories and reference trajectories generated by the primary controller.

3.2 Layer 2: MILP and Auxiliary Controller

The purpose of the second layer is to restore the integral feasibility and to seek a sub-optimal switching solution which tracks the state and input references from the primary controller. As illustrated in figure (1), a portion $\bar{\omega}_{n_r}(\cdot)$ of the relaxed trajectories is injected into an MILP to compute the closest switching counterpart within the specified upper limit of switches σ_{max}. Note that chattering is rejected by design due to the discretization in time within the employed direct method even if σ_{max} is not specified. consequentially the number of switches is upper bounded by n_r for each binary multiplier, but it can be further constrained for practical purposes.

In addition, an auxiliary unconstrained controller is included which aims at providing the needed feedback in order to minimize the error between the relaxed optimal trajectories and the computed binary solution. By discretization in time, the OCP that is solved within the auxiliary controller is given in (15). It can be observed that the switching times of the binary rounded multipliers $\omega^b_{n_r}(\cdot)$ are optimized in the first period Δt while fixing their levels to the output of the MILP ω^b_k, however the rest of the multipliers trajectories is still relaxed.

$$\min_{x(\cdot),u(\cdot),h(\cdot),\omega(\cdot)} V_{aux} \tag{15}$$
$$\begin{aligned}
\textbf{s.t.}\ x_{k+1} &= \sum_{j=1}^{\mu\cdot n_v} h_k \cdot F^j(x_k, u_k, v^j)\cdot \omega^b_{j,k}, && 0 \le k \le n_r - 1\\
x_{k+1} &= \sum_{j=1}^{\mu\cdot n_v} F^j(x_k, u_k, v^j)\cdot \omega_{j,k}, && n_r \le k \le n_p - 1\\
x_0 &= x(t_0), &&\\
1 &= \sum_{j=1}^{\mu.n_v} \omega_{j,k}, \quad \omega_{j,k} \in [0,1], && n_r \le k \le n_p - 1\\
\Delta t &= \sum_{k=0}^{n_r-1} h_k, && 0 \le k \le n_r - 1
\end{aligned}$$

where h_k are the interval lengths of the binary multipliers with dimension n_r. The tracking objective of the auxiliary controller is formulated as follows,

$$V_{aux} = \beta\varphi(x_{n_p}) + \sum_{k=0}^{n_p-1}(u_k - \bar{u}_k)^T R_1(u_k - \bar{u}_k) + \tag{16}$$
$$\sum_{k=n_r}^{n_p-1}(x_k - \bar{x}_k)^T Q(x_k - \bar{x}_k) + (\omega_k - \bar{\omega}_k)^T R_2(\omega_k - \bar{\omega}_k).$$

where R_1, R_2 and Q are positive definite symmetric matrices with suitable sizes. The terminal cost is replaced by $\beta\varphi(x_{n_p})$, which introduces an implicit terminal constraint for sufficiently large penalization β (Rawlings and Mayne (2009)). The auxiliary controller solves a continuous nonlinear MPC problem which uses the same model as the primary controller. The added degrees of freedom of the switching times within the rounding horizon n_r lead to further reduction of the integer approximation error and better tracking of the reference trajectories.

Remark 2. Parameter n_r represents a compromise between computational complexity and the integer approximation error. Therefore the bigger n_r is, the smaller the error bound between the relaxed and the switching solutions but the higher the computational effort. Similarly limiting the number of allowed switches ($\sigma_{max} < n_r$) within the MILP further increases the approximation error but may be necessary for practical implementations.

Remark 3. The switching sequence is determined by the primary controller and the auxiliary controller seeks the optimal solution within the provided mode sequence by optimizing the duration of each mode.

Remark 4. The bound (14) is considered conservative due to the expected improvement that results from the optimization within the auxiliary controller.

3.3 Recursive Feasibility

In this part, we discuss briefly how recursive feasibility of the presented scheme can be established.

Assumption 5. The set $\mathcal{X} \subset \mathbb{R}^n$ is closed, and $\mathcal{X}_f \subseteq \mathcal{X} \subset \mathbb{R}^n$ is compact and contains the origin.

Assumption 6. $\mathcal{X}_f$ is control invariant for the system (3) and there exists a stabilizing terminal control law (u_f, w_f).

Primary Controller: We require the solution of the primary controller to be strictly in the interior of the feasible region, therefore the state dependent constraints are tightened such that the propagation of the integer approximation error (14) along the prediction horizon does not lead to violation of the state constraints. It follows that, the solution generated by the auxiliary controller is feasible with respect to the state constraints and satisfies $x_{n_p} \in \mathcal{X}_f$ due to the implicit terminal constraint (16).

After sending the first portion (n_r steps) of the combined input trajectory $(u_{0:n_r}, w_{0:n_r})$ generated by the auxiliary controller to the process, the rest of the trajectory $(u_{n_r:n_p-1}, w_{n_r:n_p-1})$ is a feasible solution for the primary controller at the next iteration after appending the terminal control law. Therefore a feasible solution of the primary controller always exists if the initial solution was feasible. Based on the above discussion, recursive feasibility of the primary controller follows.

Auxiliary Controller: Given a feasible initial solution from the primary controller and due to the absence of the state constraints, recursive feasibility of the auxiliary controller follows by construction.

4. SUPERMARKET REFRIGERATION SYSTEM

In this section, the proposed nonlinear MPC approach is applied to a supermarket refrigeration system with 3 display cases and 3 compressors. It is required that the control system keeps the states within their operational bounds despite the endogenous and exogenous switching behavior of the process. In the following section, the considered system will be shortly described, then the nonlinear MPC is formulated and finally, the simulation results are shown.

4.1 Process Description

Refrigeration systems are the core storage and distribution components in the food industry and are responsible for significant consumption of electric power. Therefore, advanced control methods are sought to improve the process efficiency and to minimize power consumption. The main components of the system are the display cases, the suction manifold, the compressors rack and the condensing unit. The rack of compressors is the central element that maintains the flow of the refrigerant to the condensing unit and further to the display cases in order to preserve the edible goods within the specified temperature range. The low pressure refrigerant vapor is collected by the suction manifold from the display cases. Therefore the pressure inside the suction manifold must be held by the compressors within specified bounds in order to realize the required evaporation temperature. For a complete description of the process dynamics including the model equations and the parameter values, please consult (Larsen et al. (2007)).

In our case, the condensing unit is not relevant for the control of the process, therefore the dynamical model is comprised of only 3 sub-models. Each display case is described by 4 states, namely the temperatures of the goods T_g, the evaporator wall T_w, the air curtain T_a and the mass of the refrigerant inside the evaporator m_{ref}. The suction manifold is modeled by the suction pressure P_{suc} dynamical equation. Finally the rack of compressors is modeled by the total volume flow of the refrigerant F_{comp} it draws out of the suction manifold. By combining all the sub-models, the complete model of the supermarket refrigeration system that is considered here has $n = 13$ states. The discrete input variables are the 3 binary expansion valves signals ($v_{exp}(t) \in \{0,1\}^3$) and the compression level ($U_{comp}(t) \in \{0,1,2,3\}$) that indicates how many compressors are active at any time instant.

4.2 Nonlinear MPC Controller

The dynamic model is embedded into a family of continuous systems as shown in (3). The considered switching dynamics were reformulated in Ebrahim et al. (2018) in order to avoid synchronization of the expansion valves of the display cases. It was proposed to restrict the switching of the display valves to one single valve at any time instant, which results in $n_\alpha = 4$ assignments for $v_{exp}(t)$ including the case of switching off all the valves. Additionally, only incrementing or decrementing the current compression level is considered rather than searching for the optimal level directly. This is achieved by augmenting a relaxed version of the number of active compressors U_{comp} to the state vector which increments or decrements based on the value of P_{suc}. Thus the number of feasible assignments is limited to $n_\omega = 3$ regardless of total number of compressors, provided that they have similar capacities as formulated in (17). The choice of preserving the current level is denoted by $\omega^b_{1,k}$, therefore it appears only in the SOS1 constraint.

$$U_{comp,k+1} = U_{comp,k} + \omega^b_{2,k} - \omega^b_{3,k}, \tag{17}$$

$$\omega^b_{i,k} \in \{0,1\}, \quad \sum_{i=1}^{3} \omega^b_{i,k} = 1, \quad i \in \{1,2,3\}.$$

Moreover, separability of the dynamics was exploited in order to further reduce the number of binary multipliers to only 7. The function $F_1^j(x_k, v^j_{exp})$ represents the display cases, while the function $F_2^i(x_k, U^i_{comp})$ contains the right hand sides of the differential equations for P_{suc} and the augmented state U_{comp}:

$$x_{k+1} = \sum_{j=1}^{n_\alpha} \alpha_{j,k} \cdot F_1^j(x_k, v^j_{exp}) + \sum_{i=1}^{n_\omega} \omega_{i,k} \cdot F_2^i(x_k, U^i_{comp}),$$
$$\alpha_{j,k} \in [0,1], \quad \omega_{i,k} \in [0,1], \tag{18}$$

where $\sum_{j=1}^{n_\alpha} \alpha_j = 1$ and $\sum_{i=1}^{n_\omega} \omega_i = 1$, given $x_0 = x(t_0)$. The main function of the refrigeration system is to realize the desired temperatures of the goods inside the display cases while reducing the compressors switching frequency, which can be interpreted as a traditional tracking objective along the prediction horizon n_p.

$$\varphi(x_k) = \frac{1}{n_r \Delta t} \sum_{k=0}^{n_r - 1} (\underline{T}_{g,k} - T_{g,ref})^T Q (\underline{T}_{g,k} - T_{g,ref})$$
$$+ \frac{1}{\Delta t} \sum_{k=n_r}^{n_p - 1} (\underline{T}_{g,k} - T_{g,ref})^T Q (\underline{T}_{g,k} - T_{g,ref}), \tag{19}$$

where $\underline{T}_{g,k}$ is the vector of the goods temperatures in all three display cases. As a result of the non-uniform time discretization grid due to the rounding horizon ($n_r = 5$), there are two tracking terms with different weights. The controller was tuned by choosing the prediction horizon $n_p = 20$, the sampling interval $\Delta t = 30s$, and Q is a identity matrix. It is noted that penalizing the control moves mostly favors the solutions in the interior of the feasible region over the bang-bang solutions which in turn adds to the rounding error. For economic operation of the process, the minimization of the energy consumption of the compressors is considered as an alternative objective function.

$$\varphi_{eco}(x_k) = F_{comp} \cdot \rho_{suc} (h^o_{comp} - h^i_{comp}), \tag{20}$$

where ρ_{suc} is the density of the refrigerant inside the suction manifold which is dependent on P_{suc} and h^i_{comp}, h^o_{comp} are the enthalpies of the refrigerant going in and out of the compressor rack, respectively.

In addition to the process dynamical equations, operational bounds are imposed on some of the state variables, e.g., P_{suc} and T_a. Moreover combinatorial constraints arise as a result of the logical decisions for incrementing and decrementing the number of active compressors:

$$\underline{x} \le x_k \le \overline{x}, \qquad k \in \{0, 1, \cdots, n_p\}, \tag{21}$$
$$\omega_{i,k} \cdot g^i(x_k) \le \tau_{vc}, \quad k \in \{0, 1, \cdots, n_p\},\ i \in \{1, 2, \cdots, n_\omega\},$$

where, $\underline{x}$ and $\overline{x}$ are the lower and the upper bounds of the state vector and $g^i(x_k)$ contains P_{suc} dependent switching conditions. For the combinatorial constraints a regulation parameter $\tau_{vc} = 0.001$ is added to avoid the loss of constraint qualifications associated with combinatorial constraints (Kirches and Lenders (2016)). A suitable back off from the bounds is considered to achieve constraint satisfaction in the primary controller. It was found by extensive simulations that T_a bounds should be tightened by $0.25°C$ from both sides as well as P_{suc} by 0.1 bar.

Next, the rounding horizon section ($n_r = 5$) of the optimal relaxed binary multipliers trajectory is passed to the MILP to generate the binary trajectories. Then the switching times of rounded trajectories are optimized within the auxiliary controller in order to track the relaxed reference solution from the primary controller. The weighting matrices Q, R_2 in (16) are chosen such that P_{suc} is given 10 times more weight than other state variables and R_2 was set to the identity matrix.

4.3 Simulation Results

It is assumed that full state information is available. As described in the previous section, the process model was adapted in order to minimize the switching of the compressors by using the complete range of the suction pressure. Moreover synchronization assignments of the display valves are avoided by construction which then avoids the main problem of the traditional refrigeration controllers, see Larsen et al. (2005). The simulation of the closed loop is performed for 2 hours in the day and 2 hours in the night. During the night, the external air load is reduced by 40% and the temperature reference changes from $T_{g,ref} = 3.5°C$ to $4.0°C$, also the upper bound for P_{suc} is increased. Figure (2) shows the simulation results

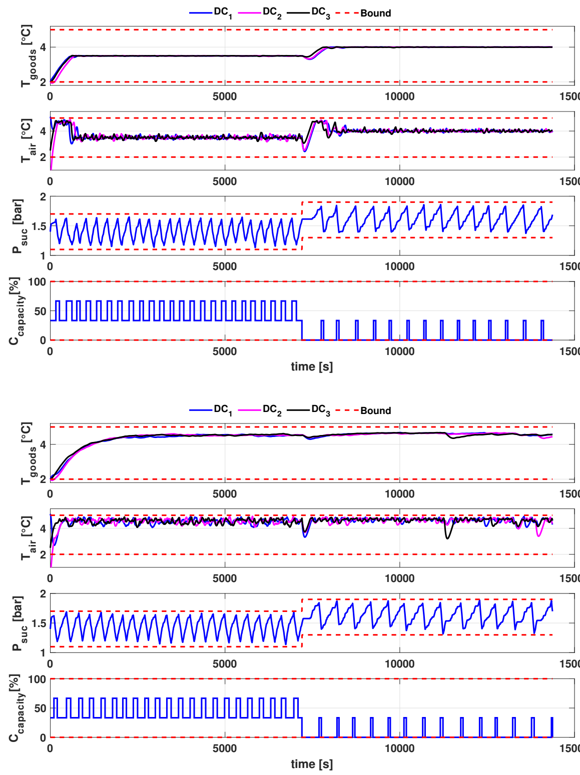

Fig. 2. Tracking (upper) and economic (lower) results of using the Bi-level nonlinear MPC approach

of the controlled process under tracking and economic nonlinear MPC, respectively. The trajectories of T_g, T_a, P_{suc} as well as the compression capacity $C_{capacity}\%$ are shown. It is found that only 2 compressors are needed for the operation of 3 display cases during the day and only one in the night mode. It can be observed that the references for the goods temperature are tracked well and all the process variables are kept within their operational bounds. Moreover, the switching of the compressors is effectively minimized by enforcing a kind of hysteresis behavior with a width equivalent to operational range of P_{suc}, which leads to prolongation of their life time. In this case, active compressors are increased by one only when P_{suc} reaches its upper bound and decreased when the pressure is at its lower bound. Moreover by considering the economic MPC, it was possible to further save about 2% of the power consumption by increasing the goods temperature T_g for all display cases slightly above the tracked value.

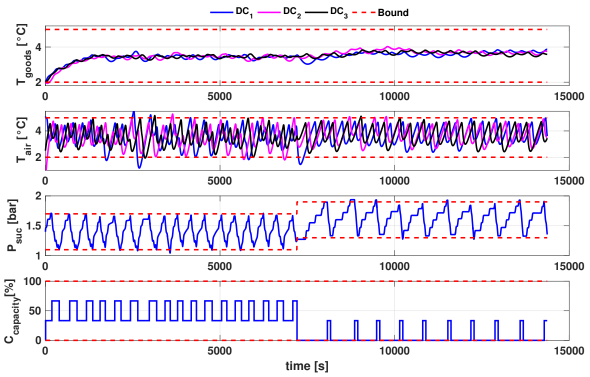

Fig. 3. Tracking Nonlinear MPC (One level)

The average CPU-time for the MPC iteration was 1.8 s, and the worst case was 5.1 s on a standard desktop PC, which is suitable for online application. In order to point out the advantage of implementing the additional layer of optimization, figure 3 shows the simulation results when the process is controlled by a single layer nonlinear MPC controller. The controller solves a single OCP over the embedding model, and the resulting relaxed trajectories are directly applied to the process after being rounded using the sum-up rounding scheme. It can be observed that the proposed Bi-level nonlinear MPC scheme achieves much tighter control with a more regular input pattern while respecting all the process constraints and states bounds. Furthermore, the proposed scheme is more computationally efficient in comparison to MPC schemes optimizing directly over the discrete degrees of freedom even those that use a linearized model, e.g., using the MLD approach (Larsen et al. (2007)) with the advantage of wider operating range.

5. CONCLUSION

A new Bi-level nonlinear MPC scheme for switching dynamical systems was proposed and validated for a nontrivial benchmark problem for both tracking and economic based operation. The scheme determines a tight attainable lower-bound relaxed solution and subsequently restores the integrality constraints and seeks a close sub-optimal switching solution within the specified bounds. Computationally efficiency and recursive feasibility are achieved considering only the nominal case. Future work focuses on stability analysis as well as robustness against plant-model mismatch and parametric uncertainties.

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