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On the Stability and Stabilization of Nonlinear Non-Stationary Discrete Systems ⋆

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Abstract: The study of many biological, economic and other processes leads to its modeling based on discrete equations. Currently, the need to develop a mathematical apparatus for the qualitative analysis of discrete equations is caused by the creation of digital control systems, processors and microprocessors as well as discrete methods of signal transmission in automatic control systems and other theoretical and technical problems. One of the important areas of the qualitative analysis of discrete equations is the stability problem. The main method for studying the stability of nonlinear differential, discrete, and other types of equations is the direct Lyapunov method. The aim of this paper is to develop the direct Lyapunov method in the study of the limiting behavior and asymptotic stability of nonlinear nonstationary discrete equations using the comparison principle. New theorems are proved that are applied in the stability problem of the well-known epidemiological model as well as in solving the stabilization problem of a nonlinear discrete controlled system. An example is shown illustrating a qualitative difference in the conditions of stabilization of non-stationary and stationary discrete systems.

Keywords: Asymptotic stabilization, Lyapunov methods, Parameter-varying systems, Stability of nonlinear systems, Systems biology, Passivity-based control.

1. INTRODUCTION

The latest active researches in fundamental and applied control theory and design of new controlled systems are stimulated by the rapid industrial robotics development as well as high growth of introduction of controlled energy, industrial and other complex processes.

For a quite long period the process of different controlled systems modelling was studied on the basis of time-continuous models. However, the creation of modern digital controlled systems, processors and microprocessors requires the development of an appropriate mathematical and computing apparatus. One of the most accurate in the methods of signal transmission and conversion are discrete-time modelling. These problems include modeling of continuous-time systems with discrete-time control and the further researches on the stabilization of such systems.

This paper considers nonlinear controlled systems with discrete control. The linearity and stationarity of the systems make it possible to apply linear equations for its analysis which has been and still remains the subject of numerous studies (Mickens, 2000). However, the wide class consisting of non-linear systems includes both continuous and discrete parts. In addition, the solution to the control problem of non-stationary systems involves extra complexity in its analysis. The most effective method for studying the stable functioning of such systems is the Lyapunov method.

To analyze the continuous-discrete structure of an automatic control system, difference equations are used in a reasonable way. This paper considers the problem of developing the direct Lyapunov method for studying the stability and stabilization of systems modeled by nonlinear difference equations.

The first results in this field were obtained in (Tsypkin, 1963a,b; Hurt, 1967; LaSalle, 1986a). Later numerous studies have been devoted to this problem. We'd like to sign out from the corresponding monographs the following ones (Halanay and Wexler, 1968; Elaydi, 2004; LaSalle, 1976, 1977, 1986b), the review (Martynyuk, 2000) and modern papers on related researches (Diblik et al., 2016; Nam et al., 2016).

The development of the direct Lyapunov method is presented in the first section. It contains the application of the vector Lyapunov functions and comparison equations. Further, new theorems on the localization of a positive limit set and on the study of stability using Lyapunov functions are proposed. The novelty of the authors theorems

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consists in the weakening of the conditions sufficient to determine the limiting properties of solutions of nonlinear non-stationary discrete systems.

The absence of universal construction methods for Lyapunov functions for solving stability and stabilization problems stimulates intensive research on finding effective algorithms for constructing them for certain classes of systems. The very effective way for solving the problems of trajectory tracking and position stabilization for nonlinear systems is the algorithm for its passification. Among the numerous works in this area, we sing out (Byrnes and Isidori, 1991; Byrnes and Lin, 1993; Polushin et al., 2000) which are directly related to the results of Section 3. In this section the asymptotic stability theorems from Section 2 are applied to the problem of a discrete-time system stabilization. Stabilization results are formulated which are direct consequences of the mentioned theorems. New results are obtained on the construction of a stabilizing control for passive systems.

Throughout this paper the following mathematical notations are used. The symbol Z^+ denotes the set of non-negative integers. The symbol R^m denotes as usual, the real linear space of the $m-$dimensional vectors x with some norm $\|x\|$. $R^+ = [0, +\infty)$. A continuous function $a : R^+ \to R^+$ is said to be of class $\mathcal{K}$, if $a(0) = 0$ and a is strictly increasing. The function $a \in \mathcal{K}$ is said to be of class $\mathcal{K}_\infty$, if $a(s) \to \infty$ as $s \to \infty$.

2. STABILITY ANALYSIS OF NONLINEAR NON-AUTONOMOUS DIFFERENCE EQUATIONS

Consider the non-autonomous system of nonlinear difference equations given by

$$x(n+1) = g(n, x(n)) \tag{1}$$

where $x \in R^m$, $n \in Z^+$, the function $g : Z^+ \times R^m \to R^m$ is continuous in x for each $n \in Z^+$ and $g(n, 0) \equiv 0$.

2.1 Preliminaries

We use the well-known definitions of stability and attractivity of the zero solution $x = 0$ of (1) (Lacschmikantham and Trigiante, 2002; LaSalle, 1976, 1986b).

Definition 1. Vector function $V : Z^+ \times R^p \to R^k$ is said to be positive definite if there exists the function a_1 of class $\mathcal{K}$ such that the following inequality holds

$$\bar{V}(n, x) \geq a_1(\|x\|) \tag{2}$$

where the scalar function $\bar{V}(n, x)$ is defined as follows

$$\begin{aligned} \bar{V}(n, x) = &\max(V_1(n, x), V_2(n, x), \ldots, V_k(n, x)) \text{or} \\ \bar{V}(n, x) = &\sum_{j=1}^{k} V_j(n, x). \end{aligned} \tag{3}$$

Theorem 2. (Hurt, 1967) Let one can find the Lyapunov vector function $V : Z^+ \times R^p \to R^k$ such that

$$V(n, x) = (V_1(n, x), V_2(n, x), \ldots, V_k(n, x))^T$$

and the following inequalities hold

$$\bar{V}(n, x) \geq a_1(\|x\|), \quad V(n+1, g(n, x)) \leq Q(n, V(n, x)) \tag{4}$$

where the function $a_1 : R^+ \to R^+$ is such that $a_1 \in \mathcal{K}$, the function $Q(n, v)$, $Q : Z^+ \times R^k \to R^k$ is nondecreasing in $v \in R^k$ and $Q(n, 0) = 0$.

Let also the zero solution $v = 0$ of the comparison system

$$v(n+1) = Q(n, v(n)) \tag{5}$$

be stable.

Then, the zero solution $x = 0$ of (1) is stable.

If in addition, the zero solution $v = 0$ of (5) is uniformly stable and the Lyapunov vector function $V(n, x)$ satisfies the inequalities $V_j(n, x) \leq a_2(\|x\|)$ $\forall j \in 1, 2, \ldots, k$, then the zero solution $x = 0$ of (1) is uniformly stable.

2.2 The Limiting Equations and Positive Limit Sets for Non-Autonomous Systems of Difference Equations

Let G be the set of the functions $g : Z^+ \times R^m \to R^m$ which are continuous in x. Introduce the following convergence on the set G.

Definition 3. The sequence $\{g_i \in G\}_{i=1}^{\infty}$ converges to g, if $\forall \varepsilon > 0$, $\forall N \in Z^+$ and for each compact set $S \subset R^m$ there exists $N_0 \in Z^+$ such that for all $i \geq N_0$ the following holds

$$\|g_i(n, x) - g(n, x)\| < \varepsilon \quad \forall (n, x) \in [0, N] \times S \tag{6}$$

Note that the convergence defined in Definition 3 is metrizable, if one can introduce the following metric in G.

Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of compact sets S_i in R^m such that

$$S_1 \subset S_2 \subset \ldots \subset S_i \subset \ldots, \quad \bigcup_{i=1}^{\infty} S_i = R^m \tag{7}$$

For all $g^{(1)}, g^{(2)} \in G$ define the metric:

$$\rho(g^{(1)}, g^{(2)}) = \sum_{i=1}^{\infty} 2^{-i} \times \frac{\sup(\|g^{(1)}(n, x) - g^{(2)}(n, x)\| : (n, x) \in [0, i] \times S_i)}{1 + \sup(\|g^{(1)}(n, x) - g^{(2)}(n, x)\| : (n, x) \in [0, i] \times S_i)}$$

Assumption 4. Assume that the right-hand side of (1) satisfies the following two conditions:

a) the function $g(n, x)$ is uniformly bounded on the set $Z^+ \times S$ for each compact set $S \subset R^m$, i.e. the following holds

$$\|g(n, x)\| \leq l = l(S) \quad \forall (n, x) \in Z^+ \times S; \tag{8}$$

b) the function $g(n, x)$ is uniformly continuous in x on each compact set $S \subset R^m$, i.e. $\forall S \subset R^m$ and $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon, S) > 0$ such that for all $n \in Z^+$ and $x_1, x_2 \in S$: $\|x_2 - x_1\| < \delta$ the following inequality holds

$$\|g(n, x_2) - g(n, x_1)\| < \varepsilon. \tag{9}$$

Lemma 5. Let Assumption 4 hold. Then, the family of translates $\{g_i(n, x) = g(i + n, x),\ i \in Z^+\}$ is contained in some compact set $G_0 \subset G$.

Remark 6. Note that the properties (8) and (9) are the precompactness conditions for the function $g(n, x)$.

Definition 7. The function $g^* : Z^+ \times R^m \to R^m$ is said to be a limiting one to g, if there exists a sequence $n_i \to \infty$ such that the sequence of translates

$\{g_i(n,x) = g(n_i+n,x)\}$ converges to the function g^* in the metrizable space G. Accordingly, the system

$$x(n+1) = g^*(n, x(n)) \tag{10}$$

is said to be a limiting one to (1).

Definition 8. Let the solution $x = x(n, n_0, x_0)$ of (1) be defined for all $n \geq n_0$. The vector $q \in \mathrm{R}^m$ is said to be a positive limit point of that solution, if there exists the sequence $n_i \to \infty$ such that $x(n_i, n_0, x_0) \to q$. The set of all limit points of the solution $x = x(n, n_0, x_0)$ is said to be a positive limit set $\Omega^+(n_0, x_0)$.

Note that since for each $n_0 \in \mathrm{Z}^+$, the translate $g_0(n,x) = g(n_0+n,x)$ is defined on the set $[-n_0, +\infty) \times \mathrm{R}^m$ so the definition domain of the limiting function g^* can be extended to the set $\mathrm{Z}^- \times \mathrm{R}^m$. Therefore, one can define the solutions of the system (10) for all initial points $(n_0, x_0) \in \mathrm{Z} \times \mathrm{R}^m$. Accordingly, one can define the following function $x^*(n, n_0, x_0)$, $x^* : \mathrm{Z} \times \mathrm{Z} \times \mathrm{R}^m \to \mathrm{R}^m$.

Definition 9. The set $D \subset \mathrm{R}^m$ is said to be quasi-invariant, if for each $x_0 \in D$ there exist both the limiting system (10) and its solution $x = x^*(n)$, $x^*(0) = x_0$ such that $x^*(n) \in D \;\forall n \in \mathrm{Z}$.

Theorem 10. (LaSalle, 1976) Let the solution $x = x(n, n_0, x_0)$ of (1) be bounded for all $n \in \mathrm{Z}^+$. Then, the positive limit set $\Omega^+(n_0, x_0)$ is bounded and quasi-invariant. Moreover, the solution $x(n, n_0, x_0)$ of (1) asymptotically tends to $\Omega^+(n_0, x_0)$ as $n \to \infty$.

2.3 The Nonlinear Variation-of-Constants Formula of V.M. Alekseev for Nonlinear Difference Equations

Consider the nonlinear difference equation

$$y(n+1) = Q(n, y(n)) + R(n, y(n)), \tag{11}$$

where the function $Q : \mathrm{Z}^+ \times \mathrm{R}^k \to \mathrm{R}^k$ is continuously differentiable in $y \in \mathrm{R}^k$ for each $n \in \mathrm{Z}^+$, and the function $R(n, y)$ is continuous in $y \in \mathrm{R}^k$ for each $n \in \mathrm{Z}^+$.

Let $z = z(n, n_0, z_0)$ be a solution of the unperturbed system

$$z(n+1) = Q(n, z(n)). \tag{12}$$

The matrix defined as (Tsypkin, 1963a)

$$\Phi(n, n_0, z_0) = \frac{\partial z(n, n_0, z_0)}{\partial z_0} \tag{13}$$

is a fundamental one of the linear variational system

$$Z(n+1) = H(n)Z(n),\; H(n) = \frac{\partial Q}{\partial z}(n, z)\Big|_{z=z(n,n_0,z_0)}. \tag{14}$$

In other words, the matrix (13) satisfies the difference equation (14) and $\Phi(n_0, n_0, z_0) = E$, where E is the identity matrix.

Theorem 11. Let $y = y(n, n_0, y_0)$ and $z = z(n, n_0, y_0)$ be the solutions of the systems (11) and (12) respectively, defined for all $n \geq n_0$. Then, for these solutions one can easily find the following relationship

$$y(n, n_0, y_0) = z(n, n_0, y_0) + \sum_{j=n_0}^{n-1} \int_0^1 \Phi(n, j+1, Q(j, y[j]) + sR(j, y[j]))ds\; R(j, y[j]), \tag{15}$$

where $y[j] = y(j, n_0, y_0)$.

Proof. For each $j = n_0, n_0+1, \ldots, n-1$ one can find the following

$$\begin{aligned} &z(n, j+1, y[j+1]) - z(n, j, y[j]) \\ &= z(n, j+1, y[j+1]) - z(n, j+1, z[j+1]), \end{aligned} \tag{16}$$

where $z[j+1] = z(j+1, j, y[j])$.

Applying the Mean Value Theorem, from (16) one can obtain

$$\begin{aligned} &z(n, j+1, z[j+1]) - z(n, j, y[j]) \\ &= \int_0^1 \Phi(n, j+1, sy[j+1] \\ &+(1-s)z[j+1])(y[j+1] - z[j+1])ds \\ &= \int_0^1 \Phi(n, j+1, Q(j, y[j]) + sR(j, y[j]))ds\; R(j, y[j]). \end{aligned} \tag{17}$$

Summarizing the equalities (17) for j from n_0 to n, one can get

$$\begin{aligned} &z(n, n_0, y[n]) - z(n, n_0, y[n_0]) \\ &= \sum_{j=n_0}^{n} \int_0^1 \Phi(n, j+1, Q(j, y[j]) + sR(j, y[j]))ds R(j, y[j]). \end{aligned} \tag{18}$$

Since $z(n, n, y[n]) = y[n] = y(n, n_0, y_0)$ and $z(n, n_0, y[n_0]) = z(n, n_0, y_0)$ so one can obtain the formula (15). This completes the proof.

Remark 12. The relationship (15) represents V.M. Alekseev's variation-of-constants formula (Alekseev, 1961) for nonlinear difference equations. Note that (15) differs from the other forms of the discrete version of V.M. Alekseev's formula obtained in (Allen, 1994; Drici and Dontha, 2004).

2.4 Asymptotic Stability problem for non-autonomous systems of nonlinear difference equations

In this subsection, we give the solution to the asymptotic stability problem for the system (1) using the formula (15) in combination with a comparison method.

Assumption 13. Assume that there exists Lyapunov vector function candidate $V = V(n, x)$, $V : \mathrm{Z}^+ \times \mathrm{R}^m \to \mathrm{R}^k$, such that it is continuous in x for each $n \in \mathrm{Z}^+$ and the following equality holds

$$\begin{aligned} &V(n+1, x(n+1)) \\ &= Q(n, V(n, x(n))) + R(n, x(n), V(n, x(n))), \end{aligned} \tag{19}$$

where $x(n) = x(n, n_0, x_0)$ is a solution of (1), the functions $Q : \mathrm{Z}^+ \times \mathrm{R}^k \to \mathrm{R}^k$ and $R : \mathrm{Z}^+ \times \mathrm{R}^m \times \mathrm{R}^k \to \mathrm{R}^k$ satisfy the conditions:

(1) The function $Q = Q(n, w)$ is quasi-monotonically nondecreasing and continuously differentiable in $w \in \mathrm{R}^k$.

(2) The functions $Q = Q(n,w)$ and $R = R(n,x,w)$ satisfy the precompactness criteria of Assumption 4 such as (8) and (9).
(3) The inequality $R(n,x,w)\leq 0$ holds for all $(n,x,w) \in Z^+ \times R^m \times R^k$.

Using Assumption 13, one can easily obtain that $V(n,x)$ is a comparison vector function and (12) is a comparison system (Martynyuk, 2000).

Lemma 14. Let Assumption 13 hold. Let also $w(n) = w(n,n_0,V_0)$ $(V(n_0,x_0) = V_0)$ be a solution of (12) defined in the interval $[n_0,N]$. Then, for all $n \in [n_0,N]$ one can get the following estimation

$$V(n,x(n,n_0,x_0))\leq w(n,n_0,V_0). \tag{20}$$

Since the comparison system (12) satisfies the precompactness criteria so one can find the family of limiting comparison systems

$$w(n+1) = Q^*(n,w(n)), \quad Q^* \in G_Q. \tag{21}$$

Using the properties of the function $Q = Q(n,x)$, one can get that all the solutions $w = w(n,n_0,w_0)$ of (12) are differentiable in $w_0 \in R^k$. Moreover, since the function $w(n,n_0,w_0)$ is nondecreasing in w_0, one can easily obtain that the matrix

$$\Phi(n,n_0,w_0) = \frac{\partial w(n,n_0,w_0)}{\partial w_0} \tag{22}$$

is positive semi-definite and normalized, i.e. $\Phi(n_0,n_0,w_0) \geq 0$ and $\Phi(n_0,n_0,w_0) = E$. Besides, $\Phi(n_0,n_0,w_0)$ is the fundamental matrix for the linear variational system (14).

Assumption 15. Assume that for each compact set $S \subset R^k$ there exist positive reals $M(S)$ and $m(S)$ such that for all $(n,n_0,w_0) \in Z^+ \times Z^+ \times S$ the matrix $\Phi(n,n_0,w_0)$ satisfies the following conditions

$$\|\Phi(n,n_0,w_0)\|\leq M(S), \quad \det\Phi(n,n_0,w_0)\geq m(S). \tag{23}$$

In the following theorem, a so called quasi-invariance principle for non-autonomous systems of difference equations is proposed which is a generalization of a well-known La-Salle's invariance principle for autonomous systems (LaSalle, 1986b).

Theorem 16. Let Assumptions 13 and 15 hold. Let also the solutions $x(n,n_0,x_0)$ and $w(n) = w(n,n_0,V_0)$ $(V_0 = V(n_0,x_0))$ of the systems (1) and (12) respectively, be bounded for all $n \geq n_0$. Then, for each positive limit point $q \in \Omega^+(n_0,x_0)$ there exists the set of the limiting functions (g^*,V^*,Q^*,R^*) such that for the solution $x = x^*(n,q)$ of the system (10) satisfying the initial condition $x^*(0,q) = q$, $\forall n \in Z$ the following holds

$$\begin{gathered} x^*(n,q) \in \Omega^+(n_0,x_0), \\ x^*(n,q) \in \{V^*(n,x) = w^*(n)\}\cap\{R^*(n,x,w^*(n)) = 0\}, \end{gathered} \tag{24}$$

where $w^*(n)$ is the solution of the limiting comparison system (21) such that $w^*(0) = V^*(0,q)$.

Proof. Using the equality (19) and Alekseev's nonlinear variation of parameters formula (15), one can obtain the following relationship between the functions $V[n] = V(n,x[n]) = V(n,x(n,n_0,x_0))$ and $w = w[n] = w(n,n_0,V_0)$ $(V_0 = V(n_0,x_0))$:

$$\begin{aligned} V(n,x[n]) = w[n] + \sum_{j=n_0}^{n-1} \int_0^1 \Phi(n,j+1,sV[j] \\ +(1-s)w[j])dsR(j,x[j],V[j]). \end{aligned} \tag{25}$$

One can easily see that the function $V(n,x(n))$ is lower bounded on the set $Z^+ \times D$ as well as the solution $w[n]$ of the system (12) is bounded for all $n \geq n_0$. Using (23), one can obtain that there exist the positive reals α_0 and β_0 such that for each $n \geq n_0$ the following inequality holds

$$\begin{aligned} \beta_0 &\geq \sum_{i=1}^{k}(w^i[n] - V^i[n]) \\ &\geq -\alpha_0 \sum_{j=1}^{k}\sum_{r=n_0}^{n-1} R^j(r,x[r],V[r]) \geq 0. \end{aligned} \tag{26}$$

In order to prove this, we consider the equality (25) which can be written as follows

$$\begin{aligned} V^i(n,x[n]) = w^i[n] + \sum_{r=1}^{k}\sum_{j=n_0}^{n-1} \int_0^1 \Phi^{ir}(n,j+1,sV[j] \\ +(1-s)w[j])ds\ R^r(j,x[j],V[j]), \quad i = 1,2,\ldots,k. \end{aligned} \tag{27}$$

Summarizing the equalities (27) over $i = 1,2,\ldots,k$, one can get

$$\begin{aligned} &\sum_{i=1}^{k}(w^i[n] - V^i[n]) \\ &= -\sum_{r=1}^{k}\sum_{j=n_0}^{n-1}\int_0^1 \sum_{i=1}^{k} \Phi^{ir}(n,j+1,sV[j] + (1-s)w[j])ds \\ &\quad \times R^r(j,x[j],V[j]). \end{aligned} \tag{28}$$

Using the second inequality of (23), one can find that there exists a positive real α_0 such that for each $n \geq n_0$ and for each $j = 1,2,\ldots,k$ the following inequality holds

$$\int_0^1 \sum_{i=1}^{k} \Phi^{ir}(n,j+1,sV[j] + (1-s)w[j])ds \geq \alpha_0. \tag{29}$$

Using Assumption 13, from (29) one can obtain that there exists $\beta_0 > 0$ such that the inequality (26) holds. So, each function series constructed by the partial sums from (26) converges. Therefore, the following holds

$$\lim_{n\to+\infty} R(n,x[n],V(n,x[n])) = 0. \tag{30}$$

Let $q \in w^+(n_0,x_0)$ be a positive limit point defined by the sequence $n_j \to +\infty$, i.e. $x(n_j,n_0,x_0) \to q$ as $n_j \to +\infty$. Choose the subsequence $n_{ji} \to +\infty$ such that $g(n_{ji} + n,x) \to g^*(n,x)$, $Q(n_{ji} + n,x) \to Q^*(n,x)$ and $R(n_{ji} + n,x,w) \to R^*(n,x,w)$ as $n_{ji} \to +\infty$. This implies that for each $\beta > 0$ the following holds

$$\begin{gathered} x[n_{ji} + n] \to x^*[n], \quad w[n_{ji} + n] \to w^*[n] \\ \text{uniformly in } n \in [-\beta,\beta] \text{ as } n_{ji} \to +\infty, \end{gathered} \tag{31}$$

where $x^*[n] = x^*(n,0,q)$ and $w^*[n] = w^*(n,0,q)$ $(w^* = V^*(0,q))$ are the solutions of the systems (10) and (21) respectively.

From (26) and (30) one can obtain that for all $t \in R$ the following equalities hold

$$V^*[n, x^*[n]] = w^*[n], \qquad R^*[n, x^*[n], V^*[n]] = 0. \quad (32)$$

This completes the proof.

Remark 17. Theorem 16 represents the solution to the positive limit set localization problem for non-autonomous systems of difference equations. The main feature of our result in comparison with some known ones (Krabs, 2002; Lacschmikantham and Trigiante, 2002) is that we use a comparison method with Lyapunov vector functions which is a generalization of the direct Lyapunov method with scalar Lyapunov functions.

The following theorem represents the development of the authors's previous results obtained for non-autonomous systems of differential equations (Andreyev and Peregudova, 2006) to the non-autonomous systems of difference equations.

Theorem 18. Assume that the Lyapunov vector function candidate $V = V(n, x)$ exists such that

(1) The function $V(n, x)$ satisfies the conditions of Assumptions 4, 13, 15, where $R = R(n, x)$;
(2) The scalar function $\bar{V}(n, x)$ defined as (3) is positive definite and radially unbounded;
(3) $V_i(n, x) \to 0$ uniformly in n as $\|x\| \to 0$, $i = 1, 2, \ldots, k$;
(4) The zero solution $w = 0$ of comparison system (12) is uniformly globally stable;
(5) For any limiting pair (g^*, R^*) there are no solutions of (10) which stay forever in the set $\{R^*(n, x) = 0\}$ except for the zero solution $x = 0$.

Then, the zero solution $x = 0$ of (1) is uniformly globally asymptotically stable.

Proof. Using the comparison principle, from the conditions 1–4 of Theorem 18 one can obtain that the zero solution $x = 0$ of (1) is uniformly globally stable.

The next step of our proof is to show the uniform global attractiveness property of the solution of (1). In other words, we have to prove that for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathrm{Z}^+$ such that $\forall \Delta > 0$ and $\forall n_0 \in \mathrm{Z}^+$ the following inequality holds

$$\|x(n, n_0, x_0)\| < \varepsilon \quad \forall n \geq n_0 + N(\varepsilon), \quad \forall \|x_0\| < \Delta. \quad (33)$$

To this end, let us suppose the contrary that there exists $\varepsilon_0 > 0$ such that $\forall \Delta > 0$ and $\forall \{N_k \to +\infty\}$ one can find a sequence (n_k, x_k), $k \to \infty$ $(n_k \geq 0, \|x_k\| < \Delta)$ such that

$$\|x(n_k + N_k, n_k, x_k)\| \geq \varepsilon_0. \quad (34)$$

Without loss of generality, we assume that $n_k \to +\infty$ and $x_k \to x_0^*$ as $k \to +\infty$ and the sequence $n_k \to +\infty$ defines the limiting pair (g^*, R^*) since otherwise we can take both $n_k + N_k/2$ and $x(n_k + N_k/2, n_k, x_k)$ instead of n_k and x_k, respectively, and then pass to the convergent subsequences if it is necessary.

Choose a real $\delta_0 = \delta_0(\varepsilon) > 0$ from the condition of the global uniform stability of the zero solution $x = 0$ of (1). One can easily see that $\forall n \in Z^+$ the following inequality holds

$$\|x(n + n_k, n_k, x_k)\| \geq \delta_0 > 0. \quad (35)$$

If we pass to the limit in (35) as $k \to +\infty$, then we obtain the following

$$\|x^*(n, 0, x_0^*)\| \geq \delta_0 > 0 \quad \forall n \in \mathrm{Z}^+. \quad (36)$$

For the solution $x^*(n, 0, x_0^*)$ of (10) define a positive limit point x_0^{**} using some sequence $n_m \to +\infty$. Without loss of generality, we assume that the sequence $n_m \to +\infty$ defines a limiting pair (g^{**}, R^{**}). Let $x^{**}(n, 0, x_0^{**})$ be a solution of $x(n+1) = g^{**}(n, x(n))$. Using Theorem 16, one can easily see that $x^{**}(n, 0, x_0^{**}) \in \{R^{**}(n, x) = 0\}$ $\forall n \in \mathrm{Z}^+$. Then, using the condition 5 of Theorem 18, we obtain the following equality

$$x^{**}(n, 0, x_0^{**}) \equiv 0. \quad (37)$$

It is obviously that the equality (37) contradicts the inequality (36). This completes the proof.

Remark 19. The advantage of Theorem 18 over the well-known results Lacschmikantham and Trigiante (2002) is that in order to derive the asymptotic stability property of the solutions of non-autonomous difference systems it is not necessary that the corresponding comparison system has the asymptotically stable zero solution.

2.5 Stability analysis of the discrete-time epidemic model

In this subsection, an epidemic model for the spread of gonorrhea or chlamydia is investigated in discrete time. The population is divided into two heterosexual groups, females and males. The infected members of one group can infect the healthy members of the other one. A discrete model of the disease course is given by

$$\begin{cases} I_1(n+1) = (1 - r_1 h) I_1(n) \\ +\dfrac{c_{12} h M}{W} I_2(n)(1 - I_1(n)), \\ I_2(n+1) = (1 - r_2 h) I_2(n) \\ +\dfrac{c_{21} h W}{M} I_1(n)(1 - I_2(n)), \end{cases} \quad (38)$$

where $n \in \mathrm{Z}^+$, I_1 and I_2 are the fractions of the infected members of the groups, respectively; $0 \leq I_1 \leq 1$, $0 \leq I_2 \leq 1$; W and M are the sizes of the groups, respectively; c_{jk} $(j, k = 1, 2,\ j \neq k)$ and r_i $(i = 1, 2)$ are the coefficients which characterize the process of the infection spread (contact and recovery rates), for which the following inequalities hold $0 \leq c_{jk} h \leq W/M$ and $0 \leq r_i h \leq 1$; the constant $h > 0$ is the unit time interval.

Assume that the contact and recovery rates can vary with the season during the year, i.e. $c_{jk} = c_{jk}(n)$ and $r_i = r_i(n)$, where $j, k = 1, 2,\ j \neq k$ and $i = 1, 2$.

It is easy to see that the set $L = \{0 \leq I_1 \leq 1, \quad 0 \leq I_2 \leq 1\}$ is invariant with respect to the solutions $I(n, n_0, I_0) \in L$ of (38) for all initial points $I(n_0) = I_0$, where $(n_0, I_0) \in \mathrm{Z}^+ \times L$ and $\forall n \geq n_0$.

The system (38) satisfies the precompactness criteria (8) and (9). Therefore, the following limiting system can be obtained

$$\begin{cases} I_1(n+1) = (1 - r_1^*(n) h) I_1(n) \\ +\dfrac{c_{12}^*(n) h M}{W} I_2(n)(1 - I_1(n)), \\ I_2(n+1) = (1 - r_2^*(n) h) I_2(n) \\ +\dfrac{c_{21}^*(n) h W}{M} I_1(n)(1 - I_2(n)), \end{cases} \quad (39)$$

where $c^*_{ij}(n) = \lim_{k\to+\infty} c_{ij}(n_k + n)$ and $r^*_i(n) = \lim_{k\to+\infty} r_i(n_k + n)$, $i, j = 1, 2$, $i \neq j$.

Choose the Lyapunov vector function candidate such as $V = (I_1, I_2)^T$. One can easily obtain the comparison system

$$\begin{aligned} w_1(n+1) &= (1 - r_1(n)h)w_1(n) + \frac{c_{12}(n)hM}{W} w_2(n), \\ w_2(n+1) &= \frac{c_{21}(n)hW}{M} w_1(n) + (1 - r_2(n)h)w_2(n). \end{aligned} \tag{40}$$

The vector $R = (R_1, R_2)^T$ is given by

$$R_1 = -\frac{c_{12}(n)hM}{W} I_1 I_2, \quad R_2 = -\frac{c_{21}(n)hW}{M} I_1 I_2. \tag{41}$$

The limiting functions R^*_1 and R^*_2 are defined as follows

$$R^*_1 = -\frac{c^*_{12}(n)hM}{W} I_1 I_2, \quad R^*_2 = -\frac{c^*_{21}(n)hW}{M} I_1 I_2. \tag{42}$$

The zero solution $w_1 = w_2 = 0$ of the comparison system (40) is uniformly stable if for each $k_0 \in \mathrm{Z}^+$ and for all $k \geq k_0$ the following holds

$$\|A(k_0) \cdot A(k_0 + 1) \cdot \ldots \cdot A(k)\| \leq a_0 = \text{constant}, \tag{43}$$

where the matrix $A(j) \in \mathrm{R}^{2\times 2}$ is defined as follows

$$A(j) = \begin{pmatrix} 1 - r_1(j)h & \dfrac{c_{12}(j)hM}{W} \\ \dfrac{c_{21}(j)hW}{M} & 1 - r_2(j)h \end{pmatrix}. \tag{44}$$

Since the comparison system (40) is linear so the uniform stability of its zero solution $w_1 = w_2 = 0$ is global.

The set $\{R^*_1 = R^*_2 = 0\}$ doesn't contain the solutions of (39) except for $I_1 = I_2 = 0$. Using Theorem 18, one can obtain the uniform asymptotic stability "in large" property for the zero solution $I_1 = I_2 = 0$ of (38) if the inequality (43) holds. Then, there is no epidemic and the disease dies out.

Note that the inequality (43) is true if the following holds

$$\max_{i,j\in \mathrm{Z}^+} (c_{12}(i)c_{21}(j))/(r_1(i)r_2(j)) \leq 1. \tag{45}$$

Note that from (45) one can obtain the well-known inequality proposed by Martin et al. (1996) for autonomous case of the system (38), which determines whether the infection dies out. The extinction diseases condition consists in introducing the basic reproduction number $\mathcal{R}_0$ which is not more than one, i.e.

$$\mathcal{R}_0 = (c_{12}c_{21})/(r_1 r_2) \leq 1.$$

3. DISCRETE-TIME CONTROLLERS FOR CONTROLLED SYSTEMS

3.1 Solution to the Stabilization Problem of Passive Systems

Consider a discrete-time controlled system given by

$$x(n+1) = f(n, x(n), u(n)), \tag{46}$$

$$y(n) = h(n, x(n), u(n)), \tag{47}$$

where $x \in \mathrm{R}^m$, $u \in \mathrm{R}^p$, $y \in \mathrm{R}^p$ are the state, input and output vectors of the system respectively; $f : \mathrm{Z}^+ \times \mathrm{R}^m \times \mathrm{R}^p \to \mathrm{R}^m$ and $h : \mathrm{Z}^+ \times \mathrm{R}^m \times \mathrm{R}^p \to \mathrm{R}^p$ are continuous functions in (x, u) satisfying the precompact conditions.

Assume that $f(n, 0, 0) \equiv 0$, $h(n, 0, 0) \equiv 0$ such that the system (46) – (47) has the zero equilibrium state $x = 0$, $y = 0$.

For (46) – (47), the limiting system can be defined in the following form

$$x(n+1) = f^*(n, x(n), u(n)), \tag{48}$$

$$y(n) = h^*(n, x(n), u(n)), \tag{49}$$

where (f^*, h^*) is any limiting pair.

Definition 20. The system (46) – (47) is called strictly observable in the zero state if for any limiting pair (f^*, h^*) the set $\{h^*(n, x, 0) = 0\}$ does not contain the solutions of the limiting system (48), except for $x = 0$.

Definition 21. Any system of a type (46) is called passive, if there exists a scalar function $V = V(n, x)$ called a storage function such that

$$V(n+1, f(n, x, u)) \leq W(n, V(n, x)) + y^T u, \tag{50}$$

where $W(n, w) \geq 0$, $W(n, 0) \equiv 0$ is a continuous, monotonic in w function such that the zero solution of the corresponding comparison equation

$$w(n+1) = W(n, w(n)) \tag{51}$$

is uniformly stable.

Theorem 22. Let for the system (46) – (47) the following conditions be hold:

(1) the system is passive with a positive definite margin-tolerant storage function $V(n, x)$;
(2) the system is strictly observable in the zero state.

Then, the controller $u = u(n, y)$ such that $y^T u(n, y) \leq -\alpha(\|y\|)$, where $\alpha \in \mathcal{K}$ solves the stabilization problem of the zero state $x = 0$ of the system (46) – (47).

Proof. Let us use the storage function $V(n, x)$ as the Lyapunov function for the following closed-loop system:

$$\begin{aligned} &x(n+1) = f(n, x, u(n, y)), \\ &y = h(n, x). \end{aligned} \tag{52}$$

Using (50) one can get

$$\begin{aligned} &V(n+1, x(n+1)) = V(n+1, f(n, x(n), u(n, y))) \\ &\leq W(n, V(n, x)) + y' u(n, y) \leq W(n, V(n, x(n))) \\ &-\alpha(\|y\|). \end{aligned} \tag{53}$$

From (53) one can obtain that for the function $V(n, x)$ there exists a comparison equation (51). Due to the strict observability of (46) – (47), the set $\{\alpha(\|y\|) = 0\} = \{y = 0\} = \{h^*(n, x) = 0\}$ does not contain the solutions of (48), except for the zero state $x = 0$. Accordingly to Theorem 22, one can get the proof.

Note that the scope of Theorem 22 can be expanded by converting the non-passive systems to the passive ones.

Consider the special case of the system (48) as

$$x(n+1) = f(n, x(n)) + B(n, x(n))u. \tag{54}$$

Assume that there exists a strictly positive definite quadratic form $V(x)$ of x such as

$$\begin{aligned}&V(x) = x^T C x \geq c_0 \|x\|^2, \ C \in \mathrm{R}^{m\times m} - \text{constant matrix},\\ &c_0 = \text{constant} > 0, \quad \|x\|^2 = x_1^2 + \ldots + x_m^2,\\ &V(x(n+1)) = f^T(n, x(n)) C f(n, x(n))\\ &\leq W(n, V(n, x(n))),\end{aligned}$$

where W is the function from the equation (51).

Assume that the following holds

$$y = 2B^T C f + B^T C B u. \tag{55}$$

Then, one can get

$$\begin{aligned}&V(f(n,x) + B(n,x)u) = (f + Bu)^T C (f + Bu)\\ &= f^T C f + 2 f^T C B u + u^T B^T C B u\\ &= f^T(n,x) C f(n,x) + y^T u \leq W(n, V(n,x)) + y^T u.\end{aligned}$$

In such a way the system (54) with the output y defined by (55) turns out to be passive. Therefore, the conditions of Theorem 22 hold.

Note that the feedback can be used to ensure some system to be passive. In this case, if for the system (54) there exists a feedback change

$$u = u_0(n,x) + u_1(n,x)v \tag{56}$$

and the output $y = h(n,x)$ is such that the system

$$\begin{aligned}&x(n+1) = f(n, x(n)) + B(n, x(n)) u_0(n, x)\\ &+B(n, x(n)) u_1(n, x) v, \quad y = h(n, x(n))\end{aligned}$$

satisfies Theorem 22, then, the zero state $x = 0$ is stabilizable by using the controller $v = v(n, y)$.

The class of passifiable systems can be expanded to the systems that are a cascade connection of two subsystems one of which is passive and the second is characterized by the fact that its origin is the equilibrium point of the corresponding open system such as follows

$$z(n+1) = g(n, z(n)) + B_2(n, z(n), y(n)) y(n) \tag{57}$$

$$x(n+1) = f(n, x(n)) + B_1(n, x(n)) u, \tag{58}$$

$$y(n) = h(n, x(n)), \tag{59}$$

where the functions g, f and h are such that $g(n,0) \equiv 0$, $f(n,0) \equiv 0$, $h(n,0) \equiv 0$, the matrices $B_1(n,x)$ and $B_2(n,z,y)$ are continuous in x and (y,z) respectively, and all the functions g, f and h and matrices $B_1(n,x)$ and $B_2(n,z,y)$ satisfy the precompact conditions.

System (57)-(59) can be considered as a cascade connection of the first subsystem (58), (59) and the second one (57).

Let's assume that the first subsystem (58), (59) is passive with the storage function $V_1 = V_1(n,x)$ such that the following holds

$$V_1(n+1, x(n+1)) \leq \mu(n) V_1(n, x(n)) + y^T u, \tag{60}$$

where $\mu(n)$ is a function satisfying the following condition

$$\prod_{j=n_0}^{n} \mu(j) \leq \mu_0 = \text{constant } \forall n \geq n_0. \tag{61}$$

Assume also that for the system $z(n+1) = g(n, z(n))$ there exists a positive definite quadratic form

$$V_2(z) = z^T C z \tag{62}$$

such that

$$V_2(g(n,z)) \leq \mu(n) V_2(z). \tag{63}$$

One can easily obtain that for the system (57) – (59) there exists a storage function $V(n,x,z) = V_1(n,x) + V_2(z)$ such that

$$\begin{aligned}&V(n+1, x(n+1), z(n+1)) = V_1(n+1, x(n+1))\\ &+V_2(g(n, z(n))) + B_2(n, z(n), y(n)) y(n))\\ &\leq \mu(n) V_1(n, x(n)) + y^T(n) u + V_2(g(n, z(n)))\\ &+2g^T(n, z(n)) C B_2(n, z(n), y(n)) y(n)\\ &+y^T(n) B_2^T(n, z(n), y(n)) C B_2(n, z(n), y(n)) y(n)\\ &\leq \mu(V_1 + V_2) + y^T(u + 2B_2^T C g + B_2^T C B_2 y).\end{aligned} \tag{64}$$

With the following change of feedback

$$u = -B_2^T(n,z,y) C (2g(n,z) + B_2(n,z,y) y) + v \tag{65}$$

one can get the estimation

$$\begin{aligned}&V(n+1, x(n+1), z(n+1))\\ &\leq \mu(n) V(n, x(n), z(n)) + y^T(n) v.\end{aligned} \tag{66}$$

From (66), one can obtain that the system (57) – (59), (65) is passive with the storage function V.

3.2 An Output Feedback Stabilization Problem of a One Second-Order Discrete-Time System

Consider a second-order system with a scalar controller

$$\begin{cases} x_1(n+1) = \nu(n)(-x_2(n) + x_1^2(n) u(n)),\\ x_2(n+1) = \nu(n)(x_1(n) - \sqrt{2} x_2(n)),\end{cases} \tag{67}$$

$$u(n) = x_1(n) / \sqrt{1 + x_1^4(n)}, \tag{68}$$

where the function $\nu(n)$ satisfies the conditions:

(1) for any sequence $n_k \to \infty$ such that $0 < n_{k+1} - n_k \leq N_0 > 0$ the following holds
$$\underline{\lim}_{k\to\infty} \nu(n_k) = \nu_0 > 0$$
(2) there exists a function $\mu(n) = \nu^2(n)$ satisfying (61).

For $\nu(n) \equiv 1$, the linear approximation system is given by

$$\begin{cases} x_1(n+1) = -x_2(n),\\ x_2(n+1) = x_1(n) - \sqrt{2} x_2(n).\end{cases}$$

Note that the roots of the associated characteristic equation, $\lambda_{1,2} = (\sqrt{2} \pm \sqrt{2} i)/2$, are such that $|\lambda_1| = |\lambda_2| = 1$, so for $\nu(n) \equiv 1$ there exists a critical case.

It is easy to show that the output feedback controller (68) globally stabilizes the zero state $x_1 = x_2 = 0$ of (67), (68).

Consider the Lyapunov function candidate as $V(x_1, x_2) = ((x_1 - x_2/\sqrt{2})^2 + x_2^2/2)/2$. Note that the function $V(x_1, x_2)$ is positive definite and radially unbounded.

One can easily obtain the following

$$\begin{aligned}&V(n+1) = \nu^2(n) V(n)\\ &-\frac{\nu^2(n) x_1^4(n)}{2\sqrt{1 + x_1^4(n)}} \left(\sqrt{2} - \frac{x_1^2(n)}{\sqrt{1 + x_1^4(n)}}\right).\end{aligned} \tag{69}$$

From (69), one can obtain the estimation

$$V(n+1) \leq \nu^2(n) V(n). \tag{70}$$

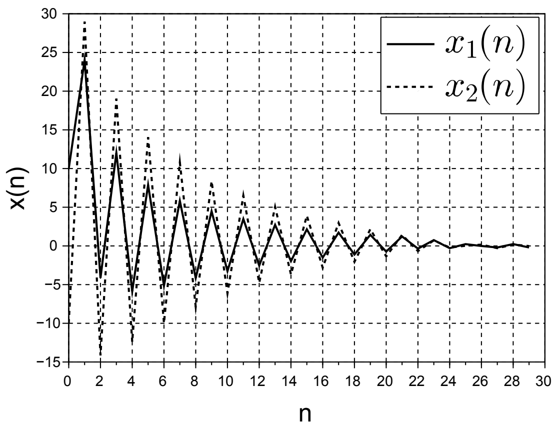

Fig. 1. The solution $x_1(n)$ and $x_2(n)$ of (67),(68)

The zero solution $w = 0$ of the comparison equation

$$w(n+1) = \nu^2(n)w(n) \tag{71}$$

is uniformly stable.

One can easily see that the set

$$\left\{ \frac{\nu^{2*}(n)x_1^4}{2\sqrt{1+x_1^4}} \left(\sqrt{2} - \frac{x_1^2}{\sqrt{1+x_1^4}} \right) = 0 \right\}$$

doesn't contain the solutions of the equation limiting to (67), (68), except for $x_1 = x_2 = 0$. Using Theorem 18, one can obtain the uniform global asymptotic stability property for the zero solution $x_1 = x_2 = 0$ of (67), (68).

From Fig. 1 one can see the numerical results of modelling the system (67), (68), where $\nu(n) = 5/6\ \forall n = 2k+1$, $k \in \mathrm{Z}^+$ and $\nu(n) = 6/5\ \forall n = 2k$, $k \in \mathrm{Z}^+$.

4. CONCLUSION

In this paper, the global asymptotic stability problem for non-autonomous systems of nonlinear difference equations has been considered using both the comparison method and the theory of limiting equations. We have proposed a so called quasi-invariance principle which is the generalization of the well-known La-Salle invariance principle to the non-autonomous systems of difference equations. The theorem on the limiting behavior of the solutions has been proved by using the variation-of-constants formula of V.M. Alekseev. Then we have proved the theorem on the uniform global asymptotic stability property without the requirement of this property for the zero solution of the comparison system. The stabilization problem of a nonlinear nonstationary discrete system has been solved using the theoretical results obtained in the paper.

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