

The AAA framework for modeling linear dynamical systems with quadratic output

Ion Victor Gosea* Serkan Gugercin**

* *Max Planck Institute for Dynamics of Complex Technical Systems,
Magdeburg, Germany, (e-mail: gosea@mpi-magdeburg.mpg.de)*

** *Department of Mathematics and Division of Computational
Modeling and Data Analytics, Virginia Tech, Blacksburg, VA 24061,
Virginia, USA (e-mail: gugercin@vt.edu)*

Abstract: We consider linear dynamical systems with quadratic output. We first define the two transfer functions, a single-variable and a multivariate one, that fully describe the dynamics of these special nonlinear systems. Then, using the samples of these two transfer functions, we extend the AAA algorithm to model linear systems with quadratic output from data.

Keywords: Model reduction, Complex systems, Time-invariant systems.

1. INTRODUCTION

Model order reduction (MOR) typically refers to class of methodologies than can be used to approximate large-scale dynamical systems with much smaller systems that ideally have similar response characteristics as the original ones. Many MOR methods have been developed in the last decades; we refer the reader to Antoulas (2005); Baur et al. (2014); Antoulas et al. (2020); Quarteroni et al. (2015); Benner et al. (2017) and to the references therein for more details on various different approaches to model reduction. The approach we consider here falls under the systems theoretical (input/output) framework for model reduction.

Many time-dependent processes modeled and studied in real-world applications exhibit nonlinear dynamics. In some cases, in order to simplify the modeling part, linearization of the dynamics is performed in many engineering branches. This is usually done around an operating point in a given domain and is restricted to local conditions. In order to obtain more general models, nonlinearities need not be omitted and, instead, need to be suitably treated. Hence, the study, analysis, and modeling of nonlinear dynamical systems have been the focus of considerable research in the last decades. Consequently, many model reduction methods that can be directly applied to nonlinear systems have been proposed. For a fairly extensive and recent review on MOR for nonlinear and linear systems, we refer the reader to Baur et al. (2014).

Considerable progress has been made by extending systems theoretic MOR methods from linear to certain classes of nonlinear systems; such as bilinear systems, quadratic-bilinear (QB) systems, and linear systems with quadratic output (LQO), which is the main focus of our work; see, e.g., Benner and Damm (2011); Antoulas et al. (2016); Benner and Breiten (2012); Flagg and Gugercin (2015) for extension of systems theoretical techniques to bilinear systems; Gosea and Antoulas (2018); Benner et al. (2018);

* The work of Gugercin was supported in parts by NSF through Grant DMS-1720257 and DMS-1819110.

Benner and Goyal (2017); Kramer and Willcox (2019) to QB systems, and Pulch and Narayan (2019); Benner et al. (2019) to LQO systems. Additional contributions to MOR of nonlinear systems were made by Kawano and Scherpen (2007) and by Astolfi (2010).

Moreover, system identification of nonlinear systems has been a popular topic for decades already. In particular, we mention here the special case of identifying linear systems with nonlinear output or input functions, e.g. the so-called Wiener and, respectively, Hammerstein models. Note also that LQO systems are a special class of Wiener models for which the nonlinear output mapping is quadratic. Significantly effort has been allocated for identification of such models; see, for example, Juditsky et al. (1995), Giri and Bai (2010) and the references therein.

We are interested data-driven methods for model reduction where one does not necessarily have access to internal degrees of freedom, i.e., the underlying large-scale state-space representation is unknown. Instead, one has access only to a collection of input/output measurements corresponding to the original system. In this work, by data we mean frequency domain samples of the input/output mapping, known as the transfer function, of the underlying system. Hence, we focus here on data-driven methods that use rational interpolation and least-square approaches to fit a rational function to the given set of frequency-domain (transfer function) measurements; such as the vector fitting approach, Gustavsen and Semlyen (1999), the Loewner framework in Mayo and Antoulas (2007); Antoulas et al. (2017), and the AAA algorithm in Nakatsukasa et al. (2018). This contribution aims at extending the AAA algorithm (Section 2) to a special class of nonlinear systems outlined (Sections 3 and 4).

2. THE AAA ALGORITHM

Given a set of measurements $h_k = H(\xi_k)$ of an underlying (transfer) function $H(\cdot)$ for $k = 1, 2, \dots, N_s$ where $\xi_k \in \mathbb{C}$ are the *sampling (support) points* and $h_k \in \mathbb{C}$ are the

sampled data, the AAA (Adaptive Antoulas-Anderson) algorithm computes a rational function $r(s)$, either to a specified accuracy or of a specified order, that approximates the sampled data. The approximating rational function $r(s)$ is written in barycentric format, a numerically stable representation of rational functions:

$$r(s) = \frac{\sum_{k=0}^n \frac{w_k h_k}{s - \xi_k}}{\sum_{k=0}^n \frac{w_k}{s - \xi_k}}, \quad (1)$$

where the *weights* $w_k \in \mathbb{C}$ are to be determined. It directly follows from the form of the rational function $r(s)$ in (1) satisfies the interpolation conditions $r(\xi_k) = h_k$ for $1 \leq k \leq n$, assuming $w_k \neq 0$. Hence, interpolation at the n support points is attained for free by means of the special representation chosen. Then, there is a freedom in choosing the weights $w_k \neq 0$ to match the remaining the $N_s - n$ data points in an appropriate measure.

Assuming enough degrees of freedom, the framework of Antoulas and Anderson (1986) chooses the weights w_k to enforce interpolation at the remaining points as well. In contrast, the AAA algorithm is iterative, and combines interpolation and least-squares fitting. Assume at step k , a rational function $r_k(s)$ in barycentric form (1) is constructed to interpolate k data points. The idea is that the next support (interpolation) point ξ_{k+1} is selected by means of a greedy algorithm, i.e., the current approximation error of $r_k(s)$ at ξ_{k+1} is maximum within the data set. Afterwards, the corresponding weights w_k are computed by solving a least-squares problem to minimize the least-square deviation in the remaining data points. For further details, we refer the reader to the original source Nakatsukasa et al. (2018). The AAA algorithm proved very flexible and effective, and has been employed in various applications such as rational approximation over disconnected domains Nakatsukasa et al. (2018), solving nonlinear eigenvalue problems Lietaert et al. (2018), modeling of parametrized dynamics Carracedo Rodriguez and Gugercin (2020), and approximation of matrix-valued functions Gosea and Güttel (2020).

3. LINEAR SYSTEMS WITH QUADRATIC OUTPUT

The main focus of this work is to study data-driven MOR of linear systems with quadratic output, which are described in state-space representation as

$$\Sigma_{\text{LQO}} : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T \mathbf{x}(t) + \underbrace{\mathbf{K}[\mathbf{x}(t) \otimes \mathbf{x}(t)]}_{\mathbf{x}^T(t)\mathbf{M}\mathbf{x}(t)}, \end{cases} \quad (2)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{K} \in \mathbb{R}^{1 \times n^2}$, and $\mathbf{M} \in \mathbb{R}^{n \times n}$ ($\mathbf{K} = \text{vec}(\mathbf{M})$). Additionally, note that in (2), the symbol \otimes denotes the Kronecker product of the vector $\mathbf{x} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$ with itself, i.e.

$$\mathbf{x} \otimes \mathbf{x} = [\mathbf{x}_1^2 \ \mathbf{x}_1 \mathbf{x}_2 \ \mathbf{x}_1 \mathbf{x}_3 \ \cdots \ \mathbf{x}_1 \mathbf{x}_n \ \cdots \ \mathbf{x}_n^2] \in \mathbb{R}^{n^2}.$$

Several MOR methodologies have been already proposed for dealing with the case of LQO systems. More precisely, balanced truncation-type methods were considered in Van Beeumen and Meerbergen (2010); Pulch and Narayan

(2019); Benner et al. (2019), while interpolation-based methods were used in Van Beeumen et al. (2012); Gosea and Antoulas (2019). In some cases, the vector \mathbf{c}^T in (2) is $\mathbf{c} = \mathbf{0}$, and thus the output has only the quadratic term.

For LQO systems in (2), the nonlinearity is present in the state-to-output equation only. Hence, one can write the input-output mapping of system Σ_{LQO} in the frequency domain using two transfer functions:

- (1) one corresponding to the linear part of the output, i.e., $y_1(t) = \mathbf{c}^T \mathbf{x}(t)$;
- (2) one corresponding to the quadratic part of the output, i.e., $y_2(t) = \mathbf{K}(\mathbf{x}(t) \otimes \mathbf{x}(t))$.

The linear transfer function corresponding to $y_1(t)$, denoted by $H_1(s)$, is as in the classical case of linear systems. It is a rational function and can be written in terms of the system matrices as

$$H_1(s) = \mathbf{c}^T (s\mathbf{I}_n - \mathbf{A})^{-1} \mathbf{b}, \quad (3)$$

where \mathbf{I}_n is the identity matrix of dimension $n \times n$. Additionally, the quadratic transfer function, corresponding to the output $y_2(t)$ is also a rational function, but of two variables. It is defined as follows

$$H_2(s, z) = \mathbf{K} \left[(s\mathbf{I}_n - \mathbf{A})^{-1} \mathbf{b} \otimes (z\mathbf{I}_n - \mathbf{A})^{-1} \mathbf{b} \right]. \quad (4)$$

We will use an extension of the barycentric representation (1) to represent $H_2(s, z)$ in a suitable barycentric-like format to extend AAA to LQO systems in (2).

4. PROPOSED FRAMEWORK FOR DATA-DRIVEN MODELING OF LQO SYSTEMS

Recall that the AAA framework for linear systems uses (transfer) function samples $h_k = H(\xi_k)$. In extending AAA to LQO systems, data (measurements) will correspond to sampling not only the single-variable transfer function $H_1(s)$ in (3) but also the two-variable transfer function $H_2(s, z)$ in (4).

Theorem 4.1. The barycentric form for approximating the first transfer function $H_1(s)$ is denoted with $r_1(s)$ and can be written in a similar manner as (1), namely

$$r_1(s) = \frac{\sum_{k=1}^n \frac{w_k h_k}{s - \xi_k}}{1 + \sum_{k=1}^n \frac{w_k}{s - \xi_k}}. \quad (5)$$

It follows that $r_1(s)$ interpolates the data at the support points $\{\xi_1, \xi_2, \dots, \xi_n\}$, i.e.,

$$r_1(\xi_k) = w_i = H_1(\xi_k), \quad (\forall) \ i = 1, 2, \dots, n. \quad (6)$$

Note that the scalar 1 was added to the denominator to make the rational approximant $r_1(s)$ strictly proper. However, this does not affect the interpolation property.

In what follows we will sketch the proof of Theorem 4.1. Define the polynomial $P(s) = \prod_{k=1}^n (s - \xi_k)$ and also $P_i(s) = \prod_{k=1, k \neq i}^n (s - \xi_k) \ \forall \ i = 1, \dots, n$. Then, by bringing all the fraction terms of $r_1(s)$ in (5) to the same common denominator P , one can rewrite the rational function as

$$r_1(s) = \frac{\sum_{k=1}^n w_k h_k P_k(s)}{P(s) + \sum_{k=1}^n w_k P_k(s)} \Rightarrow r_1(\xi_i) = \frac{w_i h_i P_i(\xi_i)}{w_i P_i(\xi_i)} = h_i \quad (7)$$

Next, we need to pick an appropriate barycentric-like form for approximating $H_2(s, z)$. The form for $H_2(s, z)$ cannot be chosen independently from that of $H_1(s)$ (e.g., as a two-variable barycentric form as in Ionita and Antoulas (2014)) because $H_1(s)$ and $H_2(s, z)$ share the same poles and the weights are related. This is easy to see from the transfer function $H_1(s)$ and $H_2(s, z)$. The eigenvalues of the same matrix \mathbf{A} contribute to the poles of both $H_1(s)$ and $H_2(s, z)$. In other words, we need to pick the barycentric(-like) forms for $r_1(s)$ and $r_2(s, z)$ to guarantee that they correspond to an underlying LQO system. We have the following result:

Theorem 4.2. Given the interpolation points ξ_k for $k = 1, 2, \dots, n$, let the data $h_k = H_1(\xi_k)$ and $h_{k,\ell} = H_2(\xi_k, \xi_\ell)$ for $k, \ell = 1, \dots, n$ result from sampling the transfer functions $H_1(s)$ and $H_2(s, z)$ corresponding to LQO system in (2). Then, $r_1(s)$ in (5) interpolates the data h_k , i.e., $r_1(\xi_k) = H_1(\xi_k)$ for $k = 1, 2, \dots, n$. Define, the two-variable function $r_2(s, z)$ in a barycentric-like form:

$$r_2(s, z) = \frac{\sum_{k=1}^n \sum_{\ell=1}^n \frac{h_{k,\ell} w_k w_\ell}{(s - \xi_k)(z - \xi_\ell)}}{1 + \sum_{k=1}^n \frac{w_k}{s - \xi_k} + \sum_{\ell=1}^n \frac{w_\ell}{z - \xi_\ell} + \sum_{k=1}^n \sum_{\ell=1}^n \frac{w_k w_\ell}{(s - \xi_k)(z - \xi_\ell)}}, \quad (8)$$

where $h_{k,\ell} := H_2(\xi_k, \xi_\ell)$. Then, $r_2(s, z)$ interpolates the data $h_{k,\ell}$, i.e., $r_2(\xi_k, \xi_\ell) = h_{k,\ell} = H_2(\xi_k, \xi_\ell)$ for $k, \ell = 1, \dots, n$. Moreover, $r_1(s)$ and $r_2(s, z)$ correspond to a reduced LQO model. In others words, there exist a LQO model as in (2) whose first (linear) transfer function is $r_1(s)$ and second transfer function is $r_2(s, z)$.

The interpolation property stated in Theorem 4.2 can be proven in a similar manner to Theorem 4.1, using polynomials in two variables. We skip those details here.

Note that the barycentric representations for the rational interpolants $r_1(s)$ and $r_2(s, z)$ have common weights given by variables w_k , and by products $w_k w_\ell$, respectively. This indeed shows that the two rational functions are connected to each other. Moreover, Theorem 4.2 reveals how to construct a reduced LQO system directly from transfer function samples. And the corresponding barycentric forms for $r_1(s)$ and $r_2(s, z)$ directly lend themselves to extending AAA to modeling LQO systems. The forms $r_1(s)$ and $r_2(s, z)$ interpolate the data by construction.

As in the original AAA algorithm for linear dynamics, we will produce the approximants iteratively. More precisely, we increase the orders of $r_1(s)$ and $r_2(s, z)$ and will automatically interpolate a subset of the data. Then, the method will choose the free parameters given by the weights w_k in order to minimize the least-squares distance in the remaining data.

REFERENCES

Antoulas, A.C. (2005). *Approximation of large-scale dynamical systems*. SIAM, Philadelphia.
 Antoulas, A.C. and Anderson, B.D.. (1986). On the scalar rational interpolation problem. *IMA Journal of Mathematical Control and Information*, 3(2-3), 61–88.
 Antoulas, A.C., Beattie, C.A., and Gugercin, S. (2020). *Interpolatory Methods for Model Reduction*. Society for Industrial and Applied Mathematics.

Antoulas, A.C., Gosea, I.V., and Ionita, A.C. (2016). Model reduction of bilinear systems in the Loewner framework. *SIAM Journal on Scientific Computing*, 38(5), B889–B916.
 Antoulas, A.C., Lefteriu, S., and Ionita, A.C. (2017). A tutorial introduction to the Loewner framework for model reduction. In *Model Reduction and Approximation*, chapter 8, 335–376. SIAM.
 Astolfi, A. (2010). Model reduction by moment matching for linear and nonlinear systems. *IEEE Transactions on Automatic Control*, 55(10), 2321–2336.
 Baur, U., Benner, P., and Feng, L. (2014). Model order reduction for linear and nonlinear systems: A system-theoretic perspective. *Archives of Computational Methods in Engineering*, 21(4), 331–358.
 Benner, P. and Breiten, T. (2012). Interpolation-based \mathcal{H}_2 -model reduction of bilinear control systems. *SIAM J. Matrix Anal. Appl.*, 33, 859–885.
 Benner, P. and Damm, T. (2011). Lyapunov equations, energy functionals and model order reduction of bilinear and stochastic systems. *SIAM J. on Control and Optimization*, 49, 686–711.
 Benner, P. and Goyal, P. (2017). Balanced truncation model order reduction for quadratic-bilinear control systems. Technical report, arXiv preprint available at <https://arxiv.org/abs/1705.00160>.
 Benner, P., Goyal, P., and Gugercin, S. (2018). \mathcal{H}_2 -quasi-optimal model order reduction for quadratic-bilinear control systems. *SIAM Journal on Matrix Analysis and Applications*, 39(2), 983–1032.
 Benner, P., Goyal, P., and Pontes Duff, I. (2019). Gramians, energy functionals and balanced truncation for linear dynamical systems with quadratic outputs. Technical report, arXiv preprint available at <https://arxiv.org/abs/1909.04597>.
 Benner, P., Ohlberger, M., Cohen, A., and Willcox, K. (2017). *Model Reduction and Approximation*. Society for Industrial and Applied Mathematics, Philadelphia, PA. doi:10.1137/1.9781611974829.
 Carracedo Rodriguez, A. and Gugercin, S. (2020). The p-AAA algorithm for data driven modeling of parametric dynamical systems. Technical report, arXiv preprint available at <https://arxiv.org/abs/2003.06536>.
 Flagg, G. and Gugercin, S. (2015). Multipoint Volterra series interpolation and \mathcal{H}_2 optimal model reduction of bilinear systems. *SIAM Journal on Matrix Analysis and Applications*, 36, 549–579.
 Giri, F. and Bai, E.W. (eds.) (2010). *Block-oriented Non-linear System Identification*. Lecture Notes in Control and Information Sciences. Springer-Verlag, London.
 Gosea, I.V. and Antoulas, A.C. (2018). Data-driven model order reduction of quadratic-bilinear systems. *Numerical Linear Algebra with Applications*, 25(6), e2200.
 Gosea, I.V. and Antoulas, A.C. (2019). A two-sided iterative framework for model reduction of linear systems with quadratic output. In *Proceedings of the 58th Conference on Decision and Control (CDC), December 11-13, Nice, France*.
 Gosea, I.V. and Güttel, S. (2020). Algorithms for the rational approximation of matrix-valued functions. Technical report, arXiv preprint available at <https://arxiv.org/abs/2003.06410>.

- Gustavsen, B. and Semlyen, A. (1999). Rational approximation of frequency domain responses by vector fitting. *IEEE Transactions on Power Delivery*, 14, 1052–1061.
- Ionita, A.C. and Antoulas, A. (2014). Data-driven parametrized model reduction in the loewner framework. *SIAM Journal on Scientific Computing*, 36(3), A984–A1007.
- Juditsky, A., Hjalmarsson, H., Benveniste, A., Delyon, B., Ljung, L., Sjöberg, J., and Zhang, Q. (1995). Nonlinear black-box models in system identification: Mathematical foundations. *Automatica*, 31(12), 1725 – 1750.
- Kawano, Y. and Scherpen, J.M.A. (2007). Model reduction by differential balancing based on nonlinear Hankel operators. *IEEE Transactions on Automatic Control*, 62(7), 3293–3308.
- Kramer, B. and Willcox, K. (2019). Balanced truncation model reduction for lifted nonlinear systems. Technical report, arXiv preprint available at <https://arxiv.org/abs/1907.12084>.
- Lietaert, P., Pérez, J., Vandereycken, B., and Meerbergen, K. (2018). Automatic rational approximation and linearization of nonlinear eigenvalue problems. Technical report, arXiv preprint available at <https://arxiv.org/abs/1801.08622>.
- Mayo, A. and Antoulas, A. (2007). A framework for the solution of the generalized realization problem. *Linear Algebra and Its Applications*, 425(2-3), 634–662.
- Nakatsukasa, Y., Sete, O., and Trefethen, L.N. (2018). The AAA algorithm for rational approximation. *SIAM Journal on Scientific Computing*, 40(3), A1494–A1522.
- Pulch, R. and Narayan, A. (2019). Balanced truncation for model order reduction of linear dynamical systems with quadratic outputs. *SIAM Journal on Scientific Computing*, 41(4), A2270–A2295.
- Quarteroni, A., Manzoni, A., and Negri, F. (2015). *Reduced basis methods for partial differential equations: an introduction*, volume 92. Springer.
- Van Beeumen, R. and Meerbergen, K. (2010). *Model reduction by balanced truncation of linear systems with a quadratic output*, 2033–2036. T.E. Simons, G. Psihoyios, Ch. Tsitouras (eds.), International Conference on Numerical Analysis and Applied Mathematics (ICNAAM). American Institute of Physics.
- Van Beeumen, R., Van Nimmen, K., Lombaert, G., and Meerbergen, K. (2012). Model reduction for dynamical systems with quadratic output. *Int. J. Numer. Meth. Engng.*, 91, 229–248.