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Some Insights on the Asymptotic Stabilization of a Class of SISO Marginally Stable Systems Using One Delay Block

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Abstract: In this paper, the stability of a class of marginally stable SISO systems is studied by applying a single one delay block as a feedback controller. More precisely, we consider an open-loop system with no zeros and whose poles are located exactly on the imaginary axis. Furthermore, a control law formed uniquely by a proportional gain and a delayed behavior is proposed for its closed-loop stabilization. The main ideas are based on a detailed analysis of the characteristic quasi-polynomial of the closed-loop system as the controller parameters (gain, delay) are varied. More precisely, by using the Mikhailov stability criterion, for a fixed delay value, we compute some gain margin guaranteeing the closed-loop stability. The particular case when the characteristic roots of the open-loop system are equidistantly distributed on the imaginary axis is also addressed. Finally, an illustrative example shows the effectiveness of the approach.

Keywords: LTI Systems, SISO systems, Delay, Mikhailov criterion, Gain margin.

1. INTRODUCTION

It is well recognized that low-order controllers are one of the most widely applied strategies to control industrial processes (see, e.g., Aström and Hägglund (2001); O'Dwyer (2009)). Probably, the most important reasons for such a “popularity” are the simplicity and the ease of implementation. Among these control schemes, PID-type controllers are known to be able to overcome parametric uncertainties and disturbances, also to achieve elimination of steady-state errors and transient response manipulation (Aström and Hägglund (1995); Méndez-Barrios et al. (2008); Ramírez et al. (2016)). Nonetheless, as reported in Aström and Hägglund (1995), there are some scenarios in which its implementation may encounter some difficulties. Probably the most common drawback lies in the tuning of the derivative term, which may amplify high-frequency noise in the control loop. In fact, as mentioned by Aström and Hägglund (2001); O'Dwyer (2009) the above arguments suggest to avoid the derivative action in most applications.

In order to overcome such a problem, the Euler approximation of the derivative:

$$y'(t) \approx \frac{y(t) - y(t-\epsilon)}{\epsilon},$$

for small $\epsilon > 0$, seems to be the simplest way to replace the derivative action by using its delay-difference approximation counterpart (Niculescu and Michiels (2004)). However, on one hand, it is important to point out that the presence of a delay in the feedback loop of continuous-time systems may induce, in some cases, oscillations, instability and bandwidth sensitivity (see for instance Niculescu (2001); Michiels and Niculescu (2014))). On the other hand, it has also been reported that there exist situations where an appropriate selection of the delay parameter may improve the behavior of the corresponding dynamical system (see for instance, Abdallah et al. (1993); Chen (1987); Sipahi et al. (2011)). Inspired by the above observations, the design of low-order controllers with delay as a *control parameter* has been addressed in several works and represent an interesting idea not sufficiently exploited in the literature (for further insights, see, e.g. Sipahi et al. (2011)).

In particular, one may cite two classical problems - the *stabilization* of *chains of integrators* and *oscillators* controlled by using delays in the feedback laws. For instance, in the case of the chain of integrators, such a problem has been studied in Niculescu and Michiels (2004) [1] and Mazenc et al. (2003) [2]. Next, the idea that inducing a delay in the control feedback may improve the stability was discussed in Abdallah et al. (1993) and Niculescu et al. (2010), where one oscillator is stabilized by using only

[1] controller represented by some chains of delay blocks
[2] considering some bounded and delayed input

one delay "block": (gain, delay) $(K, \tau) \in \mathbb{R} \times \mathbb{R}_+$. Such a controller may find some applications in robotics Abdallah et al. (1991) and flexible structures Robinett et al. (1998).

In this context, it is worthy to recall that, surprisingly, one delay block may stabilize a whole chain of oscillators Kharitonov et al. (2005). In fact, the authors of Kharitonov et al. (2005) developed appropriate *sufficient conditions* guaranteeing the stability of the chain of oscillators in closed-loop. More precisely, they have proved that if some conditions on the oscillators' frequencies and the delay are verified, such a controller may exist, but without any attempt to explicitly construct the corresponding delay "block": (gain, delay).

Inspired by these results, the proposed work further investigates explicit conditions on the delay block parameters guaranteeing the stabilization of the chain of oscillators by using the so-called Mikhailov criterion [3]. More precisely, for a fixed delay value τ_s, we compute the corresponding (positive) *gain margin* K_s, guaranteeing the closed-loop stability for all gain values $K \in (0, K_s)$.

The remaining of the paper is organized as follows: The problem formulation is stated in Section 2. Section 3 includes some discussions on the critical characteristic roots of the closed-loop system and the root crossing behavior with respect to the gain parameter. Next, Section 4 is devoted to the main results as well as to some illustrative example in the case characteristic roots equidistantly distributed on the imaginary axis. Finally, some concluding remarks end the paper.

2. PROBLEM FORMULATION

Consider the class of strictly proper single-input single-output (SISO) marginally stable open-loop systems given by the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Q(s)}, \quad Q(s) := \prod_{m=1}^{N} (s^2 + \omega_m^2), \qquad (1)$$

where $0 < \omega_1 < \omega_2 < \cdots < \omega_N$. In other words, (1) simply represents a *chain of oscillators*.

Consider the transfer function (1) and the one delay block controller:

$$C(s) = -Ke^{-\tau s}, \qquad (2)$$

where $K \in \mathbb{R}$ and τ is a positive fixed delay value. We aim to *find explicit conditions on the controller parameters pair* (K, τ) *such that the closed-loop system is asymptotically stable.* In other words, to find the set of parameters (K, τ) such that the characteristic function Δ of the closed-loop system:

$$\Delta(s) = Ke^{-\tau s} + Q(s), \qquad (3)$$

has all of its zeros on the left-half plane (LHP) of the complex plane.

3. CRITICAL CHARACTERISTIC ROOTS AND CROSSING ROOTS ANALYSIS

In the sequel, we focus on deriving the particular controller parametric settings such that at least one root of the characteristic quasi-polynomial (3) is located on the imaginary axis and, subsequently, its behavior with respect to the change of parameters. Such a root will be called *critical* or *crossing characteristic root.*

For the sake of brevity, we make the *assumption* that the crossing characteristic roots on the imaginary axis are *simple* (see, for instance, Michiels and Niculescu (2014) and the references therein). Since we are interested in constructing explicitly some gain intervals guaranteeing the stability in closed-loop such an assumption is not necessarily restrictive.

Finally, it is important to remark that, for all real ω, $Q(i\omega)$ is a real-valued function and $Q'(i\omega)$ is a purely imaginary valued function. The necessary computations supporting such an argument are left to the reader.

3.1 Stability Crossing Frequencies

Consider $s = i\omega$ in (3) and solve for K as follows:

$$K = -Q(i\omega)\Big[\cos(\tau\omega) + i\sin(\tau\omega)\Big].$$

Some simple computations show that there exists a real solution of K for this last expression only in two cases:

- first, if $\omega = \omega_m$ then $Q(i\omega) = 0$, and therefore $K = 0$ [4];
- second, given that $Q(i\omega)$ is a real-valued function of ω, a real solution of $K \neq 0$ exists iff $\omega = \tilde{\omega}_n(\tau) := n\frac{\pi}{\tau}$ for some $n \in \mathbb{Z}$ [5].

To summarize, we have the following result:

Proposition 1. Let τ be some fixed positive delay value. Then the characteristic function of the closed-loop system (3) has at least one critical characteristic root $s = i\omega$, iff:

- $K = 0$, being $\omega = \omega_m$ for any $m \in \{1, 2, \ldots, N\}$,
- $K = K(\tau, n)$, being $\omega = n\frac{\pi}{\tau}$ for some $n \in \mathbb{Z}$,

where:

$$K(\tau, n) = (-1)^{n+1} Q\left(in\frac{\pi}{\tau}\right).$$

Remark 1. As discussed in the literature (see, for instance, Niculescu (2001) and the references therein), the roots of a quasi-polynomial move continuously against continuous variation of its parameters (gain, coefficients). Let $\tau > 0$ be a fixed delay value, and since $K = 0$ or $K = K(\tau, n)$ implies root crossing, then, these values define an appropriate partition of the real K-axis in several intervals having a constant number of unstable characteristic roots inside each interval. If the number of unstable characteristic roots is 0, then the corresponding closed-loop system is asymptotically stable.

3.2 Stability Crossing Directions

We focus now on the characterization of the crossing roots deviation tendency of (3) under the assumption that the crossing characteristic roots on the imaginary axis are

[3] a criterion derived by using the well-known Cauchy's argument principle

[4] The frequencies ω_m correspond to the open-loop (system) crossing characteristic roots $i\omega_m$ located on the imaginary axis.

[5] The frequencies $\tilde{\omega}_n(\tau)$ correspond to the closed-loop (system) crossing characteristic roots $i\tilde{\omega}_n(\tau)$ located on the imaginary axis.

simple. In this case, a direct application of the implicit function theorem leads to:

$$\left[\frac{ds}{dK}\right]^{-1} = \tau K - Q'(s)e^{\tau s}. \quad (4)$$

We have the following result:

Proposition 2. Let τ be a fixed delay value and $n \in Z$. If K increases (decreases) around $K = K(\tau, n)$, then, at least one pair of critical characteristic roots moves to the RHP (LHP) of the complex plane iff $K(\tau, n) > 0 (< 0)$.

Proof 1. Consider the closed-loop crossing characteristic roots $s = i\tilde{\omega}_n(\tau)$ and let $K = K(\tau, n)$. Since $e^{i\tau\tilde{\omega}_n(\tau)} = (-1)^n$, one gets:

$$\left.\left[\frac{ds}{dK}\right]^{-1}\right|_{s=i\tilde{\omega}_n} = \tau K(\tau, n) + (-1)^{n+1}Q'(i\tilde{\omega}_n).$$

Recall that $Q'(i\omega)$ is a purely complex function of ω. Then its crossing direction is given by:

$$\tilde{R}(\tau, n) := \Re\left\{\left.\left[\frac{ds}{dK}\right]^{-1}\right|_{s=i\tilde{\omega}_n}\right\} = \tau K(\tau, n).$$

Since $\tau > 0$, then $\operatorname{sgn}\left\{\tilde{R}(\tau, n)\right\} = \operatorname{sgn}\{K(\tau, n)\}$ and the conclusions of the Proposition follow straightforwardly.

For fixed values of τ and n, compute all values of $K(\tau, n)$, arranged in such a way that: $\ldots < K_2^- < K_1^- < 0 < K_1^+ < K_2^+ < \ldots$, where the super-index is used to observe the sign of such values on the real K-axis. As mentioned in Remark 1, these values partition the real K-axis in intervals in which the characteristic equation has a constant number of roots located on the RHP of the complex plane. Let us focus now on some small neighborhood of $K = 0$ and we will analyze how the characteristic roots behave for positive, but sufficiently small K. More precisely, we have the following result:

Proposition 3. Let τ be a fixed delay value and $m \in \{1, 2, \ldots, N\}$. If K increases from $K = 0$, then, the pair of open-loop crossing roots $s = \pm i\omega_m$ move to the LHP (RHP) of the complex plane iff:

$$(-1)^{m-1}\sin(\tau\omega_m) < 0 \quad (> 0).$$

Proof 2. Consider the open-loop crossing roots, that is $s = i\omega_m$ when $K = 0$. By using (4), it follows that:

$$\left.\left[\frac{ds}{dK}\right]^{-1}\right|_{s=i\omega_m} = -Q'(i\omega_m)\Big[\cos(\tau\omega_m) + i\sin(\tau\omega_m)\Big].$$

Since $Q'(i\omega)$ is a purely complex function, then its crossing direction is given by:

$$R_m := \Re\left\{\left.\left[\frac{ds}{dK}\right]^{-1}\right|_{s=i\omega_m}\right\} = -iQ(i\omega_m)\sin(\tau\omega_m),$$

that can be rewritten as:

$$R_m = 2\omega_m\sin(\tau\omega_m)\prod_{\ell\neq m}(\omega_\ell^2 - \omega_m^2),$$

where $\ell \in \{1, 2, \ldots, N\}$. Since:

$$\operatorname{sgn}\left\{\prod_{\ell\neq m}(\omega_\ell^2 - \omega_m^2)\right\} = (-1)^{m-1},$$

the sign of R_m can be expressed as:

$$\operatorname{sgn}\{R_m\} = (-1)^{m-1}\operatorname{sgn}\{\sin(\tau\omega_m)\},$$

this last expression concludes the proof.

Bearing in mind Proposition 3 shown above, it is clear that if there exists a fixed value τ such that:

$$(-1)^{m-1}\sin(\tau\omega_m) < 0, \quad \forall m \in \{1, 2, \ldots, N\}, \quad (5)$$

then, all open-loop crossing roots cross to the LHP of the complex plane as K is varied increasingly from zero. Such conditions may imply closed-loop asymptotic stability for some positive values of the gain K. It is worthy to mention that since the parameter τ is assumed to be chosen in advance, then we drop its notation on the closed-roots crossing frequencies $\tilde{\omega}_n(\tau)$.

4. VECTOR INTERPRETATION AND MAIN RESULTS

In this section, we develop a stability analysis based on the Mikhailov criterion (see Appendix A) for the characteristic quasi-polynomial (3). As shown in Fig. 1a, the vector interpretation of $\Delta(i\omega)$ consists in the addition of a purely real vector $Q(i\omega)$ with the complex vector $Ke^{-i\tau\omega}$.

On one hand, $Ke^{-i\tau\omega}$ is a rotatory vector describing a path on a circle with radius K centered in $Q(i\omega)$ with a clockwise direction. More precisely, let $n \in \mathbb{Z}$ and $K > 0$. As illustrated in Fig. 1d, $Ke^{-i\tau\omega}$ has the following behavior:

- It stays on the real axis with positive direction for any value $\omega = \tilde{\omega}_{2n}$ (even closed-loop cross frequencies).
- It moves in a clockwise direction through the lower half-plane of the complex plane for $\omega \in (\tilde{\omega}_{2n}, \tilde{\omega}_{2n+1})$.
- It stays on the real axis with negative direction for any value $\omega = \tilde{\omega}_{2n+1}$ (odd closed-loop cross frequencies).
- It moves in a clockwise direction through the upper half plane of the complex plane for $\omega \in (\tilde{\omega}_{2n+1}, \tilde{\omega}_{2n+2})$.

On the other hand, since $Q(i\omega)$ is a polynomial on ω of degree $2N$ which changes sign as ω varies through the open-loop crossing frequencies ω_m. More precisely, let $\omega_0 := 0$, if $\omega \in (\omega_m, \omega_{m+1})$, then $\operatorname{sgn}\{Q(i\omega)\} = (-1)^m$ and let $\omega \geq \omega_N$ then $\operatorname{sgn}\{Q(i\omega)\} = (-1)^N$.

4.1 General Case

In this section, we establish stabilizing conditions on the gain K such that τ satisfies the conditions (5). In other words, such a value of τ implies that the following inequalities hold simultaneously:

$$\sin(\tau\omega_m) < 0, \text{ for odd } m, \quad \sin(\tau\omega_m) > 0, \text{ for even } m.$$

Subsequently, there exist natural numbers n_m odd (even) if m is odd (even) such that:

$$n_m\pi < \tau\omega_m < (n_m + 1)\pi, \quad n_m = \left\lfloor \omega_m\frac{\tau}{\pi}\right\rfloor,$$

straightforwardly, it is clear that:

$$n_m\frac{\pi}{\tau} < \omega_m < (n_m + 1)\frac{\pi}{\tau} \to \tilde{\omega}_{n_m} < \omega_m < \tilde{\omega}_{n_m+1}. \quad (6)$$

It is worth mentioning that this last argument shows a particular interlacing condition between the open- and closed-loop crossing roots frequencies. Such an observation is used in the proof of the following Proposition. Next, note that (6) can be rewritten as follows:

$$n_m < \omega_m \frac{\tau}{\pi} < (n_m + 1).$$

Then, by means of the definition of the floor and ceiling functions, it is clear that the computation of such upper/lower integer bounds can be done as:

$$n_m = \left\lfloor \omega_m \frac{\tau}{\pi} \right\rfloor, \quad n_m + 1 = \left\lceil \omega_m \frac{\tau}{\pi} \right\rceil.$$

Proposition 4. Consider the open-loop system (1) with N distinct simple roots on the imaginary axis and the one delay block controller (2). Let $\tau = \tau_s > 0$ be a fixed delay value such that:

$$(-1)^{m-1} \sin(\tau_s \omega_m) < 0, \quad \forall m \in \{1, 2, \dots, N\}, \tag{7}$$

and K be a positive real gain. Then, the closed-loop system is asymptotically stable if:

$$K \in (0, K_s).$$

where the gain margin K_s is given by:

$$K_s := \min_{m \in \{1,2,\dots,N\}} \left\{ \left| Q\left(i \left\lfloor \omega_m \frac{\tau}{\pi} \right\rfloor \frac{\pi}{\tau} \right) \right|, \left| Q\left(i \left\lceil \omega_m \frac{\tau}{\pi} \right\rceil \frac{\pi}{\tau} \right) \right| \right\}.$$

Proof 3. The proof is based in the geometric vector analysis schematically presented in Fig. 1 by means of the Mikhailov criterion. As mentioned in Appendix A, this result establishes that the asymptotic stability is achieved iff the complex vector $\Delta(i\omega)$ has an accumulative phase of $N\pi$ as $\omega \in [0, \infty)$. Assume that $K < |Q(i\tilde{\omega}_n)|$ for all $n \in \mathbb{Z}$.

Consider now the intervals $\omega \in (\omega_m, \omega_{m+1})$ for any $m \in \{0, 1, \dots, N-1\}$. Recall that $\omega_0 := 0$. First, consider m even, then $\operatorname{sgn}\{Q(i\omega)\} = 1$ and $Q(i\omega)$ is a positive real vector. As illustrated in Fig. 1b, since $Ke^{-i\tau\omega}$ is a rotatory vector with a clockwise direction, $\Delta(i\omega)$ has a tendency to encircle the origin with a clockwise direction (negative accumulated phase). More precisely, it achieves it if $K > |Q(i\tilde{\omega}_n)|$ for some odd n. Second, consider m odd, then $\operatorname{sgn}\{Q(i\omega)\} = -1$ and $Q(i\omega)$ is a negative real vector. Similarly, as illustrated in Fig. 1c, $\Delta(i\omega)$ encircles the origin with a clockwise direction (negative accumulated phase) if $K > |Q(i\tilde{\omega}_n)|$ for some even n. Finally, consider $\omega \in (\omega_N, \infty)$ then $\operatorname{sgn}\{Q(i\omega)\} = (-1^N)$, depending on the number N this match any of the two scenarios presented above. In general, since $K < |Q(i\tilde{\omega}_n)|$ for any n then $\Delta(i\omega)$ does not encircle the origin if $\omega \in (\omega_m, \omega_{m+1})$ for $m \in \{0, 1, \dots, N-1\}$ or $\omega \in (\omega_N, \infty)$.

We now study the behavior of $\Delta(i\omega)$ as ω varies through the open-loop crossing frequencies ω_m. To this end, we analyze the intervals $\omega \in [\tilde{\omega}_{n_m}, \tilde{\omega}_{n_m+1}]$. First ,consider m odd, for this case n_m is an odd number and, as illustrated in Fig. 1c, $Q(i\omega)$ changes its sign from positive to negative and $Ke^{-i\tau\omega}$ rotates in a clockwise direction through the upper half-plane. Since $K < |Q(i\tilde{\omega}_{n_m})|$ and $K < |Q(i\tilde{\omega}_{n_m+1})|$, this behavior starts in the positive real axis and traverses the upper half-plane describing a path accumulating its phase in π radians. Second, consider m even, for this case n_m is an even number and as illustrated in Fig. 1f, $Q(i\omega)$ changes sign from negative to positive and $Ke^{-i\tau\omega}$ rotates in a clockwise direction through the lower half-plane. Similarly, this behavior starts in the negative real axis and traverses the lower half-plane describing a path accumulating its phase in π radians. In general, we can state that:

$$\underset{\omega \in [\tilde{\omega}_{n_m}, \tilde{\omega}_{n_m+1}]}{\theta \arg \Delta(i\omega)} = \pi,$$

and bearing in mind the above observations, then:

$$\underset{\omega \in [0,\infty)}{\theta \arg \Delta(i\omega)} = \sum_{m=1}^{N} \left[\underset{\omega \in [\tilde{\omega}_{n_m}, \tilde{\omega}_{n_m+1}]}{\theta \arg \Delta(i\omega)} \right] = N\pi,$$

if $K < |Q(i\tilde{\omega}_n)|$ for all $n \in \mathbb{N} \cup \{0\}$, which implies asymptotic stability according to the Mikhailov Theorem.

4.2 Equidistant Distribution

Consider the particular case of an equidistant distribution of the open-loop crossing roots (located on the imaginary axis). More precisely, such a case implies that $\omega_m = m\omega_b$ for some base frequency $\omega_b \in \mathbb{R}$ and for all $m \in \{1, 2, \dots, m\}$.

We have the following result:

Corollary 1. Consider the open-loop system (1) with N distinct single roots on the imaginary axis such that $\omega_m = m\omega_b$, where $\omega_b > 0$ and for all $m \in \{1, 2, \dots, m\}$. Let $\tau = \tau_s$ be a fixed delay value:

$$\tau_s \in \left(\frac{\pi}{\omega_b}, \frac{N+1}{N} \frac{\pi}{\omega_b} \right),$$

for some $j \in \mathbb{N}$, and K be a positive real gain. Then, the chain of oscillators can be stabilized by one delay block controller (2) for all gains K, where

$$K \in (0, K_s).$$

and the gain margin K_s is given by:

$$K_s := \min_{m \in \{1,2,\dots,N\}} \left\{ \left| Q\left(im \frac{\pi}{\tau_s} \right) \right|, \left| Q\left(i\,(m+1) \frac{\pi}{\tau_s} \right) \right| \right\}.$$

Proof 4. The proof of this result makes use of Proposition 4. First, in order to construct a solution τ for conditions (7) we propose the following distribution of the values $\omega_m = m\omega_b$:

$$m\pi < \tau m\omega_b < (m+1)\pi, \quad \forall m \in \{1, 2, \dots, N\}$$

It is clear that a solution τ for these inequalities can be computed as:

$$\frac{\pi}{\omega_b} < \tau < \frac{m+1}{m} \frac{\pi}{\omega_b}, \quad \forall m \in \{1, 2, \dots, N\}, \tag{8}$$

since $m \leq N$ it follows directly:

$$\frac{1}{m} \geq \frac{1}{N} \to 1 + \frac{1}{m} \geq 1 + \frac{1}{N} \to \frac{m+1}{m} \geq \frac{N+1}{N},$$

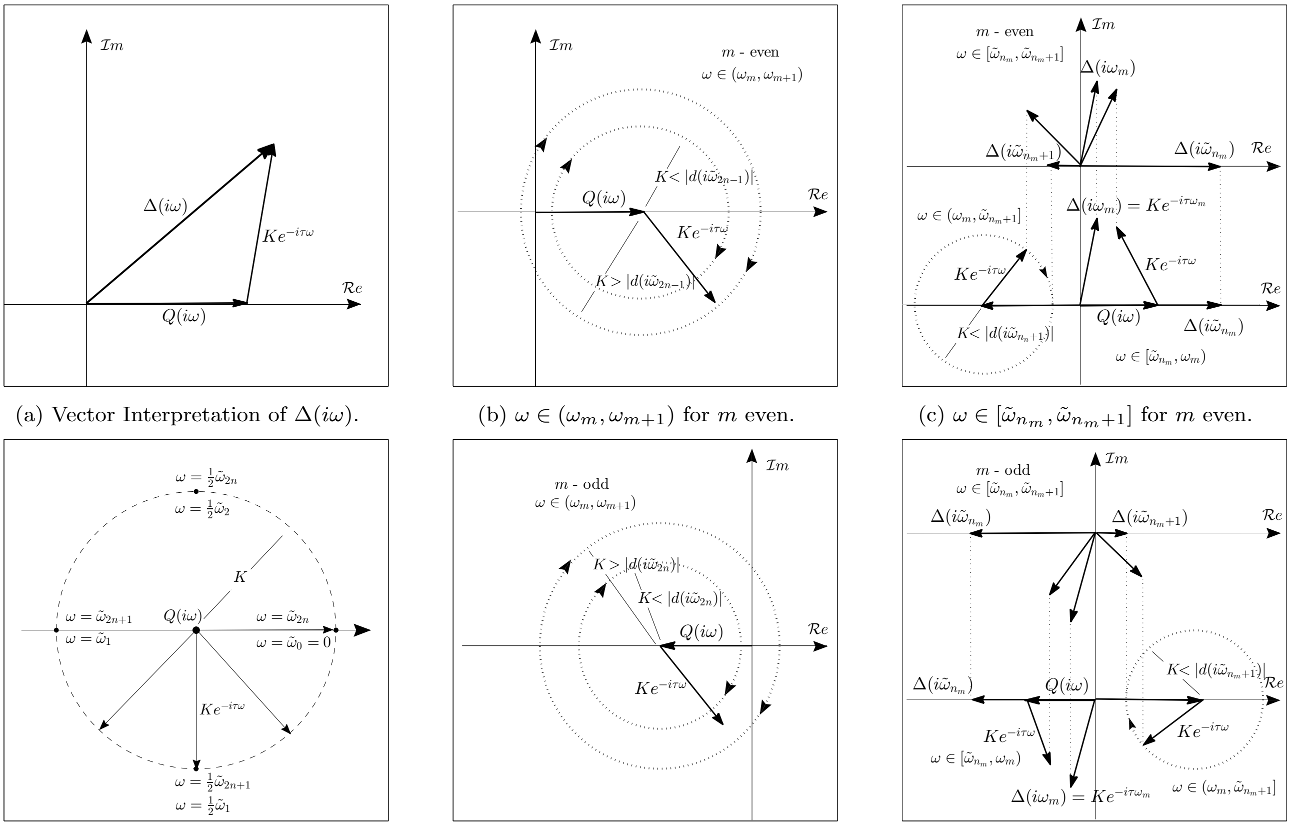
then, the intersection of all intervals solving (8) can be computed explicitly as:

$$\frac{\pi}{\omega_b} < \tau < \frac{N+1}{N} \frac{\pi}{\omega_b}.$$

The following step is to compute the proper stabilizing value of K defined by Proposition 4. Since it is evident that:

$$\left\lfloor \omega_m \frac{\tau}{\pi} \right\rfloor = \left\lfloor \frac{m\omega_b \tau}{\pi} \right\rfloor = m,$$
$$\left\lceil \omega_m \frac{\tau}{\pi} \right\rceil = \left\lceil \frac{m\omega_b \tau}{\pi} \right\rceil = m + 1,$$

then, accordingly to Proposition 4, the conclusion follows straightforwardly.



(a) Vector Interpretation of $\Delta(i\omega)$. (b) $\omega \in (\omega_m, \omega_{m+1})$ for m even. (c) $\omega \in [\tilde{\omega}_{n_m}, \tilde{\omega}_{n_m+1}]$ for m even.

(d) Vector Interpretation of $Ke^{-\tau\omega}$. (e) $\omega \in (\omega_m, \omega_{m+1})$ for m odd. (f) $\omega \in [\tilde{\omega}_{n_m}, \tilde{\omega}_{n_m+1}]$ for m odd.

Fig. 1. Accumulative argument analysis of the complex vector $\Delta(i\omega)$.

Remark 2. It is worth mentioning that the results in Corollary 1 concern the construction of the gain K_s if τ_s belongs to some predefined delay interval not including the origin (in fact, the first one). The results above can be extended to cover other delay intervals of the form:

$$\tau_s \in \left((2\ell - 1)\frac{\pi}{\omega_b}, \frac{(2\ell-1)N+1}{N}\frac{\pi}{\omega_b} \right),$$

for some $\ell \in \mathbb{N}$ such that the interval above is well-defined. Then, for appropriate positive integers ℓ, the chain of oscillators can be stabilized by one delay block controller (2) for all gains K, where

$$K \in (0, K_s).$$

and the gain margin K_s is given by:

$$K_s := \min_{m\in\{1,2,\ldots,N\}} \left\{ \left| Q\left(i(2\ell-1)m\frac{\pi}{\tau_s} \right) \right|, \right.$$
$$\left. \left| Q\left(i\left((2\ell-1)m+1\right)\frac{\pi}{\tau_s} \right) \right| \right\}.$$

4.3 Illustrative Example - Equidistant Distribution

In this section, we present an illustrative example of an equidistant distribution of the open-loop crossing roots.

For the simplicity, consider $\omega_b = \pi$ and $N = 3$. In other words, a sixth order open-loop system with poles located exactly in $s = \pm i\pi, \pm i2\pi, \pm i3\pi$. Using the results above and the Remark 2, we construct the stability regions shown in Fig 2a, particularly for $\ell = 1, 2, 3, 4$. Let us take a look at the case $\ell = 2$ correspondent to the stability region shown in Fig. 2b. According to Corollary 1, its stabilizing interval of τ is directly computed as $\tau_s \in (3, \frac{3N+1}{N}) = (3, \frac{10}{3})$.

Consider, for instance, $\tau = 3.31$ a value of the interval $(3, 3.333)$. Then, the corresponding gain margin $K_s = 0.446$ defines the stabilizing gain interval $(0, 4460)$. We test this scenario by using DDE-BIFTOOL, a Matlab package for bifurcation analysis of delay-differential equations (se for instance, Engelborhs et al. (2002)).The results presented in Fig. 2c illustrate the way the characteristic roots move as the controller parameters are varied.

5. CONCLUDING REMARKS

To summarize,this paper presented some insights in the computation of the parameters of one delay block (gain, delay) able stabilizes a chain of oscillators. More precisely, for a fixed delay value, and for a positive gain, we computed some lower bounds of the gain margin, such that the closed-loop system is asymptotically stable. Such an idea was further exploited in the case when the characteristic roots of the open-loop system are equidistant on the imaginary case. In this case, the method allowed to compute several delay intervals for which the closed-loop stability can be guaranteed as well as the corresponding gain intervals. This last result was illustrated by an example.

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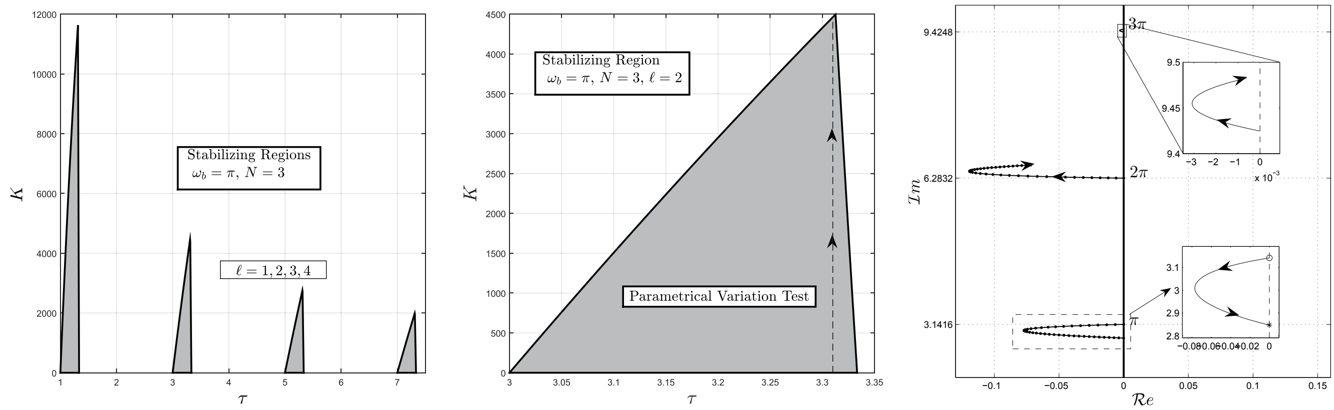

(a) Stability Regions (b) Parametrical Variation Test (c) Roots Behavior.

Fig. 2. Illustrative example.

Appendix A. MIKHAILOV STABILITY CRITERION

Theorem 1. (Mikhailov Stability Criterion, Pekar et al. (2010); Gorecki et al. (1989)) Consider the retarded quasi-polynomial with single delay:

$$\Delta_r(s) = P(s) + Q(s)e^{-\tau s},$$

where $\tau > 0$ and $k := \deg\{P(s)\} > \deg\{Q(s)\}$. The characteristic quasi-polynomial has all of its zeros located on the LHP of the complex plane) iff:

$$\underset{\omega\in[0,\infty)}{\theta \arg} \{\Delta_r(i\omega)\} = k\frac{\pi}{2}.$$

Remark 3. Consider the particular quasi-polynomial (3) which corresponds to the closed-loop system of a chain of oscillators subject to one delay block. It is evident that Theorem 1 implies that the asymptotic stability is achieved iff:

$$\underset{\omega\in[0,\infty)}{\theta \arg} \{\Delta(i\omega)\} = N\pi.$$

REFERENCES

Abdallah, C., Dawson, D., Dorato, P., and Jamshidi, M. (1991). Survey of robust control for rigid robots. *IEEE Control Systems Magazine*, 11, 24–30.

Abdallah, C., Dorato, P., Benitez-Read, J., and Byrne, R. (1993). Delayed Positive Feedback can Stabilize Oscillatory Systems. In *Proc. American Contr. Conf.*, 3106–3107.

Aström, K.J. and Hägglund, T. (1995). *PID Controllers: Theory, Design, and Tuning.* Instrument Society of America, Research Triangle Park, NC, 2 edition.

Aström, K.J. and Hägglund, T. (2001). The Future of PID Control. *Chem. Eng. Progress*, 9(11), 1163–1175.

Chen, Y.H. (1987). New type of controller: the proportional integral minusdelay controller. *International Journal of Systems Science*, 18(11), 2033–2041.

Engelborhs, K., Luzyanina, T., and Roose, D. (2002). Numerical bifurcation analysis of delay differential equations using dde-biftool. *ACM Transactions on Mathematical Software*, 28(1), 1–21.

Gorecki, H., Fuksa, S. Grabowsky, P., and A., K. (1989). *Analysis and Synthesis of Time-Delay Systems.* New York: John Wiley & Sons.

Kharitonov, V., Niculescu, S.I., Moreno, J., and Michiels, W. (2005). Static output feedback stabilization: Necessary conditions for multiple delay controllers. *IEEE Transactions On Automatic Control*, 50(1), 82–86.

Mazenc, F., Mondié, S., and Niculescu, S.I. (2003). Global asymptotic stabilization for chains of integrators with a delay in the input. *IEEE Transactions on Automatic Control*, 48(1), 57–63.

Méndez-Barrios, C.F., Niculescu, S.I., Morărescu, I.C., and Gu, K. (2008). On the Fragility of PI Controllers for Time-Delay SISO Systems. In *16th Mediterranean Conference on Control and Automation*, 529–534. IEEE, Ajaccio, France.

Michiels, W. and Niculescu, S.I. (2014). *Stability, Control, and Computation for Time-Delay Systems. An Eigenvalue-Based Approach.* SIAM, Philadelphia.

Niculescu, S.I. (2001). *Delay Effects on Stability: A Robust Control Approach.* Heidelberg: Springer.

Niculescu, S.I. and Michiels, W. (2004). Stabilizing a chain of integrators using multiple delays. *IEEE Trans. Aut. Control*, 49(5), 802–807.

Niculescu, S.I., Michiels, W., Gu, K., and Abdallah, C. (2010). Delay Effects on Output Feedback Control of Dynamical Systems. In F. Atay (ed.), *Complex Time-Delay Systems*, 63–84. Berlin: Springer-Verlag.

O'Dwyer, A. (2009). *Handbook of PI and PID Controller Tuning Rules.* Imperial College Press (ICP), London, 3rd edition.

Pekar, L., R., P., and Matusu, R. (2010). Stability conditions for a retarded quasipolynomial and their applications. *International Journal of Mathematics and Computers in Simulation.*

Ramírez, A., Mondié, S., Garrido, R., and Sipahi, R. (2016). Design of Proportional-Integral-Retarded (PIR) Controllers for Second-Order LTI Systems. *IEEE Transactions On Automatic Control*, 61(6), 1688–1693.

Robinett, R.D., Petterson, B.J., and Farenholtz, C. (1998). Lag-stabilized force feedback damping. *Journal of Intelligent and Robotic Systems*, (21), 277–285.

Sipahi, R., Niculescu, S.I., Abdallah, C.T., Michiels, W., and Gu, K. (2011). Stability and stabilization of systems with time delay. *IEEE Control Systems Magazine*, 31(1), 38–65.