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On imitation dynamics for potential population games over networks with community patterns ⋆

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Abstract: We study the asymptotic behavior of deterministic, continuous-time imitation dynamics for potential population games. In these dynamics, which is a general class of learning protocols that encompasses the replicator equation, players exchange information through pairwise interactions, whereby getting aware of the actions played by the others and the corresponding rewards. The pattern of interactions that regulates the learning process is determined by a community structure. We characterize the set of equilibrium points of the dynamics. Then, for the class of potential games and community networks that are undirected and connected, we prove global asymptotic convergence to the set of Nash equilibria of game.

Keywords: Nonlinear systems; Learning; Game theory

1. INTRODUCTION

Imitation dynamics are a game-theoretical learning paradigm that has been used to model and study the evolution of behaviors and strategies in social, economic, and biological network systems Weibull (1995); Hofbauer and Sigmund (2003); Govaert et al. (2017). Their success relies in their simplicity, whereby players need minimal information of the structure of the game: it is only required that they can observe the action they are currently playing and the corresponding reward. Players interact on a communication network with fellow players, exchanging information on the action played by them and on the associated reward. Then players may use the information obtained to revise their action, adopting the one of the contacted fellow players.

We focus on the asymptotic behavior of such imitation dynamics in the scenario of a communication network characterized by the presence of a community structure that governs the pattern of interactions. Imitation dynamics have been widely studied in the literature Nachbar (1990); Hofbauer (2000); Sandholm (2001, 2010). In particular, Sandholm (2010) offers an extensive study of local stability and instability for the different kinds of rest points of such dynamics. These results, however, deal only with local stability, therefore one can not conclude global asymptotic stability. Indeed, only for specific dynamics, such as the replicator equation, and for some specific classes of games, a global analysis has been carried on. See, for instance, Bomze (2002); Shamma and Arslan (2005); Fox and Shamma (2012); Cressman and Tao (2014); Barreiro-Gomez et al. (2016).

Here, we extend these results by studying the asymptotic behavior of imitation dynamics for the important class of potential population games. Some preliminary results in this direction can be found in Zino et al. (2017). Therein, global stability of Nash equilibria has been proved for the fully-mixed community-free scenario. Here, we expand that preliminary analysis by considering a general network of interaction driven by a community structure. The presence of a non-fully-mixed network of interactions poses several technical issues. In fact, we show through a set of examples that convergence to the set of Nash equilibria is in general not guaranteed.

Our asymptotic analysis focuses on two main contributions. First, we provide a general characterization of the equilibrium points, shedding light on the role of the reward function and on the network architecture on determining the asymptotic behavior of the system. Specifically, in the case of connected, undirected networks, we shows that the set of equilibria is independent of the network topology. Second, we specialize our analysis to the class of potential games, founding sufficient conditions for the network structure to guarantee global convergence to Nash equilibria. Specifically, through a Lyapunov-based argument, we study the stability of the equilibrium points for undirected and connected networks, proving global convergence of the imitation dynamics to the set of Nash equilibria. We further present a simple example of a nonsymmetric network of interactions to exhibit the emergence of new rest points for the imitation dynamics which are not Nash equilibria.

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2. PROBLEM STATEMENT

We study imitation dynamics in continuous population games. In such setting, a continuum of players of unitary mass is divided into a set $\mathcal{H}$ of mutual exclusive communities. The interactions are driven by the community structure which is represented through a weighted network $\mathcal{G} = (\mathcal{H}, W)$. Specifically, W_{hk} measures the rate of interactions of members of the hth community with members of the kth community, so that $W_{hk} = 0$ models the case community h does not communicate with community k. We define a vector $\eta \in \mathbb{R}^{\mathcal{H}}_{>0}$ such that η_h is fraction of players that belong to the hth community. Clearly, it holds that $\mathbb{1}^T\eta = 1$, where $\mathbb{1}$ is the all-1 vector.

Players choose actions from a finite set $\mathcal{A}$ and x_{ih} denotes the fraction of players that play $i \in \mathcal{A}$ and belongs to the hth community. We gather the quantities x_{ih}, $i \in \mathcal{A}$, $h \in \mathcal{H}$, into a matrix $x \in \mathcal{X} = \{x \in \mathbb{R}^{\mathcal{A}\times\mathcal{H}}_{\geq 0} : \mathbb{1}^T x = \eta^T\}$, called *configurations* of the population. Given a configuration $x \in \mathcal{X}$, we define its *type* as $y = x\mathbb{1}$, that is, y_i is the overall fraction of the players that play action i, for $i \in \mathcal{A}$. Formally, the type $y \in \mathcal{Y}$, where $\mathcal{Y} = \left\{x \in \mathbb{R}^{\mathcal{A}}_{\geq 0} : \mathbb{1}^T y = 1\right\}$ is the unitary simplex over the action set $\mathcal{A}$.

We assume that the reward r_i of all those players playing action $i \in \mathcal{A}$ depends on the configuration only through its type. Hence, we define the *reward vector function* $r : \mathcal{Y} \to \mathbb{R}^{\mathcal{A}}$, whose entry $r_i(y)$ represent the reward received by any player playing action $i \in \mathcal{A}$ when the population is in a configuration with type $y \in \mathcal{Y}$. Throughout, we assume the reward vector function $r(y)$ to be Lipschitz-continuous over the space of types $\mathcal{Y}$. Let

$$r_*(y) := \max_{i\in\mathcal{A}} r_i(y)\,, \qquad \bar{r}(y) := \sum_{i\in\mathcal{A}} y_i r_i(y), \qquad (1)$$

stand for the maximum and the average rewards when the population is in a configuration with type $y \in \mathcal{Y}$, respectively.

The set of *Nash equilibria* of the considered population game is then given by

$$\mathcal{Y}^* := \{y \in \mathcal{Y} : \, y_i > 0 \Rightarrow r_i(y) = \bar{r}(y) = r^*(y)\}\,, \qquad (2)$$

in which no player has a strict incentive to deviate from her current action. We also define

$$\mathcal{X}^* = \{x \in \mathcal{X} : x\mathbb{1} \in \mathcal{Y}^*\}, \qquad (3)$$

that is, the set of configuration whose type is a Nash equilibrium; and

$$\mathcal{X}^*_\eta = \{x \in \mathcal{X}^* : x\mathbb{1} = \eta\}\,, \qquad (4)$$

the set of balanced configuration whose type is a Nash. In these configurations, the fraction of players of a given action $i \in \mathcal{A}$ is the same in every community $h \in \mathcal{H}$. Finally, we define the set of *critical configurations*, as

$$\mathcal{Z} = \{x \in \mathcal{X} : \sum_{h,k\in\mathcal{H}} W_{hk}x_{ih}x_{jk} > 0 \Rightarrow r_i(x\mathbb{1}) = r_j(x\mathbb{1})\}\,, \qquad (5)$$

and

$$\mathcal{Z}_\eta = \{x \in \mathcal{Z} : x\mathbb{1} = \eta\}\,. \qquad (6)$$

We assume that communities are fully mixed and homogeneous. Hence, upon a possible renormalization, the overall frequency of members of the hth community with action i that interacts with players in the kth community that play action j can be assumed equal to the product $W_{hk}x_{ih}x_{jk}$. When a player interacts with another, the former receive information from the latter about the action that it is playing and the corresponding reward. Then, depending on the difference between the two rewards, the player may decide to update its action to the one of the other player. This dynamics yields the following system of ordinary differential equations

$$\dot{x}_{ih} = \sum_{j\in\mathcal{A}} x_{jh} f_{ji}(y) \sum_{k\in\mathcal{H}} W_{hk}x_{ik} - x_{ih} \sum_{j\in\mathcal{A}} f_{ij}(y) \sum_{k\in\mathcal{H}} W_{hk}x_{jk} \qquad (7)$$

where $y = x\mathbb{1}$ and, for $i, j \in \mathcal{A}$, the function, called *imitation rate*, $f_{ij}(y)$ is Lipschitz-continuous on $\mathcal{Y}$ and satisfies

$$\mathrm{sign}\left(f_{ij}(y) - f_{ji}(y)\right) = \mathrm{sign}\left(r_j(y) - r_i(y)\right), \quad y \in \mathcal{Y}, \quad (8)$$

which models the natural tendency of players to seek to increase they reward. In fact, (8) guarantees that the imitation rate in the direction of the increasing reward is grater than the one in the opposite direction. The class of imitation dynamics contains the well studied replicator equation (for which f_{ij} is simply proportional to the reward difference between the two actions). See, for instance, Schuster and Sigmund (1983). Other examples of imitation dynamics can be found in Zino et al. (2017).

The second part of this work deal with the global asymptotic behavior of the imitation dynamics (7) for potential population games. Therefore, before presenting our results, we briefly recall the notion of potential game Monderer and Shapley (1996) in the context of continuous population games. A population game with action set $\mathcal{A}$ and Lipschitz-continuous reward function vector $r : \mathcal{Y} \to \mathbb{R}^{\mathcal{A}}$ is a *potential population game* if there exists a *potential function* $\Phi : \mathcal{Y} \to \mathbb{R}$ of class $\mathcal{C}^1$ on $\mathcal{Y}$, such that

$$r_j(y) - r_i(y) = \frac{\partial}{\partial y_j}\Phi(y) - \frac{\partial}{\partial y_i}\Phi(y)\,, \qquad (9)$$

for $i, j \in \mathcal{A}$, and almost every $y \in \mathcal{Y}$.

The potential is function of the type $y \in \mathcal{Y}$, which, in turn, depends on the configuration, being $y = c\mathbb{1}$. Hence, with a slight abuse of notation, we can write the potential as a function of $x \in \mathcal{X}$ as $\Phi(x) = \Phi(x\mathbb{1})$. It is a standard fact that, in the interior of $\mathcal{X}$, the set of critical points of the potential coincides with the set of Nash equilibria of the game Sandholm (2010).

3. MAIN RESULTS

3.1 Equilibrium points of the imitation dynamics

Here, we study the equilibrium study the equilibrium points of the imitation dynamics (7). Such an analysis is performed along the following steps, whose technical details are omitted due to space constraints.

(1) We prove that, in all configurations $x \in \mathcal{Z}_\eta$, all players play actions giving the same reward. If the network $\mathcal{G}$ is connected, than the same property holds for all configurations $x \in \mathcal{Z}$. Hence, for connected networks, $\mathcal{Z}$ coincides with the union of all the Nash equilibria of restricted games (i.e., all the games with action set equal to any subset of $\mathcal{A}$).

(2) We use matrix theory to show that, if the network is connected, then a generic equilibrium point x of the imitation dynamics (7) necessarily belongs to the set $\mathcal{Z}$.
(3) Using a similar argument and the properties of stochastic matrices with positive entries, we refine the previous result under the assumption that all imitation rates are strictly positive, fully characterizing the set of equilibrium points as $\mathcal{Z}_\eta$.

The following theorem summarizes the main result of this section.

Theorem 1. Consider a population game on a connected community network and some imitation dynamics (7) satisfying (8). Then

(i) every equilibrium point of the imitation dynamics (7) belongs to $\mathcal{Z}$.

Moreover, if $f_{ij}(y) > 0$ for every $y \in \mathcal{Y}$ and $i, j \in \mathcal{A}$, then

(ii) the set of equilibrium points of the imitation dynamics (7) coincides with $\mathcal{Z}_\eta$.

3.2 Asymptotic behavior in potential population games

The asymptotic analysis of imitative dynamics for potential population games is performed along the following steps, whose technical details are omitted due to space constraints.

(1) We observe that actions that are not played in the initial condition, will never be played, and actions that are played in the initial condition will be played for any $t \geq 0$. Hence, without any loss in generality, we assume that all the actions are played in the initial condition. Otherwise, we can simply study the game on the restricted set of actions played in the initial condition.
(2) We show that, for undirected and connected community networks, the set $\mathcal{Z}$ is invariant for the imitation dynamics (7).
(3) If the imitation rates are always nonzero, for undirected and connected community networks, we prove convergence to the points in $\mathcal{Z}_\eta$.
(4) We demonstrate that the configurations in $\mathcal{Z}$ that does not correspond to Nash equilibria are always locally repulsive for any imitation dynamics that imitation dynamics (7) satisfying (8), for undirected and connected community networks.
(5) For undirected connected networks, we prove that the potential function $\Phi(x)$ is never decreasing along trajectories of the imitation dynamics (7) and it is strictly increasing whenever x does not belong to the set $\mathcal{Z}$. An intuitive consequence of this first result is that every imitation dynamics on a symmetric network has ω-limit set contained in the set $\mathcal{Z}$.

The following theorem summarizes our main results.

Theorem 2. Consider a potential population game on an undirected connected community network $\mathcal{G}$, and some imitation dynamics (7) satisfying (8). Then, for every initial configuration $x(0) \in \mathcal{X}$ in which all actions in $\mathcal{A}$ are played, the solution $x(t)$ of the imitation dynamics (7) satisfies

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), \mathcal{X}^*) = 0. \qquad (10)$$

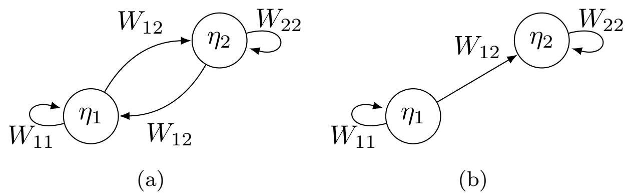


Fig. 1. Examples of the networks analyzed in this paper.

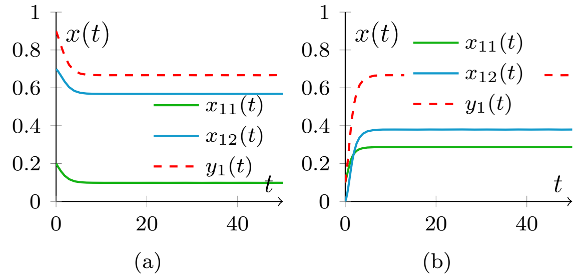


Fig. 2. Sample paths (solid curves) and temporal evolution of the type (red dashed curve) of the imitation dynamics in Example 1 for two different initial conditions.

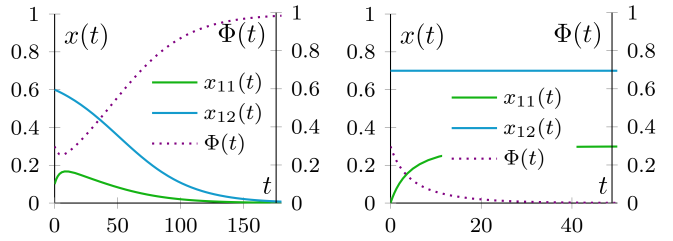
Finally, we observe that Theorem 2 yields as a corollary that global convergence of the imitation dynamics to Nash equilibria is guaranteed for potential population games on undirected connected graphs if the Nash are isolated points.

4. EXAMPLES

We conclude this extended abstract by presenting two examples. In the first one, we consider a scenario in which the network is undirected and connected and the hypotheses of our main results are verified. We validate our analytical predictions through numerical simulations, showing convergence to Nash equilibria of the game. In the second one, we consider instead a network of interactions that does not verify the conditions required for applying our results. In this second scenario, we exhibit the emergence of novel rest points for the dynamics (and corresponding asymptotic behavior), which depends on the structure of the network of interactions.

Example 1. We consider a binary game ($\mathcal{A} = \{1, 2\}$) on a symmetric connected network with two communities, as the one illustrated in Fig. 1a. The reward functions are $r_1(y) = -2y_1$ and $r_2(y) = -4y_2$, resulting into the potential $\Phi(y) = 3 - y_1^2 - 2y_2^2$. The Nash equilibria of the game coincides with the critical points of the potential, that is, all the configurations with $y_1 = \frac{2}{3}$. The trajectories of the imitation dynamics (replicator equation) integrated numerically in Fig. 2 converge to a Nash equilibrium, consistently with the analytical predictions from Theorem 2 and the following observation.

Example 2. Let us consider a binary game with two communities connected by a directed link, as illustrated in Fig. 1b. Let us consider the reward functions $r_1(y) = 0$ and $r_2(y) = 1$, yielding the potential $\Phi(y) = y_2$, which implies that the only Nash equilibrium is the configuration in which all the players play action 2. We set the imitation



(a) (b)

Fig. 3. Sample paths (solid curves) and potential (purple dotted curve) of the imitation dynamics in Example 2 for two different initial conditions.

rates equal to $f_{12}(y) = 1$ and $f_{21}(y) = 1/2$. Then, since $x_{1h} + x_{2h} = \eta_h$, (7) reduces to

$$\begin{cases} \dot{x}_{11} &= -\frac{1}{2}W_{11}x_{11}(\eta_1 - x_{11}) + \frac{1}{2}W_{12}(\eta_1 - x_{11})x_{12} \\ & \quad -W_{12}x_{11}(\eta_2 - x_{12}) \\ \dot{x}_{12} &= -\frac{1}{2}W_{22}x_{12}(\eta_2 - x_{12}), \end{cases} \tag{11}$$

from which we observe that, if $\eta_2 W_{12}/W_{11} < \eta_1$, then the configuration $\bar{x}_{11} = \eta_1 - \eta_2 W_{12}/W_{11}$, $\bar{x}_{12} = \eta_2$ is an equilibrium point of (11), but is not a configuration in $\mathcal{Z}$. From the numerical integration of the system in 3, we observe that, depending on the initial condition, the system may converge to the Nash equilibrium, as in (a), or to the other rest point, as in (b), and that the potential does not exhibit a monotonic increasing behavior (even when the imitation dynamics converges to the Nash equilibrium).

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