

Range-based Navigation and Target Localization: Observability Analysis and Guidelines for Motion Planning ^{*}

Nguyen T. Hung ^{*}, António M. Pascoal ^{*}

^{*} *ISR/IST, University of Lisbon, Lisbon, Portugal*
 {hungnguyen,antonio}@isr.ist.utl.pt.

Abstract: This paper addresses the problem of target localization with a single or multiple mobile trackers using range measurements from the trackers to the target. We consider three scenarios: i) the target is fixed, ii) the target's velocity vector is unknown but constant, and iii) the target's acceleration vector is unknown but constant. The main contributions of the paper are twofold: i) we derive a set of necessary and sufficient conditions on the motion of the trackers under which the target's state, that might include the target's position, velocity and acceleration vectors is globally observable, and ii) we show how the conditions derived lend themselves to an intuitive geometric interpretation that yields valuable guidelines to plan the tracker's motion. Numerical simulations are included to confirm the conditions derived.

Keywords: Range-based Navigation, Target localization, Observability, Mobile Robots, Autonomous Vehicles.

1. INTRODUCTION

The problems of range-based navigation and target localization have been studied extensively in recent years. In what follows, by range-based navigation we mean the problem of having a vehicle estimate its own state (position, possibly velocity and acceleration) using measurements of the distances of the vehicle to a single or multiple known beacons Bayat et al. [2016]. Target localization (or tracking), on the other hand, is defined for one or multiple trackers as the problem of tracking the state of a fixed or moving target using range measurements from the tracker(s) to the target, Crasta et al. [2018]. The two problems are dual and impose the same fundamental issues on observability analysis. Namely, for the case of target tracking, to find under what conditions of the relative motion of the tracker(s) with respect to the target is the state of the latter observable. This problem is challenging due to the fact that range measurements are nonlinear function of the target's position, thus making the observability of the resulting system hard to analyze.

One of the earliest results on the observability of target localization can be found in Song [1999], where the authors conclude that: "the tracker maneuver should include a nonzero jerk motion to track a target with a constant acceleration vector while a nonzero acceleration motion is required to track a target with a constant velocity vector". Although this conclusion sounds logical it can be shown that even those conditions are satisfied, the target might not be localizable (trackable). To illustrate this, we consider a target starting at an initial position $\mathbf{p}_T(t_0)$ and moving along a straight line with velocity vector \mathbf{u}_T , as shown in Fig 1.1. We also consider a tracker that can be started anywhere and moves with a velocity

vector $\alpha(t)\mathbf{u}_T$, with $\ddot{\alpha}(t) \neq 0$ for all t , so that it satisfies the conditions stated above. Note that with this velocity vector, the tracker will move along a straight line parallel to the trajectory of the target. With this motion of the tracker, as shown in the figure, there exists a virtual target moving with the same velocity vector \mathbf{u}_T reflected about the tracker's trajectory, such that the ranges from the tracker to the true target and its mirror image are the same. This, obviously, makes it impossible to distinguish the target and its mirror image and brings attention to the need to study the problem of target observability in a rigorous setting.

Because of the nonlinearity of the map from linear po-

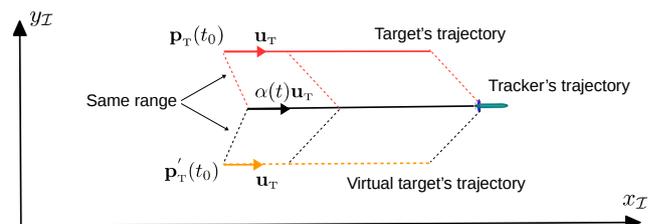


Fig. 1.1. A counterexample: the target is not localizable if the tracker moves parallel to the target.

sitions to ranges, the range-based observability problem must be addressed in a nonlinear system setting. This can be done by resorting to the tools of differential algebraic geometry described in Hermann and Krener [1977], where a sufficient condition for local observability of a nonlinear system is given in terms of an observability rank condition. In this context, the local observability for an AUV modeled by an integrator is studied in Arrichiello et al. [2013]. This work was extended in Palma et al. [2017] for an AUV modeled as a double integrator system.

Another approach to study local observability of a given nonlinear system involves the use of the Fisher information

^{*} This research was supported in part by the Marine UAS project under the Marie Curie Skłodowska grant agreement No 642153, the H2020 EU Marine Robotics Research Infrastructure Network (Project ID 731103), and the FCT Project UID/EEA/5009/2013.

matrix (FIM) to measure the amount of information that the range measurements carry about the target's motion. With this approach, the observability problem is converted into that of finding conditions on the tracker's trajectory so as to guarantee that the FIM is non-singular, thus ensuring that the target state is at least locally observable. See for example Crasta et al. [2018], Masmitja et al. [2018], Ristic et al. [2002], Song [1999] and the references therein for related work in the area.

Other interesting results were reported in Batista et al. [2011], where the authors considered the problem of localizing a source (fixed-target) using single range to a tracker. The underlying idea behind the work was to transform the original nonlinear system into a higher dimensional linear time varying (LTV) system via an appropriate state augmentation. Conditions on the tracker's motion to localize the target were then derived for the LTV system, which were proved to be sufficient for the original nonlinear system. Work along the same lines is reported in Indiveri et al. [2016] where a different state argumentation is proposed to avoid the singularity that might happen in Batista et al. [2011], Palma et al. [2017] when the range is closed to zero. Motivated by the above considerations, this paper addresses the observability problem of range-based target localization with one or two trackers. We analyze three scenarios where: i) the target is fixed, ii) the target's is unknown but constant, iii) and the target's acceleration vector is unknown but constant. The key contributions of the paper are the following.

- (i) We propose a novel approach to derive conditions on the motion of the tracker(s) under which the target's state is globally observable. The method adopted uses simple mathematical tools to characterize the linear independence of a set of functions of time. The results in this paper extends those in Batista et al. [2011], where only the case of a single tracker and a fixed target is considered. For this case, our method yields conditions identical to the ones Batista et al. [2011]. However, the method adopted is much simpler.
- (ii) We also propose the results for the case where the target is localized with two trackers. With two range measurements the trackers' motion required for observability is less demanding than in the case of a single tracker.
- (iii) We also show how the observability conditions derived lend themselves to intuitive geometric interpretations that yield valuable guidelines to plan the trackers' motions.

The paper is organized as follows. The problem of interest is formulated in Section II. Section III summarizes the main tools for observability analysis used in the paper. Section IV derives the condition for the motion of single tracker to localize a single target. Section V extends the results in Section IV for the case of two trackers-single target. Illustrative simulations are presented in Section VI. Section VII contains the main conclusions.

2. PROBLEM FORMULATION

For the sake of simplicity and transparency in the notation, in this section we start by formulating the problem of range-based target localization for a single tracker-single target pair.

2.1 System Model

Consider a tracker that attempts to localize an unknown fixed or moving target. In what follows, $\{\mathcal{I}\} = \{x_{\mathcal{I}}, y_{\mathcal{I}}, z_{\mathcal{I}}\}$ denotes an inertial frame and $\{\mathcal{B}\} = \{x_{\mathcal{B}}, y_{\mathcal{B}}, z_{\mathcal{B}}\}$ denotes a body frame attached to the tracker. In what follows we described the tracker and target models adopted.

Tracker's model: We consider the cases where the tracker's motion is described by the equations

$$\begin{aligned}\dot{\mathbf{p}}(t) &= R_{\mathcal{B}}^{\mathcal{I}}(\boldsymbol{\eta}(t))\mathbf{v}(t) + \mathbf{v}_c(t), \\ \dot{\mathbf{v}}_c(t) &= \mathbf{0},\end{aligned}\quad (1)$$

where $\mathbf{p} = [p_x, p_y, p_z]^T \in \mathbb{R}^3$ is the inertial position vector of the tracker expressed in $\{\mathcal{I}\}$; $R_{\mathcal{I}}^{\mathcal{B}}(\boldsymbol{\eta}) \in \mathbb{R}^{3 \times 3}$ is the rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$, parameterized by a vector $\boldsymbol{\eta} \triangleq [\phi, \theta, \psi]^T$ of Euler angles: denoted roll (ϕ), pitch (θ) and yaw (ψ); \mathbf{v} is the tracker's velocity vector expressed in the body frame $\{\mathcal{B}\}$; and \mathbf{v}_c is an *unknown* external disturbance vector assumed to be constant in the inertial frame, e.g. wind velocity vector in air or ocean current velocity vector in the marine environment.

Targets' model: We consider three practical scenarios for the motion of the target.

Scenario A: *Target is fixed.*

Let $\mathbf{p}_T = [x_T, y_T, z_T]^T \in \mathbb{R}^3$ be the position vector of the target in the inertial frame $\{\mathcal{I}\}$. The target's model is given by

$$\dot{\mathbf{p}}_T(t) = \mathbf{0}. \quad (2)$$

The target state is defined as $\mathbf{x}_T \triangleq \mathbf{p}_T \in \mathbb{R}^3$.

Scenario B: *Target moving with unknown velocity vector.*

In this case, we assume that the target's velocity vector changes slowly but is *unknown*. An approximate target model is given by

$$\begin{aligned}\dot{\mathbf{p}}_T(t) &= \mathbf{u}_T(t) + \mathbf{v}_c(t), \\ \dot{\mathbf{u}}_T(t) &= \mathbf{0}.\end{aligned}\quad (3)$$

Because $\dot{\mathbf{v}}_c(t) = \mathbf{0}$ the model (3) is equivalent to $\dot{\mathbf{p}}_T(t) = \mathbf{v}_T(t)$ and $\dot{\mathbf{v}}_T(t) = \mathbf{0}$, where $\mathbf{v}_T(t) \triangleq \mathbf{u}_T(t) + \mathbf{v}_c(t)$ is unknown. However, later we will show that by splitting \mathbf{v}_T as in (3), the analysis will be more convenient. In this scenario, the target's state vector is defined as $\mathbf{x}_T \triangleq [\mathbf{p}_T^T, \mathbf{u}_T^T]^T \in \mathbb{R}^6$.

Scenario C: *Target moving with unknown acceleration vector.*

We now consider the most challenging cases where target's acceleration vector is *unknown*. The target model is given by

$$\begin{aligned}\dot{\mathbf{p}}_T(t) &= \mathbf{u}_T(t) + \mathbf{v}_c(t), \\ \dot{\mathbf{u}}_T(t) &= \mathbf{a}_T(t), \\ \dot{\mathbf{a}}_T(t) &= \mathbf{0}.\end{aligned}\quad (4)$$

The model in (3) is equivalent to: $\dot{\mathbf{p}}_T(t) = \mathbf{v}_T(t)$, $\dot{\mathbf{v}}_T(t) = \mathbf{a}_T(t)$, and $\dot{\mathbf{a}}_T(t) = \mathbf{0}$ where, $\mathbf{v}_T(t) \triangleq \mathbf{u}_T(t) + \mathbf{v}_c(t)$. In (4), the target's state vector is defined as $\mathbf{x}_T \triangleq [\mathbf{p}_T^T, \mathbf{u}_T^T, \mathbf{a}_T^T]^T \in \mathbb{R}^9$.

Range measurement model: We assume that the tracker is equipped with a sensor unit capable of measuring its range to the target according to the model

$$r(t) = \|\mathbf{p}(t) - \mathbf{p}_T(t)\|. \quad (5)$$

2.2 Problem Statements

Problem 1 [Single tracker-single target localization].
 Consider a tracker moving in the presence of an unknown

external constant disturbance described by (1), the target's model given in (2)–(4) depending on different scenarios considered, and the range measurement model given by (5). Assume further that the measurements of the tracker's position, velocity, and Euler angles are available. Derive conditions for the tracker's motion (either in terms of its input profile $\mathbf{v}(t)$ or its trajectory $\mathbf{p}(t)$) under which the target's state is completely observable, i.e. the initial target's state $\mathbf{x}_T(t_0)$ is uniquely determined.

The dual problem of the target localization problem defined above is the navigation problem. That is, given all information of a single or multiple fixed or moving beacon(s), find conditions on the trajectory of the vehicle under which its position (and possibly velocity and acceleration) vectors can be determined. In the literature, this is also referred as the positioning problem. Since the navigation and target localization are dual, we only propose the solution for the later. The former can be obtained analogously.

3. TOOLS FOR OBSERVABILITY ANALYSIS

To solve **Problem 1**, we consider the extended dynamical system consisting of the tracker's model (1), the target's model (2)–(4) that depends on the target model adopted, and the range measurement model (5) in the following form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}(t)),\end{aligned}\quad (6)$$

where, $\mathbf{x} \triangleq [\mathbf{p}^T, \mathbf{v}_c^T, \mathbf{x}_T^T]^T$ is the state vector, $\mathbf{u} = [u_x, u_y, u_z]^T \triangleq R_B^T(\boldsymbol{\eta})\mathbf{v} \in \mathbb{R}^3$ is the input vector, and $\mathbf{y} = [\mathbf{p}^T, r]^T \in \mathbb{R}^4$ is the output vector. Notice that the dimension of \mathbf{x} depends on the dimension of \mathbf{x}_T , which in turn depends on the target model adopted, and the maps $\mathbf{f}(\cdot)$ and $\mathbf{h}(\cdot)$ can be obtained directly from equation (1)–(5). We use the following definition of observability for the system (6) that is an extension of the Definition 5-5 in Chen [1984] as follows.

Definition 1. (Observability). The dynamical system described by (6) is said to be (completely state) observable at t_0 if there exists a finite time $t_f > t_0$ such that for any initial state $\mathbf{x}(t_0)$, the knowledge of the input $\mathbf{u}_{[t_0, t_f]}$ and the output $\mathbf{y}_{[t_0, t_f]}$ suffices to determine the initial state $\mathbf{x}(t_0)$. Otherwise, the system is said to be unobservable at t_0 .

In the state vector \mathbf{x} of system (6), \mathbf{p} is known because it is a part of the measurement vector \mathbf{y} . In addition, the disturbance \mathbf{v}_c is observable with any tracker's trajectory. This can be verified by considering the sub-system given by (1) with the state $\mathbf{z} \triangleq [\mathbf{p}^T, \mathbf{v}_c^T]^T$, the input \mathbf{u} , and the output \mathbf{p} . The sub-system (1) is linear and can be rewritten as

$$\dot{\mathbf{z}} = \begin{bmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{bmatrix} \mathbf{z} + \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix} \mathbf{u}, \quad \mathbf{p} = [I_3 \quad 0_3] \mathbf{z}. \quad (7)$$

where $I_3 \in \mathbb{R}^{3 \times 3}$ is the identity matrix. Using the linear observability rank condition, it is easy to check that the system (7) is observable. Hence, the disturbance \mathbf{v}_c can be determined with the knowledge of the tracker's position \mathbf{p} and its input \mathbf{u} . Therefore, system (6) is completely state observable if and only if the target state \mathbf{x}_T is observable. Because the system (6) is nonlinear, its state observability depends on the system input \mathbf{u} . The purpose of this paper is to derive the set of conditions on the tracker's motion

under which the state \mathbf{x} of system (6) or equivalently, the target's state is observable, i.e. the initial target's state $\mathbf{x}_T(t_0)$ is uniquely determined. The essential tool to derive the conditions is linear independence of functions over a compact interval of time is stated as follows.

Lemma 1. Let $\mathbf{f}_i(t)$, for $i = 1, 2, \dots, n$ be $1 \times p$ vector real-valued continuous functions of t defined on the interval $[t_0, t_f]$. Let $F = [\mathbf{f}_1, \dots, \mathbf{f}_n]$ be the $n \times p$ matrix with \mathbf{f}_i as its i^{th} row. Define

$$W(t_0, t_f) \triangleq \int_{t_0}^{t_f} F(t)F^T(t)dt.$$

Then, $W(t_0, t_f)$ is non-singular if and only if the $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ are linear independent on $[t_0, t_f]$.

Proof. See the proof of Theorem 5-1 in Chen [1984]. For the definition of linear independence of functions, we refer the reader to section 5-2 in Chen [1984].

4. SINGLE TRACKER-SINGLE TARGET

In this section, we will derive a set of conditions on the motion of the tracker (either on its input $\mathbf{u}(t)$ or on its trajectory $\mathbf{p}(t)$) under which the target's state is observable, i.e. the initial state of target $\mathbf{x}_T(t_0)$ is uniquely determined. For the sake of convenience, let

$$\mathbf{q}(t_0) \triangleq \mathbf{p}(t_0) - \mathbf{p}_T(t_0). \quad (8)$$

Because $\mathbf{p}(t_0)$ is known, $\mathbf{p}_T(t_0)$ is uniquely determined if and only if $\mathbf{q}(t_0)$ is uniquely determined. Furthermore, from (1) we obtain

$$\mathbf{p}(t) = \mathbf{p}(t_0) + \boldsymbol{\lambda}(t) + (t - t_0)\mathbf{v}_c(t_0), \quad (9)$$

where

$$\boldsymbol{\lambda}(t) = [\lambda_x(t), \lambda_y(t), \lambda_z(t)]^T \triangleq \int_{t_0}^t \mathbf{u}(\tau)d\tau. \quad (10)$$

We start by considering the simplest case when the target is fixed in the inertial frame $\{\mathcal{I}\}$.

4.1 Target is fixed

From (2), it follows that $\mathbf{p}_T(t) = \mathbf{p}_T(t_0)$ for all $t \geq t_0$. Inserting the above equality and (9) in (5) yields

$$\begin{aligned}r^2(t) &= \|\mathbf{p}(t) - \mathbf{p}_T(t)\|^2 \\ &= \|\mathbf{p}(t_0) + \boldsymbol{\lambda}(t) + (t - t_0)\mathbf{v}_c(t_0) - \mathbf{p}_T(t_0)\|^2.\end{aligned}\quad (11)$$

Furthermore, inserting (8) in (11), we obtain

$$\begin{aligned}r^2(t) &= \|\mathbf{q}(t_0)\|^2 + \|\boldsymbol{\lambda}(t)\|^2 + 2\boldsymbol{\lambda}^T(t)\mathbf{q}(t_0) \\ &\quad + 2(t - t_0)\boldsymbol{\lambda}^T(t)\mathbf{v}_c(t_0) + 2(t - t_0)\mathbf{q}^T(t_0)\mathbf{v}_c(t_0) \\ &\quad + (t - t_0)^2\|\mathbf{v}_c(t_0)\|^2.\end{aligned}\quad (12)$$

Notice that $\|\mathbf{q}(t_0)\|^2 = r^2(t_0)$. Let $y_A(t) \triangleq r^2(t) - r^2(t_0) - \|\boldsymbol{\lambda}(t)\|^2$, from which equation (12) can be rewritten as

$$y_A(t) = C_A(t)\mathbf{z}_A(t), \quad (13)$$

where $\mathbf{z}_A(t_0) \triangleq [\mathbf{q}^T(t_0), \mathbf{v}_c^T(t_0), \mathbf{q}^T(t_0)\mathbf{v}_c(t_0), \|\mathbf{v}_c(t_0)\|^2]^T$ and

$$C_A(t) \triangleq [2\boldsymbol{\lambda}^T(t) \quad 2(t - t_0)\boldsymbol{\lambda}^T(t) \quad 2(t - t_0) \quad (t - t_0)^2]. \quad (14)$$

Notice that $y_A(t)$ is known because $r(t)$ and $\boldsymbol{\lambda}(t)$ are known for all $t \geq t_0$. Matrix $C(t)$, which carries the information about the tracker's input trajectory $\mathbf{u}(t)$ in the interval $[t_0, t]$ is also known. Because $\mathbf{z}_A(t_0)$ contains $\mathbf{q}(t_0)$, it is clearly that knowing a solution for $\mathbf{z}_A(t_0)$ is sufficient to determine $\mathbf{q}(t_0)$, which from (8) it is also sufficient to determine $\mathbf{p}_T(t_0)$. The following results states sufficient

condition on the tracker's motion under which $\mathbf{p}_T(t_0)$ is uniquely determined.

Theorem 1. Consider the target localization problem defined in **Problem 1**, where the target is fixed. Let S_1 and S_2 be two sets of functions defined by

$$S_1 \triangleq \{\lambda_x(t), \lambda_y(t), \lambda_z(t), (t-t_0)\lambda_x(t), (t-t_0)\lambda_y(t), (t-t_0)\lambda_z(t), t-t_0, (t-t_0)^2\}. \quad (15)$$

$$S_2 \triangleq \{p_x(t) - p_x(t_0), p_y(t) - p_y(t_0), p_z(t) - p_z(t_0), (t-t_0)(p_x(t) - p_x(t_0)), (t-t_0)(p_y(t) - p_y(t_0)), (t-t_0)(p_z(t) - p_z(t_0)), t-t_0, (t-t_0)^2\}. \quad (16)$$

Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq \mathbf{p}_T(t_0)$ is uniquely determined if either the set of functions in S_1 or in S_2 are linear independent on the interval $[t_0, t_f]$, where $t_f > t_0$.

Proof. We first prove the result for the condition stated with the set S_1 . Multiplying both sides of (13) by $C^T(t)$ and integrating from t_0 to t_f , we obtain

$$\int_{t_0}^{t_f} C_A(\tau)y_A(\tau)d\tau = \left(\int_{t_0}^{t_f} C_A^T(\tau)C_A(\tau)d\tau \right) \mathbf{z}(t_0). \quad (17)$$

Let $W(t_0, t_f) \triangleq \int_{t_0}^{t_f} C_A^T(\tau)C_A(\tau)d\tau$. If $W(t_0, t_f)$ is non-singular, $\mathbf{z}(t_0) = W^{-1}(t_0, t_f) \int_{t_0}^{t_f} C_A(\tau)y_A(\tau)d\tau$ is a unique solution of (17). Using Lemma 1, $W(t_0, t_f)$ is non-singular if and only if the columns of matrix $C_A(t)$ are linear independent on $[t_0, t_f]$. From (10), (14), and noticing that by multiplying the columns of matrix $C_A(t)$ with any constant, linear independence of the columns is still preserved, we conclude that $\mathbf{z}_A(t_0)$ is uniquely determined if and only if the set of functions in S_1 are linear independent on $[t_0, t_f]$. Consequently, as explained earlier, this is a sufficient condition to determine $\mathbf{p}_T(t_0)$.

We now prove the same result for the condition stated with the functions in S_2 . From (9), we obtain

$$\lambda(t) = \mathbf{p}(t) - \mathbf{p}(t_0) - (t-t_0)\mathbf{v}_c(t_0). \quad (18)$$

Substituting (18) in (15) and using the definition of linear independence of functions (see Section 5-2 in Chen [1984]) it can be easily checked that the set of functions in S_1 are linear independent if and only if the set of functions in S_2 are also linear independent. Hence, we conclude that $\mathbf{p}_T(t_0)$ is uniquely determined if the set of functions in S_2 are linear independent on $[t_0, t_f]$. This completes the proof. ■

It is interesting to observe that the condition stated in Theorem 1 for the set S_2 is identical to the one given in Theorem 3 in Batista et al. [2011]. However, it can be seen above that the method used to derive the condition is much simpler in our paper. In Batista et al. [2011], to obtain the conditions in Theorem 1, the authors make use of a "Lyapunov state transformation and state augmentation" to transform the original nonlinear system to a LTV system. The authors then derive a condition for the LTV that later can be proved to be a sufficient condition for the original nonlinear system as well. Another advantage of our method is that it avoids the singularity that might happen in Batista et al. [2011] when the tracker is close to the target, i.e. when $y(t) = 0$ for some $t \in [t_0, t_f]$.

We now consider a special case of Theorem 1 where the external disturbance is negligible, i.e. $\mathbf{v}_c(t) \equiv \mathbf{0}$.

Corollary 1. Consider the target localization problem in Theorem 1. Assume further that the disturbance \mathbf{v}_c is negligible ($\mathbf{v}_c(t) \equiv \mathbf{0}$). Let S'_1 and S'_2 be two sets of functions defined by

$$S'_1 \triangleq \{\lambda_x(t), \lambda_y(t), \lambda_z(t)\}. \quad (19)$$

$$S'_2 \triangleq \{p_x(t) - p_x(t_0), p_y(t) - p_y(t_0), p_z(t) - p_z(t_0)\}. \quad (20)$$

Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq \mathbf{p}_T(t_0)$ is uniquely determined if and only if either the set of functions in S'_1 or in S'_2 are linear independent on the interval $[t_0, t_f]$, where $t_f > t_0$.

Proof: We first show the result for S'_1 . Substituting $\mathbf{v}_c = \mathbf{0}$ in (12), we obtain $y_A(t) = 2\lambda^T(t)\mathbf{q}(t_0)$. Using Lemma 1 we conclude that $\mathbf{q}(t_0)$ is uniquely determined if and only if the columns of matrix $\lambda^T(t) \in \mathbb{R}^{1 \times 3}$ (that are the functions in the set S'_1) are linear independent on $[t_0, t_f]$. Recall from (8) that $\mathbf{q}(t_0) = \mathbf{p}(t_0) - \mathbf{p}_T(t_0)$ where $\mathbf{p}(t_0)$ is known, hence the linear independence of functions in S'_1 is necessary and sufficient to determine $\mathbf{p}_T(t_0)$. Furthermore, substituting $\mathbf{v}_c = \mathbf{0}$ in (9), we obtain that $S'_2 \equiv S'_1$. ■

We now discuss the geometrical intuition behind the condition given in the corollary. The above necessary and sufficient condition implies that the target's position can not be determined if the tracker moves along any straight line, since this violates the linear independence condition of the functions in S'_2 . This is not surprising and can be explained intuitively from a geometrical standpoint. In fact, if the tracker moves along a straight line, then there exists an reflected image (in 2D) or a set of reflected images (in 3D) of the target about that line such that the ranges from tracker to the target and to its reflected image are the same, thus making it impossible to distinguish the true target and its reflected images. See a trajectory in Fig.4.1 (solid-black) as an illustration for the case of 2D.

In practice, it is common to use an under-actuated vehicle as the tracker to localize the target. In 2D, $x_I - y_I$ plane for example, the motion of the robot can be simply rewritten from (1) as

$$\dot{p}_x = v \cos(\psi) \triangleq u_x, \quad \dot{p}_y = v \sin(\psi) \triangleq u_y. \quad (21)$$

For this type of tracker, the condition in Corollary 1 implies that it is sufficient to localize the target if the tracker moves with non zero speed ($v \neq 0$) and changes its heading (ψ) at least one time in the interval (t_0, t_f) (see the red curve in Fig.4.1 as an example of this type of trajectory where $\dot{\psi}(t) \neq 0$ for all $t \in [t_0, t_f]$). This can be extended to 3D analogously, it is sufficient to localize the target that the tracker move with non-zero speed and change at least any two Euler angles at any different times in the interval (t_0, t_f) .

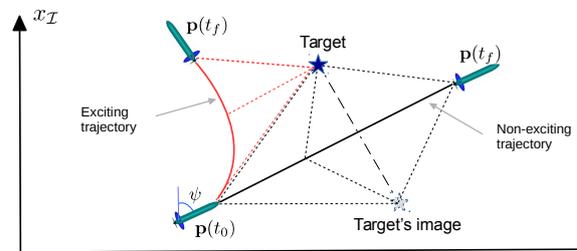


Fig. 4.1. Localization of a fixed target using single tracker under $\mathbf{v}_c = \mathbf{0}$.

4.2 Target moving with unknown velocity vector

From (3), we obtain

$$\mathbf{p}_T(t) = \mathbf{p}_T(t_0) + (t - t_0)\mathbf{u}_T(t_0) + (t - t_0)\mathbf{v}_c(t_0). \quad (22)$$

Substituting (9) and (22) in (5) yields

$$r^2(t) = \|\mathbf{p}(t_0) + \boldsymbol{\lambda}(t) - (t - t_0)\mathbf{u}_T(t_0) - \mathbf{p}_T(t_0)\|^2. \quad (23)$$

Extending (23) similarly to the case for a fixed target we obtain

$$y_c(t) = C_c(t)\mathbf{z}_c(t_0), \quad (24)$$

where $\mathbf{z}_c(t_0) \triangleq [\mathbf{q}^T(t_0), \mathbf{u}_T^T(t_0), \mathbf{q}^T(t_0)\mathbf{u}_T(t_0), \|\mathbf{u}_T(t_0)\|^2]^T$, $y_c(t) \triangleq y_A(t)$, and $C(t)$ is given by

$$C_c(t) \triangleq [2\boldsymbol{\lambda}^T(t) \quad -2(t - t_0)\boldsymbol{\lambda}^T(t) \quad -2(t - t_0) \quad (t - t_0)^2]. \quad (25)$$

Recall from (8) that $\mathbf{q}(t_0) = \mathbf{p}(t_0) - \mathbf{p}_T(t_0)$, where $\mathbf{p}(t_0)$ is known. Hence, the condition that will enable to determine $\mathbf{z}_c(t_0)$ is sufficient to determine the initial position vector $\mathbf{p}_T(t_0)$ and the initial velocity vector $\mathbf{u}_T(t_0)$ of the target. We obtain the following result.

Theorem 2. Consider the target localization problem defined in **Problem 1**, where the target moving with unknown velocity vector given by model (3). Then, the initial state of the target $\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0)]^T$ is uniquely determined at t_0 if either the set of functions in S_1 or in S_2 defined in (15) and (16), respectively are linear independent on the interval $[t_0, t_f]$, where $t_f > t_0$.

Proof: From (24) and using Lemma 1 in similar way to the proof of Theorem 1 we conclude that $\mathbf{z}_c(t_0)$ is uniquely determined on the interval $[t_0, t_f]$ if and only if the columns of matrix $C_c(t)$ are linear independent on $[t_0, t_f]$. It can be easily seen that the columns of $C_c(t)$ are linear independent if and only if the columns of $C_A(t)$ are linear independent. This condition, as shown in Theorem 1, is equivalent to having the set of functions in S_1 or in S_2 linearly independent on $[t_0, t_f]$. This completes the proof. ■

We now discuss several types of trajectory that satisfy the condition given in Theorem 2. In 2D, it can be checked (using either the definition of linear independence of functions or their Wronskian (Theorem 5-2, Chen [1984])) that every “cycloid-type” trajectory for the tracker given in the form $\mathbf{p}(t) = [p_x(t), p_y(t)]^T = [r_x \sin(\omega t) + c_x t, r_y \cos(\omega t) + c_y t]^T$ with $r_x, r_y, \omega \neq 0$ satisfies the condition in the theorem. See Fig. 4.2 as a graphical presentation of several trajectories given by the above formula. Similarly, in 3D, every “helix-type” trajectories of the form $\mathbf{p}(t) = [r_x \sin(\omega_1 t) + c_x t, r_y \cos(\omega_1 t) + c_y t, r_z \sin(\omega_2 t) + c_z t]^T$ with $\omega_1 \neq \omega_2 \neq 0$ satisfies the condition in the theorem. However, “pure helix” trajectories (obtained with $r_z = 0$) do not satisfy the condition. Thus, we can not conclude if they guarantee that the target’s states (both the position and velocity vector) are observable or not as the condition given is sufficient. From a system identification point of view, this can be explained that the full state of the target may not be “completely” observable. That is, if $c_2 = 0$ or $\omega_2 = 0$ then the relative velocity vector between the tracker and the target $\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \mathbf{u}_T(t) = [\star, \star, c_1 - c]$, where c is the third component of \mathbf{u}_T and constant. Because the third component of the relative vector $\tilde{\mathbf{u}}$ does not change over time, this information may not be “sufficient” to identify the third component of the target’s velocity vector \mathbf{u}_T , hence the full target’s state might not be completely observable with the “pure helix” trajectories.

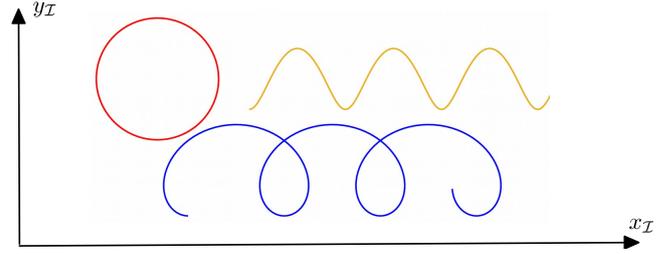


Fig. 4.2. Illustration of several “cycloid-type” trajectories.

4.3 Target moving with unknown acceleration vector

From (4), we obtain

$$\mathbf{p}_T(t) = \mathbf{p}_T(t_0) + (t - t_0)\mathbf{u}_T(t_0) + 0.5(t - t_0)^2\mathbf{a}_T(t_0) + (t - t_0)\mathbf{v}_c(t_0). \quad (26)$$

Inserting (8), (9) and (26) in (5) yields

$$r^2(t) = \|\mathbf{q}(t_0) + \boldsymbol{\lambda}(t) - (t - t_0)\mathbf{u}_T(t_0) - 0.5(t - t_0)^2\mathbf{a}_T(t_0)\|^2 \quad (27)$$

Extending equation (27) similarly to the previous cases, we obtain

$$y_D(t) = C_D(t)\mathbf{z}_D(t_0), \quad (28)$$

where $y_D(t) \triangleq y_A(t)$,

$\mathbf{z}_D(t_0) \triangleq$

$$[\mathbf{q}^T(t_0), \mathbf{u}_T^T(t_0), \mathbf{a}_T^T(t_0), \mathbf{q}^T(t_0)\mathbf{u}_T(t_0), \|\mathbf{u}_T(t_0)\|^2 - \mathbf{q}^T(t_0)\mathbf{a}_T^T(t_0), \mathbf{u}_T^T(t_0)\mathbf{a}_T^T(t_0), \|\mathbf{a}_T(t_0)\|^2]^T, \quad (29)$$

and $C_D(t)$ is given by

$$C_D(t) \triangleq [2\boldsymbol{\lambda}^T(t) \quad -2(t - t_0)\boldsymbol{\lambda}^T(t) \quad -(t - t_0)^2\boldsymbol{\lambda}^T(t) \quad -2(t - t_0) \quad (t - t_0)^2 \quad 2(t - t_0)^3 \quad 0.25(t - t_0)^4]. \quad (30)$$

At this stage, we obtain the following result.

Theorem 3. Consider the target localization problem defined in **Problem 1**, where the target is moving with an unknown acceleration vector given by model (4). Let S_3 and S_4 be two sets of functions defined by

$$S_3 \triangleq S_1 \cup \{(t - t_0)^3, (t - t_0)^4, (t - t_0)^2\lambda_x(t), (t - t_0)^2\lambda_y(t), (t - t_0)^2\lambda_z(t)\}, \quad (31)$$

$$S_4 \triangleq S_2 \cup \{(t - t_0)^3, (t - t_0)^4, (t - t_0)^2(p_x(t) - p_x(t_0)), (t - t_0)^2(p_y(t) - p_y(t_0)), (t - t_0)^2(p_z(t) - p_z(t_0))\}. \quad (32)$$

Then, the initial target’s state

$$\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0), \mathbf{a}_T^T(t_0)]^T$$

is uniquely determined at t_0 if either the set of functions in S_3 or in S_4 are linear independent on the interval $[t_0, t_f]$, where $t_f > t_0$.

Proof: The proof for the set S_3 is similar to the proof for the set S_1 in Theorem 1. Similarly substituting (18) in (31) and noticing that $\mathbf{v}_c(t_0)$ is constant, it can be shown that the functions in S_3 are linear independent if the functions in S_4 also are. ■

Clearly, Theorem 2 is a special case of Theorem 3 when the target velocity vector is constant, i.e. $\mathbf{a}_T(t) \equiv \mathbf{0}$. Note also that the “cycloid-type” trajectories for 2D and the “helix-type” trajectories for 3D discussed in Section 4.2 satisfy the condition in Theorem 3 as well.

5. TWO TRACKERS- SINGLE TARGET

We now consider the cases when the target is localized using two trackers. Assume that the two trackers have the same kinematics model given by (1) and each tracker can measure range itself to the target. For the sake of consistency, we keep using the notation in the previous section with an extra subscript i ; $i \in \{1, 2\}$ to index the trackers. Specifically, for each tracker i , \mathbf{p}_i denotes its position vector; $\mathbf{u}_i \triangleq R_B^T(\boldsymbol{\eta}_i)$ denote its velocity vector respect to wind (in air) or fluid (in water) expressed in the inertial frame; $\mathbf{q}_i = \mathbf{p}_i - \mathbf{p}_T$ is the relative position vector from the tracker to the target; and r_i is the range measurement from the tracker to the target. We now derive the conditions for the motion of the trackers under which the target's state is observable.

5.1 Target is fixed

We start by considering the simplest case where the target is fixed. Let $\mathbf{d}_i(t) \triangleq \mathbf{p}_i(t_0) + \boldsymbol{\lambda}_i(t)$. From (11) we obtain

$$\begin{aligned} r_i^2(t) = & \|\mathbf{d}_i(t)\|^2 + (t - t_0)^2 \|\mathbf{v}_c(t_0)\|^2 + \|\mathbf{p}_T(t_0)\|^2 \\ & + 2(t - t_0)\mathbf{d}_i^T(t)\mathbf{v}_c(t_0) - 2\mathbf{d}_i^T(t)\mathbf{p}_T(t_0) \\ & - 2(t - t_0)\mathbf{v}_c^T(t_0)\mathbf{p}_T(t_0) \end{aligned} \quad (33)$$

for all $i = \{1, 2\}$. Subtracting $r_1^2(t) - r_2^2(t)$ yields

$$\begin{aligned} r_1^2(t) - r_2^2(t) = & \|\mathbf{d}_1(t)\|^2 - \|\mathbf{d}_2(t)\|^2 \\ & - 2(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{p}_T(t_0) \\ & - 2(t - t_0)(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{v}_c(t_0). \end{aligned} \quad (34)$$

Let $\bar{y}_A(t) \triangleq (r_1^2(t) - r_2^2(t) - \|\mathbf{d}_1(t)\|^2 + \|\mathbf{d}_2(t)\|^2)/2$, and $\bar{\mathbf{z}}_A(t_0) = [\mathbf{p}_T^T(t_0), \mathbf{v}_c^T(t_0)]^T$. Equation (34) can be rewritten as

$$\bar{y}_A(t) = \bar{C}_A(t)\bar{\mathbf{z}}_A(t_0), \quad (35)$$

where

$$\bar{C}_A(t) \triangleq [\mathbf{d}_2^T(t) - \mathbf{d}_1^T(t) \quad (t - t_0)(\mathbf{d}_2^T(t) - \mathbf{d}_1^T(t))], \quad (36)$$

Recall that $\mathbf{d}_i(t) = \mathbf{p}_i(t_0) + \boldsymbol{\lambda}_i(t)$. Thus, from (9) it follows that $\mathbf{d}_i(t) = \mathbf{p}_i(t) - (t - t_0)\mathbf{v}_c(t_0)$ for all $i \in \{1, 2\}$. Substituting this in (36), $\bar{C}_A(t)$ can be rewritten as

$$\bar{C}_A(t) \triangleq [\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t) \quad (t - t_0)(\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t))]. \quad (37)$$

The main result for this case is stated next.

Theorem 4. Consider the target localization problem defined in **Problem 1**, where the target is fixed and two trackers are used. Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T(t_0)]$ is uniquely determined if the columns of $\bar{C}_A(t)$ given in (37) are linear independent on $[t_0, t_f]$, where $t_f > t_0$.

Proof: The proof can be done similarly to that of Theorem 1. ■

We now consider a special case of Theorem 4 where the external disturbance is neglected, i.e. $\mathbf{v}_c(t) \equiv \mathbf{0}$.

Corollary 2. Consider the set-up stated in Theorem 4. Assume further that the disturbance \mathbf{v}_c is negligible, i.e. $\mathbf{v}_c(t) \equiv \mathbf{0}$. Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T(t_0)]$ is uniquely determined at t_0 if and only if the columns of matrix

$$D(t) \triangleq [\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t)] \in \mathbb{R}^{1 \times 3} \quad (38)$$

are linear independent on $[t_0, t_f]$, where $t_f > t_0$.

Proof: Substituting $\mathbf{v}_c = 0$ in (34), we obtain $\bar{y}_A(t) = [\mathbf{d}_2^T(t) - \mathbf{d}_1^T(t)]\mathbf{p}_T(t_0) = [\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t)]\mathbf{p}_T(t_0) \triangleq D(t)\mathbf{p}_T(t_0)$.

Following Lemma 1 and the methodology adopted in Theorem 1 we conclude that the solution for $\mathbf{p}_T(t_0)$ is uniquely determined if and only if the columns of matrix $D(t)$ are linearly independent on $[t_0, t_f]$ for any $t_f > t_0$. ■

We now discuss geometrical intuition behind the condition stated in the corollary. For the sake of clarity, we consider the case where one of the trackers is stationary. Without loss of generality, fix tracker 2, i.e. $\mathbf{p}_2(t) = \mathbf{p}_2(t_0)$ for all $t \geq t_0$. This makes $D(t) = [\mathbf{p}_2(t_0) - \mathbf{p}_1(t)]^T$. Hence, for 2D, the necessary and sufficient condition for the columns of $D(t)$ to be independent on $[t_0, t_f]$ implies that the trajectory of tracker 1 must not move along the line that connects two points $\mathbf{p}_1(t_0)$ and $\mathbf{p}_2(t_0)$. This is illustrated in Fig. 5.1 where it can be seen that the ranges from the trackers to the target and the target's reflected image via the line are the same, making it impossible to distinguish the true target and its reflected image. However, if tracker 1 does not go along with that line (the red line in the figure for example) then the position of target can be uniquely determined.

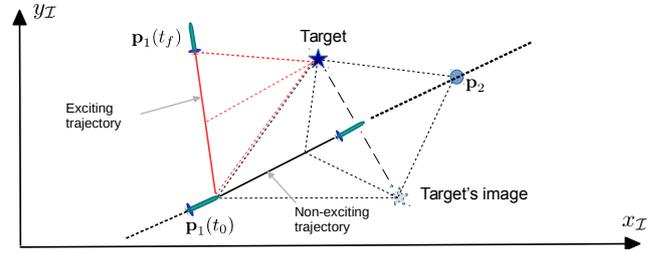


Fig. 5.1. Localization of a fixed target using two trackers under $\mathbf{v}_c = \mathbf{0}$.

5.2 Target moving with unknown velocity vector

Recall again that $\mathbf{d}_i(t) \triangleq \mathbf{p}_i(t_0) + \boldsymbol{\lambda}_i(t)$. From (23), for each $i \in \{1, 2\}$ we have

$$\begin{aligned} r_i^2(t) = & \|\mathbf{d}_i(t) - (t - t_0)\mathbf{u}_T(t_0) - \mathbf{p}_T(t_0)\|^2 \\ = & \|\mathbf{d}_i(t)\|^2 - 2(t - t_0)\mathbf{d}_i^T(t)\mathbf{u}_T(t_0) - 2\mathbf{d}_i^T(t)\mathbf{p}_T(t_0) \\ & + \delta_c(t), \end{aligned} \quad (39)$$

where,

$$\delta_c(t) \triangleq (t - t_0)^2 \|\mathbf{u}_T(t_0)\|^2 + \|\mathbf{p}_T(t_0)\|^2 + 2(t - t_0)\mathbf{u}_T^T(t_0)\mathbf{p}_T(t_0). \quad (40)$$

Thus, subtracting $r_2^2(t)$ from $r_1^2(t)$ yields

$$\begin{aligned} r_1^2(t) - r_2^2(t) = & \|\mathbf{d}_1(t)\|^2 - \|\mathbf{d}_2(t)\|^2 \\ & - 2(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{p}_T(t_0) \\ & - 2(t - t_0)(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{u}_T(t_0). \end{aligned} \quad (41)$$

Recall that for this scenario $\mathbf{x}_T(t_0) = [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0)]^T$ is the vector of the initial target's state. Equation (41) can be rewritten as

$$\bar{y}_c(t) = \bar{C}_c(t)\mathbf{x}_T(t_0), \quad (42)$$

where, $\bar{y}_c(t) = \bar{y}_A(t)$, $\bar{C}_c(t) = \bar{C}_A(t)$ and $\bar{C}_A(t)$ is given by (37). We obtain the following result.

Theorem 5. Consider the target localization problem defined in **Problem 1**, where the target is moving with unknown velocity vector given by model (3) and is localized by two trackers. Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0)]^T$ is uniquely determined at t_0 if and only

if the columns of matrix $\bar{C}_c(t)$ are linear independent on $[t_0, t_f]$, where $t_f > t_0$.

Proof: From (42), using Lemma 1 and the methodology adopted in the proof of Theorem 1, we conclude that $\mathbf{x}_T(t_0)$ is uniquely determined if and only if the columns of matrix $\bar{C}(t)$ are linear independent on $[t_0, t_f]$, where $t_f > t_0$. ■

5.3 Target moving with unknown acceleration vector

Recall that $\mathbf{d}_i(t) \triangleq \mathbf{p}_i(t_0) + \boldsymbol{\lambda}_i(t)$. From (27) we obtain

$$\begin{aligned} r_i^2(t) &= \|\mathbf{d}_i(t_0) - (t - t_0)\mathbf{u}_T(t_0) \\ &\quad - 0.5(t - t_0)^2 \mathbf{a}_T(t_0) - \mathbf{p}_T(t_0)\|^2 \\ &= \|\mathbf{d}_i(t)\|^2 - 2(t - t_0)\mathbf{d}_i^T(t)\mathbf{u}_T(t_0) - 2\mathbf{d}_i^T(t)\mathbf{p}_T(t_0) \\ &\quad - (t - t_0)^2 \mathbf{d}_i^T(t)\mathbf{a}_T(t_0) + \delta_D(t) \end{aligned} \quad (43)$$

for all $i \in \{1, 2\}$ where

$$\delta_D(t) \triangleq \delta_C(t) + (t - t_0)^2 ((t - t_0)\mathbf{u}_T(t_0) + \mathbf{p}_T(t_0))^T \mathbf{a}_T(t_0).$$

Subtracting $r_2^2(t)$ from $r_1^2(t)$ yields

$$\begin{aligned} r_1^2(t) - r_2^2(t) &= \|\mathbf{d}_1(t)\|^2 - \|\mathbf{d}_2(t)\|^2 \\ &\quad - 2(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{p}_T(t_0) \\ &\quad - 2(t - t_0)(\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{u}_T(t_0) \\ &\quad - (t - t_0)^2 (\mathbf{d}_1(t) - \mathbf{d}_2(t))^T \mathbf{a}_T(t_0). \end{aligned} \quad (44)$$

Recall that $\mathbf{x}_T(t_0) = [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0), \mathbf{a}_T^T(t_0)]^T$ is the vector of the initial target's state. Recall also that $\mathbf{d}_1(t) - \mathbf{d}_2(t) = \mathbf{p}_1(t) - \mathbf{p}_2(t)$. Then, equation (44) can be rewritten as

$$\bar{y}_D(t) = \bar{C}_D(t)\mathbf{x}_T(t_0), \quad (45)$$

where, $\bar{y}_D(t) = \bar{y}_A(t)$, and $\bar{C}_D(t)$ is given by

$$\bar{C}_D(t) \triangleq [\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t) \quad (t - t_0)(\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t)) \quad 0.5(t - t_0)^2(\mathbf{p}_2^T(t) - \mathbf{p}_1^T(t))]. \quad (46)$$

We obtain the following result.

Theorem 6. Consider the target localization problem defined in **Problem 1**, where the target moving with unknown acceleration vector given by model (4) and is localized by two trackers. Then, the initial target's state $\mathbf{x}_T(t_0) \triangleq [\mathbf{p}_T^T(t_0), \mathbf{u}_T^T(t_0), \mathbf{a}_T^T(t_0)]^T$ is uniquely determined at t_0 if and only if the columns of matrix $\bar{C}_D(t)$ are linear independent on $[t_0, t_f]$, where $t_f > t_0$.

Proof: The proof follows from equation (45) and Lemma 1.

6. ILLUSTRATIVE EXAMPLES

In this section, we present simulation results for the most challenging scenario, i.e. localizing a target moving with unknown acceleration vector (scenario D). The simulation setup is given in Table 1. Two simulations are made. In the first one, the target is localized using only a single range from tracker 1 to the target. In the second simulation, the target is localized using two ranges from both tracker 1 and tracker 2 to the target. In both simulations tracker 1 move along the same trajectory defined by its velocity vector $\mathbf{u}_1(t)$. In the second simulation, tracker 2 is fixed in the inertial frame with a known position. It can be checked that with the parameters set-up in the table, the motions of the trackers satisfy the observability conditions in Theorem 3 and Theorem 6. Thus, the target's state will

be observable.

We assume that the measurement of tracker1's position and ranges from the two trackers to the target are disturbed by Gaussian noises with zero means and covariance of $0.5I_3$ (m) for \mathbf{p}_1 and standard deviations of 0.1(m) for r_1 and r_2 . To estimate the disturbance \mathbf{v}_c and the target's state \mathbf{x}_T , an EKF is set up to estimate the state of system (6) where, $\mathbf{x} = [\mathbf{p}_1, \mathbf{v}_c, \mathbf{x}_T]^T$ and $\mathbf{x}_T = [\mathbf{p}_T, \mathbf{u}_T, \mathbf{a}_T]^T$. With a single range measurement from tracker 1, the output vector of system (6) is $\mathbf{y} = [\mathbf{p}_1^T, r_1]^T \in \mathbb{R}^4$. Whereas, with the second range from tracker 2, $\mathbf{y} = [\mathbf{p}_1^T, r_1, r_2]^T \in \mathbb{R}^5$. The covariance matrices for the process noise was chosen as $Q = \text{diag}(10I_3, 0.1I_3, 10I_3, 0.1I_3, 0.01I_3)$ while the covariance matrix for measurement noises were chosen as $R = \text{diag}(10I_3, 100)$ for the case of a single tracker and, $R = \text{diag}(10I_3, 100I_2)$ for the case of two trackers. The initial guess of the state \mathbf{x} and its covariance is set as $\hat{\mathbf{x}}(0) = [\hat{\mathbf{p}}^T(0), \hat{\mathbf{v}}_c^T(0), \hat{\mathbf{p}}_T^T(0), \hat{\mathbf{u}}_T^T(0), \hat{\mathbf{a}}_T^T(0)]^T = [\mathbf{0}_3, \mathbf{0}_3, 50, -70, 35, \mathbf{0}_3, 0.08, -0.15, 0.09]^T$ and $\hat{P}(0) = \text{diag}(10I_3, 0.5I_3, 100I_3, 0.1I_3, 0.01I_3)$, respectively. The simulation results are plotted in Fig. 6.1 and Fig.6.2.

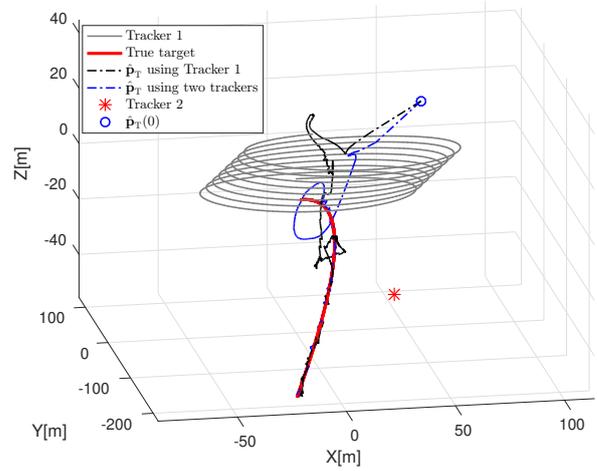


Fig. 6.1. Trajectories of the trackers, target and the target's estimates.

It can be observed in Fig. 6.1 that even though it was initialized considerably far from the true target, the EKF estimates converge to the true target trajectory. This observation is enforced in Fig.6.2 where it shows that all estimation errors of disturbance \mathbf{v}_c , target's position \mathbf{p}_T , target velocity \mathbf{u}_T and acceleration \mathbf{a}_T converge asymptotically to zero. This implies that with the trajectory of the trackers, both the target's state and the disturbance are fully observable. Note also that with measurement of the

Table 1. Simulation Setup

	Parameters
Target	Initial position: $\mathbf{p}_T(0) = [-0, -20, -5]$ (m)
	Initial velocity $\mathbf{u}_T(0) = [0.5, 0.5, -0.2]$ (m/s)
	Initial acceleration $\mathbf{a}_T(0) = [-0.02, -0.05, -0.01]$ (m/s ²)
Trackers	$\mathbf{p}_1(0) = [0, 0, 0]$ (m), $\mathbf{u}_1(t) = [25 \cos(0.5t), -25 \sin(0.5t), 0.5 \sin(t)]$ m/s
	Agent 2 is fixed with $\mathbf{p}_2(t) = \mathbf{p}_2(0) = [50, 50, -50]$ (m)
Disturbance	$\mathbf{v}_c = [0.2, -0.3, 0.2]$ m/s

second range from tracker 2, it is easy to see that with the second range from tracker 2, the convergence of the estimation errors are faster.

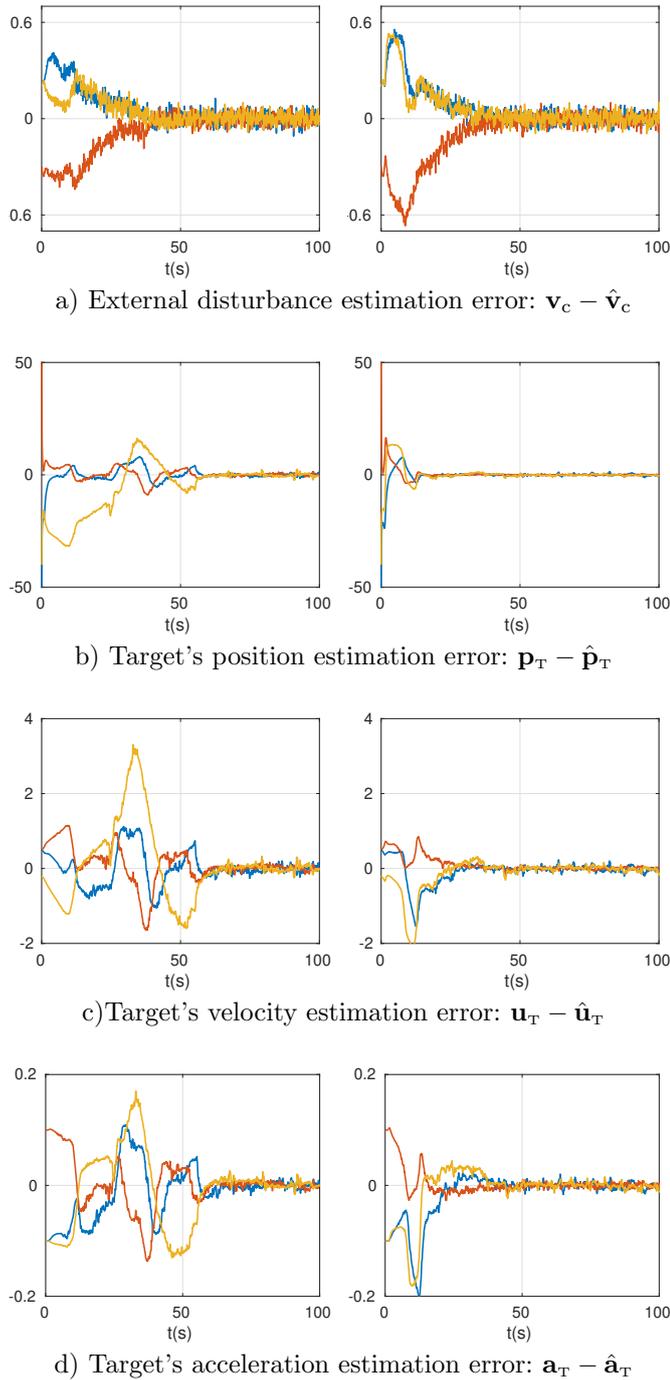


Fig. 6.2. Estimation errors with EKF. Left is with a single tracker (tracker 1) while Right is for two trackers (tracker 1 and 2).

7. CONCLUSIONS

We proposed a novel approach to study the observability problem of range-based navigation and target localization using one or two trackers. The approach uses simple tools to characterize the linear independence of a set of functions that was shown to be very efficient to derive conditions on

the motion of tracker(s) to ensure global observability of the target state. We also gave geometry interpretation of the conditions derived that can be used as guidelines to plan the motion of trackers.

REFERENCES

- Arrichiello, F., Antonelli, G., Aguiar, A., and Pascoal, A. (2013). An observability metric for underwater vehicle localization using range measurements. *Sensors*, 13(12), 16191–16215.
- Batista, P., Silvestre, C., and Oliveira, P. (2011). Single range aided navigation and source localization- observability and filter design. *Systems and Control Letters*, 60(8), 665 – 673. doi: <https://doi.org/10.1016/j.sysconle.2011.05.004>.
- Bayat, M., Crasta, N., Aguiar, A.P., and Pascoal, A.M. (2016). Range-based underwater vehicle localization in the presence of unknown ocean currents: Theory and experiments. *IEEE Transactions on Control Systems Technology*, 24(1), 122–139. doi: 10.1109/TCST.2015.2420636.
- Chen, C. (1984). Linear system theory and design.
- Crasta, N., Moreno-Salinas, D., Pascoal, A., and Aranda, J. (2018). Multiple autonomous surface vehicle motion planning for cooperative range-based underwater target localization. *Annual Reviews in Control*, 46, 326 – 342.
- Hermann, R. and Krener, A. (1977). Nonlinear controllability and observability. *IEEE Transactions on automatic control*, 22(5), 728–740.
- Indiveri, G., De Palma, D., and Parlangeli, G. (2016). Single range localization in 3-d: Observability and robustness issues. *IEEE Transactions on Control Systems Technology*, 24(5), 1853–1860. doi: 10.1109/TCST.2015.2512879.
- Masmitja, I., Gomariz, S., Del-Rio, J., Kieft, B., O'Reilly, T., Bouvet, P.J., and Aguzzi, J. (2018). Optimal path shape for range-only underwater target localization using a wave glider. *The International Journal of Robotics Research*, 37(12), 1447–1462.
- Palma, D.D., Arrichiello, F., Parlangeli, G., and Indiveri, G. (2017). Underwater localization using single beacon measurements: Observability analysis for a double integrator system. *Ocean Engineering*, 142, 650 – 665. doi:<https://doi.org/10.1016/j.oceaneng.2017.07.025>.
- Ristic, B., Arulampalam, S., and McCarthy, J. (2002). Target motion analysis using range-only measurements: algorithms, performance and application to isar data. *Signal Processing*, 82(2), 273 – 296.
- Song, T.L. (1999). Observability of target tracking with range-only measurements. *IEEE Journal of Oceanic Engineering*, 24(3), 383–387.