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Modeling of benchmark underactuated systems via different approaches

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Abstract: The presented paper deals with the area of research on multibody mechanical systems focused on underactuated systems, where the impact of the missing actuator is compensated by means of mathematical modeling. Several approaches based on the dynamics of individual bodies make it possible to obtain the mathematical model of the mechanical system. This paper focuses on summarizing and comparing Lagrange's and Kane's method, describing the steps that need to be applied to obtain a model for a particular mechanical system in each case. The methods are subsequently applied to selected benchmark underactuated systems such as cart-pole, acrobot, and reaction wheel pendulum, where the aim is to show that both produce the same equations of motion. The corresponding procedures are implemented in MATLAB as part of a custom application with graphical user interface.

Keywords: benchmark underactuated system, mathematical modeling, Lagrangian mechanics, Kane's method, equations of motion, MATLAB

1. INTRODUCTION

Although cybernetics is a young science created in the middle of the last century (Novikov, 2016), the foundations it is based on, such as system dynamics, had been studied decades to centuries before. In today's modern age of automation, when people are talking of a new industrial revolution (Vuksanovic et al., 2016), this science is an essential part of technical life and involves a lot of unanswered questions and tasks.

Modeling is a very important prerequisite for the analysis and control of nonlinear physical systems. It is based on basic principles that are valid in the field of cybernetics and system theory (Novikov, 2016). (Fantoni, 2002). The mathematical model, which replaces the physical system, is the most fundamental starting point for the analysis of nonlinear systems. It makes it easier to explore the observed system, to analyze its dynamics and to design control algorithms. Its use in simulation allows us to experiment with the model the same way as we would with a real object, but with significantly lower costs. MATLAB program environment, with its proprietary matrix-oriented language, graphical capabilities and specialized toolboxes, can be used as a suitable tool in all stages of modeling, analysis and control algorithm design.

As a significant category of physical systems, mechanical systems generally consist of n free particles with 3 degrees of freedom (Hamil, 2014; Goldstein et al., 2002). Studying underactuated systems that have fewer inputs than degrees of freedom is a broad-spectrum cybernetic and mechanical problem. Their extensive applications can be seen in the field of robotics or transport, but due to their complicated nonlinear dynamics, they are often studied on the basis of simpler benchmarks, most of which are based on pendulum dynamics (Spong et al., 2005). The pendulum is a typical nonlinear system, while also being relatively simple. Generally, it is one of the most important and most used mechanical systems that have been studied since Galileo. It was his study of the pendulum's motion that brought many questions that were later answered e.g. by Newton's laws of dynamics (Aström et al., 2007).

Mathematical modeling of underactuated systems yields equations of motion which are represented by highly nonlinear differential equations (Bremer, 2008). A number of approaches can be used to obtain these equations and although each is based on the model's dynamics, different physical laws, such as Newton's second law or d'Alembert principle, may apply (Goldstein et al., 2002) (Fantoni, 2002). One of the earliest modeling methods is the Newton's method, which has the disadvantages of inefficient computation and determination of all forces acting upon a system, including constraint forces. That is why the process is hard to algorithmize. On the other hand, energy-based methods, such as the one invented by Joseph-Louis Lagrange, do not consider these forces because they do not have a direct impact on the kinetic or potential energy of the system (Parsa, 2007). As a result, this method is easier to algorithmize, but its complexity arises with more complex systems, such as multibody spatial systems. These classical methods are complemented by Kane's method, which is rarely used because of its short history, but it has great potential in solving more complicated multibody systems due to less computational load and preserving the physical nature of the system, which we will discuss in this paper (Kane et al., 1985). Some of the example usage can be seen in (Gillespie, 2003; Chen et al., 2017) or (Sandino et al., 2011). Part of the modeling skill is to become familiar with available approaches with their

advantages and disadvantages, and to be able to choose the one which is most effective for a particular system.

The structure of this paper is as follows. Section 2 describes the process of obtaining the mathematical model for a multibody mechanical system by the Lagrangian mechanics and Kane's method, respectively. This is followed by section 3, where we present the most frequent underactuated benchmarks – cart-pole, acrobot and reaction wheel pendulum – and describe the process of obtaining their equations of motion using both methods. The final section 4 describes the implementation of methods into MATLAB using Symbolic Math Toolbox in order to generate the equations of motion, and the corresponding application to represent the results in a user-friendly manner. The paper concludes by evaluating the methods with respect to the stated objective and outlines the next research directions.

2. HOW TO OBTAIN THE EQUATIONS OF MOTION

In this section, we will analyze two approaches for obtaining the equations of motion for any mechanical system in more detail. Out of the available modeling methods, we have chosen the method based on Lagrangian mechanics (Parsa, 2007), which is frequently used to obtain motion equations, and the more recently introduced Kane's method, in which it is necessary to derive generalized active and inertia forces (Gillespie, 2003).

2.1 Lagrangian mechanics

Complexity of Newton-Euler approach was the main motivation behind the development of the Lagrangian view of mechanics, where the resulting second-order differential equations represent the dynamics of the individual degrees of freedom, each characterized by a generalized coordinate (Parsa, 2007). They define the position of a center of gravity for each system's particle in a reference frame relative to Newtonian reference frame (Gillespie, 2003).

As a fundamental principle of Lagrangian mechanics, the principle of virtual work says that the system stays in balance if the virtual work of all forces acting on the system is zero (Bremer, 2008), i.e.:

$$\delta w = \sum_{i=1}^{n} Q_i \delta q_i = 0, \tag{1}$$

where Q_i is the general force acting in the direction of the ith coordinate, q_i is the ith generalized coordinate.

We will consider the basic form of Lagrange equations of the second kind, defined as

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = Q_i \quad \text{for } i = 1, 2, \ldots, n, \tag{2}$$

where i is a degree of freedom index, L is the Lagrange's function (also referred to as Lagrangian or kinetic potential), defined as the difference of the overall kinetic and potential energy of the system (Goldstein et al., 2002):

$$L = E_k - E_p. \tag{3}$$

The total kinetic energy of the system is calculated as the sum of the kinetic energies of all mass points of the system, with the kinetic energy of the jth mass point expressed as (Spong et al., 2005)

$$E_{kj} = \frac{1}{2} m_j \big(v_{xj}^2 + v_{yj}^2\big) + \frac{1}{2} I_j \Big(\sum_{n=1}^{i} \dot{q}_n\Big)^2, \tag{4}$$

where v_{xj} and v_{yj} are the linear velocities of the center of gravity of the jth mass point in the direction of the coordinates, I_j is the moment of inertia of the jth mass point, $\dot{q}$ is the angular velocity of jth mass point.

The same principle applies to the total potential energy of the system, which is for the jth mass point obtained as (Spong et al., 2005)

$$E_{pj} = m_j g r_{yj}, \tag{5}$$

where m_j is the mass of j-th point, g is the gravitational acceleration, r_{yj} is the y-coordinate of the system's jth mass point.

2.2 Kane's method

We will next focus on the Kane's method, which is based on the derivation of physical quantities such as angular velocity and acceleration, linear velocity and acceleration, moment of all forces and generalized active and inertia forces acting on the system (Kane et al., 1985).

If the rigid body B moves within the reference frame A in such a way that there exists a unit vector $\boldsymbol{k}$ for a certain time interval t, whose orientation in A and B is independent of time, we can say that B has a *simple angular velocity* during the time interval t which can be defined as (Kane et al., 1985)

$${}^{A}\boldsymbol{\omega}^{B} = \boldsymbol{\omega}\boldsymbol{k}, \tag{6}$$

where $\boldsymbol{\omega}$ is the angle change over time

$$\boldsymbol{\omega} = \dot{\theta}, \tag{7}$$

and θ is the angle of rotation measured in radians and its orientation is fixed in both frames A and B. The θ sign is determined by the right hand rule.

The angular acceleration ${}^{A}\boldsymbol{\alpha}^{B}$ if the rigid body B in a reference frame A is defined as the first time derivative in A angular velocity B in A (Kane et al., 1983)

$${}^{A}\boldsymbol{\alpha}^{B} = \frac{{}^{A}d^{A}\boldsymbol{\omega}^{B}}{dt}. \tag{8}$$

Since the time derivative ${}^{A}\boldsymbol{\omega}^{B}$ in A and B is the same, we can say that

$${}^{A}\boldsymbol{\alpha}^{B} = \frac{{}^{B}d^{A}\boldsymbol{\omega}^{B}}{dt}. \tag{9}$$

Let $\boldsymbol{p}$ be a position vector from any point O moving in reference frame A and fixed to point P within A. The linear velocity P in A is given by the time derivative of the position vector $\boldsymbol{p}$

$${}^{A}\boldsymbol{v}^{P} = \frac{{}^{A}d\boldsymbol{p}^{P}}{dt} \tag{10}$$

and the linear acceleration of the point P within A is defined as

$${}^{A}\boldsymbol{a}^{P} = \frac{{}^{A}d\boldsymbol{v}^{P}}{dt}. \tag{11}$$

However, if two points P and Q having an angular velocity ${}^{A}\omega^{B}$ in A are attached to a rigid body B, then linear velocities ${}^{A}\boldsymbol{v}^{P}$ of the point P in A and ${}^{A}\boldsymbol{v}^{Q}$ in A are interrelated by (Kane et al., 1985)

$${}^{A}\boldsymbol{v}^{P} = {}^{A}\boldsymbol{v}^{Q} + {}^{A}\omega^{B} \times \boldsymbol{r}, \tag{12}$$

where $\boldsymbol{r}$ is the position vector from Q to P. In this case, the equation that applies for the acceleration is

$$^A\boldsymbol{a}^P = ^A\boldsymbol{a}^Q + ^A\boldsymbol{\omega}^B \times (^A\boldsymbol{\omega}^B \times \boldsymbol{r}) + ^A\boldsymbol{\alpha}^B \times \boldsymbol{r}. \quad (13)$$

Angular momentum principle of inertia

The angular momentum principle of inertia for any system S in main reference frame A defines the moment of all forces acting on the center of gravity S_m as the time derivative of system's angular momentum $^A\boldsymbol{H}^{S/S_m}$ about S_m within A (Mitiguy, 2017)

$$\boldsymbol{M}^{S/S_m} = \frac{^A d^A\boldsymbol{H}^{S/S_m}}{dt}, \quad (14)$$

while

$$^A\boldsymbol{H}^{S/S_m} = \ddot{I}^{S/S_m} \cdot ^A\boldsymbol{\omega}^{S_m}, \quad (15)$$

where $\ddot{I}^{S/S_m}$ is the inertia matrix defined as

$$\ddot{I} = \begin{bmatrix} I_{xx}ii + I_{xy}ij + I_{xz}ik \\ I_{yx}ji + I_{yy}jj + I_{yz}jk \\ I_{zx}ki + I_{zy}kj + I_{zz}kk \end{bmatrix}. \quad (16)$$

Generalized active forces

The generalized active forces depend on forces acting on the system S in the main reference frame A and the partial derivatives of angular and linear velocities. If $u_1, u_2, \ldots, u_n$ are generalized speeds for the system S in the A, then the generalized active forces are defined as (Gillespie, 2003; Kane et al., 1985)

$$F_u = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}^{P_i}}{\partial u} \cdot R_i + \frac{\partial \boldsymbol{\omega}^{P_i}}{\partial u} \cdot T_i \right), \quad (17)$$

where n is the number of particles that are part of the system S, P_i is a particular particle, $\boldsymbol{v}^{P_i}$ is the linear velocity of particle P_i, R_i is the result of all contact forces (such as frictional forces) and forces, such as gravity, magnetic, translational, and input forces, $\boldsymbol{\omega}^{P_i}$ is the angular velocity of particle P_i and T_i is the torque.

Generalized inertia forces

The generalized inertia forces depend on the forces that are derived by the 2nd Newton's law and the partial angular and linear velocities. If $u_1, u_2, \ldots, u_n$ are generalized speeds for the system S in the reference frame A, then the generalized inertia forces are defined (Gillespie, 2003; Kane et al., 1985)

$$^A F_u^S = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}^{P_i}}{\partial u} \cdot R_i^* + \frac{\partial \boldsymbol{\omega}^{P_i}}{\partial u} \cdot M_i \right), \quad (18)$$

where n is the number of particles that are part of the system S, P_i is a particular particle, v^{P_i} is the linear velocity of particle P_i, $\boldsymbol{\omega}^{P_i}$ is the angular velocity of particle P_i, M_i is the angular momentum and R_i^* is the force defined as

$$R_i^* = m^{P_i} a^{P_i}, \quad (19)$$

where m^{P_i} is the mass of the particle P_i and a^{P_i} is a linear velocity.

Equations of motion

The equations of motion derived by Kane's method are expressed from the equality of generalized active forces and inertia forces. If $u_1, u_2, \ldots, u_n$ are generalized speeds for system S in the reference frame A, then (Mitiguy, 2017):

$$F_u = ^A F_u^S. \quad (20)$$

As it can be seen from equations (2) and (20), there are several ways to construct the equations of motion of any system. In this section, we have shown Lagrange's and Kane's method. The goal for the following section is to use these methods to construct the equations of motion for basic underactuated benchmark systems and to prove that the results will be the same.

3. MATHEMATICAL MODELING OF BENCHMARK UNDERACTUATED SYSTEMS

A benchmark underactuated system is a system that may not have significant direct applications, but its features allow to analyze and verify a number of properties of complex systems on a simpler setup. Most common underactuated benchmarks include the pendulum on the cart (cart-pole), the acrobot and the reaction wheel pendulum (see Figure 1). All mentioned systems are underactuated, with two degrees of freedom but only a single actuator. The cart-pole system consists of a pendulum attached by a joint to a cart moving horizontally due to the force acting upon it. The Acrobot was inspired by the gymnast (acrobat), who hangs on the bar and moves only through the strength of the waist, and is therefore represented by a two-link planar robotic arm consisting of two pendulums that are connected by a joint with an actuator (Murray et al., 1991). The reaction wheel pendulum is composed of a pendulum with an actuated reaction wheel mounted on its end, rotating about an axis that is parallel to the pivot axis of the pendulum itself - as a result, the reaction wheel's angle acceleration is used to control the entire system (Spong et al., 2001). The most common control objective for these benchmarks involves stabilization of all pendulum links in the upward equilibrium.

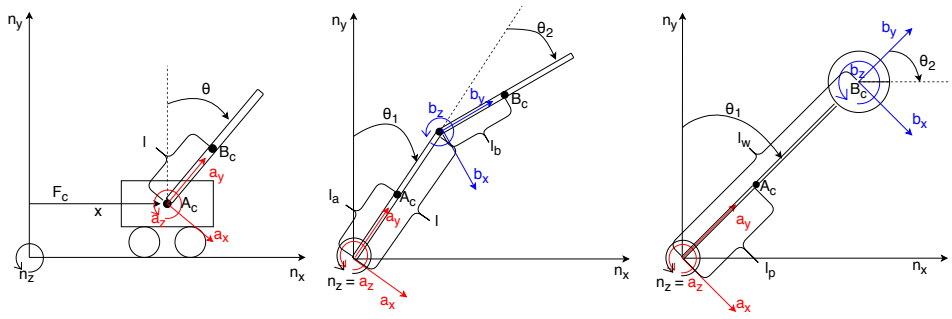

Fig. 1. Benchmark systems with their coordinate systems: cart pole, acrobot and reaction wheel pendulum, respectively

Table 1. x- and y-coordinates of mass points of benchmarks

	Cart pole	**Acrobot**	**Reaction wheel pendulum**
$\boldsymbol{r}_{xa}$	x	$l_a \sin\theta_1$	$l_p \sin\theta_1$
$\boldsymbol{r}_{ya}$	0	$l_p \cos\theta_1$	$l_a \cos\theta_1$
$\boldsymbol{r}_{xb}$	$x + l\sin\theta$	$l\sin\theta_1 + l_b \sin(\theta_1+\theta_2)$	$l_w \sin\theta_1$
$\boldsymbol{r}_{yb}$	$l\cos\theta$	$l\cos\theta_1 + l_b \cos(\theta_1+\theta_2)$	$l_w \cos\theta_1$

An accurate mathematical model is necessary for all manipulations or generalizations involving an underactuated benchmark. A set of equations of motion will now be derived for each system via both the Lagrange's and Kane's method and the results will be compared to verify the equivalence of obtained mathematical models.

Since all three benchmark systems can be described by an analogical set of parameters, let us define the nomenclature as follows:

- weight – $m[kg]$ – m_c, m_p, m_a, m_b, m_w
- length – $l[m]$ – l_p, l_a, l_b, l_w
- inertia – $I[kgm^2]$ – I_p, I_a, I_b, I_w

where indices a and b refer to the first and second pendulum respectively, index p refers to pendulum, w describes the reaction wheel and c makes reference to the cart.

3.1 Using Lagrangian mechanics

The first step of the method based on the Lagrange equations of the second kind consists of determining the x and y coordinates for the individual mass points (center of gravity) of the system. The center of gravity positions are listed in Table 1, where $\boldsymbol{r}_{xa}$ and $\boldsymbol{r}_{xb}$ represent the x-coordinate of the cart, the first pendulum, and the pendulum and $\boldsymbol{r}_{ya}$ and $\boldsymbol{r}_{yb}$ are y-coordinates of the pendulum, the second pendulum, and the reaction wheel.

To obtain the Lagrange function, kinetic and potential energies of the systems are first derived according to the formulas (4) and (5) based on the positions and their derivatives (i.e. velocities). They are all listed below for each benchmark system (E_k refers to the total kinetic energy and E_p is the total potential energy):

- Cart pole:

$$\begin{aligned} E_k &= \frac{1}{2}\dot{x}^2(m_c+m_p) + \dot{x}\dot{\theta}m_p l\cos\theta + \frac{1}{2}\dot{\theta}^2 m_p l^2 \\ E_p &= m_p l g\cos\theta \end{aligned} \tag{21}$$

- Acrobot:

$$\begin{aligned} E_k = &\frac{1}{2}\dot{\theta}_1^2(m_b l^2 + I_a + m_a l_a + 2m_b l l_b\cos(\theta_2) \\ &+ I_b + m_b l_b) + \dot{\theta}_1\dot{\theta}_2(I_b + m_b l l_b\cos(\theta_2) + m_b l_b^2) \\ &+ \frac{1}{2}\dot{\theta}_2^2(I_b + m_b l_b^2) \\ E_p = &\, m_a l_a g\cos(\theta_1) + m_b g(l\cos(\theta_1) + l_b\cos(\theta_1+\theta_2)) \end{aligned} \tag{22}$$

- Reaction wheel pendulum:

$$\begin{aligned} E_k &= \frac{1}{2}\dot{\theta}_1^2(m_p l_p^2 + m_w l_w^2 + I_p) + \frac{1}{2}(\dot{\theta}_1+\dot{\theta}_2)^2 I_w \\ E_p &= m_a l_a g\cos(\theta_1) + m_b g(l\cos(\theta_1) \\ &\quad + l_b\cos(\theta_1+\theta_2)) \end{aligned} \tag{23}$$

Based on the computed expressions for energies, the Lagrange function is formed according to (3) and the equations of motion are obtained via (2).

The resulting equations of motion for the cart-pole are given as

$$\begin{aligned} &\ddot{x}(m_c+m_p) - \dot{\theta}^2 m_p l\sin\theta + \ddot{\theta}m_p l\cos\theta = F_c \\ &\ddot{\theta}(m_p l^2 + I_p) + \ddot{x}m_p l\cos\theta = m_p l g\sin\theta, \end{aligned} \tag{24}$$

where F_c is the input force applied to cart.

Putting (2) and (3) together for the acrobot system results in:

$$\begin{aligned} &\ddot{\theta}_1(Q_1 + Q_2 + 2Q_3\cos\theta_2) + \ddot{\theta}_2(Q_2 + Q_3\cos\theta_2) \\ &-2\dot{\theta}_1\dot{\theta}_2 Q_3\sin\theta_2 - \dot{\theta}_2^2 Q_3\sin\theta_2 \\ &= Q_5 g\sin(\theta_1+\theta_2) + Q_4 g\sin\theta_1 \\ &\ddot{\theta}_1(Q_2 + Q_3\cos\theta_2) + \ddot{\theta}_2 Q_2 + \dot{\theta}_1^2 Q_3\sin\theta_2 \\ &= Q_5 g\sin(\theta_1+\theta_2) - \tau, \end{aligned} \tag{25}$$

where $Q_1 = m_b l^2 + m_a l_a^2 + I_a$, $Q_2 = I_b + m_b l_b^2$, $Q_3 = m_b l l_b$, $Q_4 = m_b l + m_a l_a$ a $Q_5 = m_b l_b$ and τ is the input torque.

The final motion equations for the reaction wheel pendulum can be written as

$$\begin{aligned} &\ddot{\theta}_1(m_p l_p^2 + m_w l_w^2 + I_p + I_w) + \ddot{\theta}_2 I_w \\ &= (m_p l_p + m_w l_w)\sin\theta_1 \\ &\ddot{\theta}_1 I_w + \ddot{\theta}_2 I_w = -\tau, \end{aligned} \tag{26}$$

where τ is the input torque.

This concludes the modeling section via the Lagrange method. For now, we can declare that these results are consistent with those found in frequently cited literature such as (Aström et al., 2007; Liu et al., 2013; Fantoni, 2002) or (Tedrake, 2012).

3.2 Using Kane's method

The starting point of the Kane's method is to correctly select positions and angular velocities. Positions were obtained based on coordinate systems in Figure 1 and are listed in Table 2 along with angular velocities derived by (6), where $\boldsymbol{r}_{A_c}$ is the position and ${}^N\boldsymbol{\omega}^{A_c}$ is the angular velocity of the cart/first pendulum/pendulum and $\boldsymbol{r}_{B_c}$ is the position and ${}^N\boldsymbol{\omega}^{B_c}$ is the angular velocity of the pendulum/second pendulum/reaction wheel. Then, by applying the formulas (9), (12) and (13) to the position vector and the angular velocity, we get the results that are represented in the Table 3. Afterwards, it is necessary to use for all angular velocities, which are nonzero, angular moment principle of inertia, to be able to derive the total moment of inertia for all system mass points.

For instance, we show how this is applied to a cart pole. Since ${}^N\boldsymbol{\omega}^{A_c}$ equals 0, it is pointless to derive ${}^N\boldsymbol{H}^{A/A_c}$ because it would lead to zero as well. On the other hand, for mass point B_c according to (14) we can write

Table 2. The center of gravity of the systems and their angular velocities

	Cart pole	**Acrobot**	**Reaction wheel pendulum**
$\boldsymbol{r}^{A_c}$	xn_x	$l_a a_y$	$l_p a_y$
$\boldsymbol{r}^{B_c}$	la_y	$l_b b_y$	$l_w a_y$
${}^N\boldsymbol{\omega}^{A_c}$	0	$\dot{\theta}_1 a_z$	$\dot{\theta}_1 a_z$
${}^N\boldsymbol{\omega}^{B_c}$	$\dot{\theta} a_z$	$(\dot{\theta}_1 + \dot{\theta}_2) b_z$	$(\dot{\theta}_1 + \dot{\theta}_2) a_z$

Table 3. Angular acceleration, linear velocity and acceleration of the pendulum system

	Cart pole	**Acrobot**	**Reaction wheel pendulum**
${}^N\boldsymbol{\alpha}^{A_c}$	0	$\ddot{\theta}_1 a_z$	$\ddot{\theta}_1 a_z$
${}^N\boldsymbol{\alpha}^{B_c}$	$\ddot{\theta} a_z$	$(\ddot{\theta}_1 + \ddot{\theta}_2) b_z$	$(\ddot{\theta}_1 + \ddot{\theta}_2) a_z$
${}^N\boldsymbol{v}^{A_c}$	$\dot{x} n_x$	$\dot{\theta}_1 l_a a_x$	$\dot{\theta}_1 l_p a_x$
${}^N\boldsymbol{v}^{B_c}$	$\dot{x} n_x + \dot{\theta} l a_x$	$\dot{\theta}_1 l a_x + (\dot{\theta}_1 + \dot{\theta}_2) l_b b_x$	$\dot{\theta}_1 l_w a_x$
${}^N\boldsymbol{a}^{A_c}$	$\ddot{x} n_x$	$\ddot{\theta}_1 l_a a_x \quad \dot{\theta}_1^2 l_a a_y$	$\ddot{\theta}_1 l_p a_x \quad \dot{\theta}_1^2 l_p a_y$
${}^N\boldsymbol{a}^{B_c}$	$\ddot{x} n_x \quad \dot{\theta}^2 l a_y + \ddot{\theta} l a_x$	$\ddot{\theta}_1 l a_x \quad \dot{\theta}_1^2 l a_y \quad b_y l_b (\dot{\theta}_1 + \dot{\theta}_2)^2 + b_x l_b (\ddot{\theta}_1 + \ddot{\theta}_2)$	$\ddot{\theta}_1 l_w a_x \quad \dot{\theta}_1^2 l_w a_y$

$$
\begin{aligned}
{}^N\boldsymbol{H}^{A/B_c} &= \begin{bmatrix} I_{xx} a_x a_x + I_{xy} a_x a_y + I_{xz} a_x a_z \\ I_{yx} a_y a_x + I_{yy} a_y a_y + I_{yz} a_y a_z \\ I_{zx} a_z a_x + I_{zy} a_z a_y + I_{zz} a_z a_z \end{bmatrix} (-\dot{\theta} a_z) \\
&= -I_{xz} \dot{\theta} a_x - I_{yz} \dot{\theta} a_y - I_{zz} \dot{\theta} a_z,
\end{aligned} \tag{27}
$$

by deriving which we obtain the moments of inertia as

$$
\begin{aligned}
&\boldsymbol{M}^{A_c} = 0 \\
&\boldsymbol{M}^{B_c} = (\dot{\theta}^2 I_{xz} - \ddot{\theta} I_{yz}) a_y - (\dot{\theta}^2 I_{yz} + \ddot{\theta} I_{xz}) a_x \\
&-\ddot{\theta} I_{zz} a_z
\end{aligned} \tag{28}
$$

For the acrobot system, the angular momentum principle of inertia yields

$$
\begin{aligned}
&{}^N\boldsymbol{H}^{A/A_c} = -I_{xz} \dot{\theta}_1 a_x - I_{yz} \dot{\theta}_1 a_y - I_{zz} \dot{\theta}_1 a_z \\
&{}^N\boldsymbol{H}^{B/B_c} = -I_{xz} (\dot{\theta}_1 + \dot{\theta}_2) b_x - I_{yz} (\dot{\theta}_1 + \dot{\theta}_2) b_y \\
&-I_{zz} (\dot{\theta}_1 + \dot{\theta}_2) b_z
\end{aligned} \tag{29}
$$

The angular momentum of all mass points of the acrobot can be expressed as:

$$
\begin{aligned}
&\boldsymbol{M}^{A_c} = (\dot{\theta}_1^2 I_{xz} - \ddot{\theta}_1 I_{yz}) a_y - (\dot{\theta}_1^2 I_{yz} + \ddot{\theta}_1 I_{xz}) a_x - \ddot{\theta}_1 I_{zz} a_z \\
&\boldsymbol{M}^{B_c} = ((\dot{\theta}_1 + \dot{\theta}_2)^2 I_{xz} - (\ddot{\theta}_1 + \ddot{\theta}_2) I_{yz}) b_y \\
&-((\dot{\theta}_1 + \dot{\theta}_2)^2 I_{yz} + (\ddot{\theta}_1 + \ddot{\theta}_2) I_{xz}) b_x - (\ddot{\theta}_1 + \ddot{\theta}_2) I_{zz} b_z
\end{aligned} \tag{30}
$$

We get similar results for the reaction wheel pendulum:

$$
\begin{aligned}
&{}^N\boldsymbol{H}^{A/A_c} = -I_{xz} \dot{\theta}_1 a_x - I_{yz} \dot{\theta}_1 a_y - I_{zz} \dot{\theta}_1 a_z \\
&{}^N\boldsymbol{H}^{A/B_c} = -I_{xz} (\dot{\theta}_1 + \dot{\theta}_2) a_x - I_{yz} (\dot{\theta}_1 + \dot{\theta}_2) a_y \\
&-I_{zz} (\dot{\theta}_1 + \dot{\theta}_2) a_z
\end{aligned} \tag{31}
$$

If we implement the angular momentum principle for reaction wheel pendulum, as the result we are given

$$
\begin{aligned}
&\boldsymbol{M}^{A_c} = (\dot{\theta}_1^2 I_{xz} - \ddot{\theta}_1 I_{yz}) a_y - (\dot{\theta}_1^2 I_{yz} + \ddot{\theta}_1 I_{xz}) a_x - \ddot{\theta}_1 I_{zz} a_z \\
&\boldsymbol{M}^{B_c} = ((\dot{\theta}_1 + \dot{\theta}_2)^2 I_{xz} - (\ddot{\theta}_1 + \ddot{\theta}_2) I_{yz}) a_y \\
&-((\dot{\theta}_1 + \dot{\theta}_2)^2 I_{yz} + (\ddot{\theta}_1 + \ddot{\theta}_2) I_{xz}) a_x - (\ddot{\theta}_1 + \ddot{\theta}_2) I_{zz} a_z
\end{aligned} \tag{32}
$$

The resulting equations of motion are obtained by calculating generalized active and inertia forces according to (17) and (18), where I_{zz} can be expressed as $I_a(I_p)$ or $I_b(I_w)$ for mass points A_c or B_c respectively. Other inertias are eliminated during the process of calculating generalized forces, because a_z and b_z are the rotating axes perpendicular to plane and time invariant both in A and B. The derived motion equations match the equations (24), (25), and (26), which resulted from the method based on Lagrange's equations.

4. MATLAB IMPLEMENTATION

In this section, an application with graphical user interface called *Modeling Selected Underactuated Systems*, which generates the motion equations for a selected system and then displays them in a user-friendly manner, is presented. We used MATLAB and its Symbolic Math Toolbox to implement the step-by-step process used by each modeling method described in previous sections.

The application features a menu in the top left corner where the user can choose a system whose equations of motion are required (the three systems described in this paper are available to choose from so far). In addition, the method (Lagrange's/Kane's) is selected by which the equations need to be derived. When the user presses the *Derive EoM* button, the interface behavior is triggered by calling either the *Lagrange_method.m* or the *Kane_method.m* function according to the user's choice. After executing the selected function, the resulting equations of motion of the system are displayed. On the right, the schematic of the system is depicted for better understanding.

Since both methods produce the same equations of motion, we also added steps to track the difference in the procedure of obtaining them. In the case of Lagrange method, we chose x and y-coordinates of the system rx and ry, and linear velocities vx and vy, from which the kinetic E_k and potential E_p energy of the whole system can be calculated. On the other hand, if the user selects Kane's method, the generated result includes the positions r, angular w and linear v velocities, from which accelerations *alpha,* a and finally moments Ma, Mb are obtained. As it can be seen from Figure 2, the final equations of motion of the presented system (acrobot) are equivalent for both methods and they are also identical to (25), as derived in Section 2.

5. CONCLUSION

The aim of this paper was to analyze two methods of mathematical modeling of mechanical systems – Lagrange's and Kane's method – and to apply them to selected underactuated benchmarks, all based on the inverted pendulum. Due to the need to compensate for a missing actuator, an accurate mathematical model is necessary for any control algorithm design involving an underactuated system.

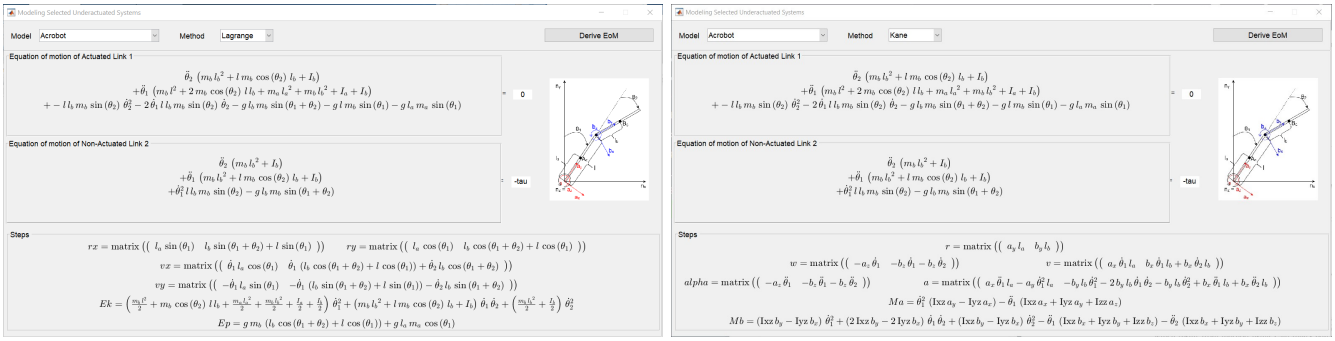

Fig. 2. Derivation of motion equations via Lagrange's and Kane's method, respectively

The conclusion of this analysis is that while the resulting equations are the same for these basic benchmarks, Kane's method is less laborious, which could prove especially useful for complex systems, such as multi-body systems constituting a spatial configuration with a higher number of degrees of freedom. We also see the advantage of the Kane's method in greater systematicity, better representation of the system's physical nature and computationally undemanding steps. In a majority of cases, computations include dot or cross product, while the Lagrange method requires mostly partial derivations or time derivations - therefore, even in terms of computational load, Kane's method is less demanding.

For all the reasons mentioned above, we consider Kane's method to be a more appropriate alternative to be used in mathematical model derivation for advanced systems. Therefore, we plan to use it in our upcoming research by applying it to spatial systems such as the rotary inverted pendulum and the three-dimensional overhead crane, followed by systems consisting of multiple bodies with a greater number of degrees of freedom, such as robotic arms or the mechatronic cube composed of three reaction wheels, balancing on the edge or the corner.

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