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Lyapunov characterizations of input-to-state stability for discrete-time switched systems via finite-step Lyapunov functions

Maryam Sharifi * **Navid Noroozi** ** **Rolf Findeisen** ***

* *School of Electrical and Computer Engineering, University of Tehran, Tehran, Iran (e-mail: sharifi.m@ ut.ac.ir).*
** *Laboratory for Systems Theory and Automatic Control, Otto-von-Guericke University Magdeburg, Germany. (e-mail: navid.noroozi@ovgu.de).*
*** *Laboratory for Systems Theory and Automatic Control, Otto-von-Guericke University Magdeburg, Germany. (e-mail: rolf.findeisen@ovgu.de).*

Abstract: This paper addresses Lyapunov characterizations of input-to-state stability for nonlinear switched discrete-time systems via finite-step Lyapunov functions with respect to *closed* sets. The use of finite-step Lyapunov functions permits not-necessarily input-to-state stable systems in the systems family, while input-to-state stability of the resulting switched system is ensured. The result is generally presented for systems under arbitrary switching. It additionally covers the case of constrained switchings. We illustrate the effectiveness of our results by application to networked control systems with periodic scheduling policies under a priori known and dwell time-based switching mechanism.

Keywords: Discrete-time switched systems, Lyapunov methods, input-to-state stability, finite-step Lyapunov functions, networked control systems

1. INTRODUCTION

A discrete-time switched system consists of a family of discrete-time systems and a switching signal orchestrating the switching between these systems, where switching signals can behave either arbitrarily or under certain constraints. For a discrete-time switched system with a given switched digraph, arbitrary switching reflects that there is no restriction on the switching instants. On the other hand, constraints on the switching signals may be due to nature of the system. For instance, in networked control systems a scheduling protocol, such as standard Round-Robin protocol (Walsh and Ye, 2001), determines certain switching times at which plant(s) and controller(s) can communicate with each other.

Discrete-time switched systems are widely used to describe engineering applications; of our particular interest, networked control systems (NCSs) (Gatsis et al., 2018; Donkers et al., 2011; Hespanha et al., 2007). In a standard NCS setup, a communication channel is usually shared between multiple physically decoupled control systems, in which at each transmission instant only a few plants and their corresponding controllers can exchange their information over the communication network; see Fig. 1.

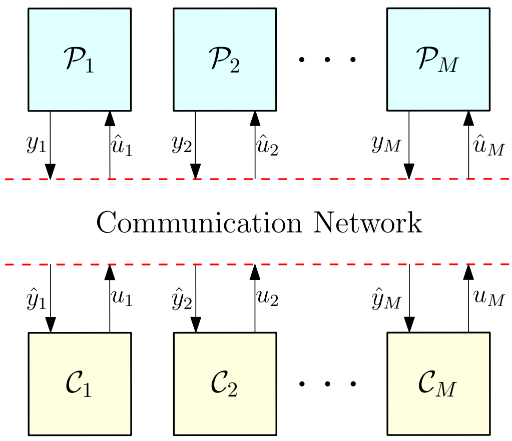

Fig. 1. The implementation of M independent control loops over a single communication channel, taken from (Noroozi et al., 2020).

The remaining plants have to operate in open-loop until they are granted access to the communication network at next transmission instants. In that way, each resulting control system can be viewed as a switched system where the control system infinitely often switches between open-loop and closed-loop operations.

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There is a large body of literature on stability analysis and stabilization of discrete-time switched systems over the last few decades; see, e.g., (Fiacchini et al., 2018; Sun and Ge, 2011; Lin and Antsaklis, 2009) and references therein. Formulating Lyapunov and input-output stability theories in a unified manner, the notion of input-to-state stability (ISS) (Sontag, 1989) plays a key role in analysis and design of nonlinear control systems. In (Kundu and Chatterjee, 2017) , a class of ISS stabilizing switching signals via a graph theoretic approach is developed, in which non-ISS systems in the systems family are permitted. A converse Lyapunov theorem of ISS for discrete-time switched systems under arbitrary switching is given in (Pepe, 2018).

Classically a Lyapunov function has to decay at every time step to guarantee stability of a system. [1] The concept of finite-step Lyapunov functions, introduced by (Aeyels and Peuteman, 1998), relaxes this requirement. A finite-step Lyapunov function needs to satisfy a dissipation condition only after some constant, but finite number of steps. In (Geiselhart et al., 2014), finite-step Lyapunov functions and their connections to classic Lyapunov functions are deeply investigated. In particular, they provide a *constructive* [2] converse Lyapunov theorem for global asymptotic stability of discrete-time systems using finite-step Lyapunov functions. Motivated by the constructiveness of converse Lyapunov theorems obtained from finite-step Lyapunov functions, the notion of fs-CLF is introduced and used for development of optimization-based control schemes in NCSs (Noroozi et al., 2020), see also further applications in (Noroozi et al., 2018b). Interestingly enough, finite-step Lyapunov functions are particularly useful to develop *non-conservative* dissipativity and small-gain theorems (Gielen and Lazar, 2015; Geiselhart and Wirth, 2016; Noroozi et al., 2018a). As regards switched systems, in (She et al., 2017; Lu et al., 2018), some sufficient conditions for asymptotic stability of discrete-time switched systems via finite-step Lyapunov functions are given, where all systems in the family are stable modes and switching rules are constrained.

In this paper, we provide Lyapunov characterizations (i.e. both *necessary and sufficient* conditions) of ISS with respect to *closed* sets via finite-step Lyapunov functions for nonlinear discrete-time switched systems. We note that ISS with respect to closed sets brings diverse variants of ISS including standard ISS with respect to the origin, independent-input-to-output stability, partial ISS, incremental ISS, ISS for time-varying systems and so on under the same umbrella and provides a large applicability of our results to various problems including adaptive control, consensus problems, observer design etc.; see (Noroozi et al., 2018a, Section V) for more details. In that way, our results extend those in (Pepe, 2018), where ISS of such systems with respect to the origin through standard Lyapunov functions are addressed. Given a switched digraph associated with the family of systems, our results are mainly developed for the case of arbitrary switching signals. As this case addresses the worst case scenario, our results also cover dwell time switching and a priori known switching cases. Interestingly enough, in the case of constrained switching our approach allows for *non*-ISS modes (i.e. zero-input unstable systems) in the systems family, while ISS of the resulting system is ensured. This is our other contribution over (Pepe, 2018) in which all systems in the family have to be ISS. To support the theoretical analyses, application to NCSs with periodic scheduling policies is provided.

The paper is organized as follows. Some relevant notation and problem statement are given in Section 2. Main results are provided in Section 3. An application of our results to networked control systems with some numerical simulations is given in Section 4. Section 5 concludes the paper.

2. PROBLEM FORMULATION

2.1 Notation

In this paper, $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ and $\mathbb{N}(\mathbb{N}_0)$ denote the nonnegative (positive) real numbers and the nonnegative (positive) integers, respectively. By $|\cdot|$, we denote an arbitrary vector norm on $\mathbb{R}^n$. Given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ and any point $x \in \mathbb{R}^n$, we denote $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$.

Given a function $\varphi \colon \mathbb{N} \to \mathbb{R}^m$, its sup-norm (possibly infinite) is denoted by $\|\varphi\| = \sup\{|\varphi(k)| : k \in \mathbb{N}\} \leq \infty$. The set of all functions $\mathbb{N} \to \mathbb{R}^m$ with finite sup-norm is denoted by ℓ_∞. A function $\rho \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive (semi-) definite if it is continuous, zero at zero and positive (nonnegative) otherwise. A positive definite function α is of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ if $s \to \infty$. A continuous function $\beta \colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$, and for each $r \geq 0$, $\beta(r, \cdot)$ is decreasing with $\beta(r, s) \to 0$ as $s \to \infty$. The identity function is denoted by id. Composition of functions is denoted by the symbol $\circ$ and repeated composition of, e.g., a function γ by γ^i. For positive definite functions α, γ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$.

2.2 System description

Consider the following switched discrete-time system

$$x(k+1) = G_{\sigma(k)}(x(k), u(k)), \quad k \in \mathbb{N}, \tag{1}$$

where $x(k) \in \mathbb{R}^n$; $u(k) \in \mathbb{R}^m$ is the input signal; $\sigma : \mathbb{N}_0 \to S$ is a function (the switching signal) for $S = (1, \ldots, p)$ a finite set with p a positive integer. We denote the set of all such switching signal by $\mathcal{S}$.

Given an initial condition $x(0) =: \xi \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$, $u \in \ell_\infty$, by $x(\cdot, \xi, \sigma, u)$ we mean a solution to system (1). While speaking about the solution in the absence of control input, we drop u from the arguments of x and simply denote a system solution with $x(\cdot, \xi, \sigma)$.

Definition 1. A continuous and positive semidefinite function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a measurement function. □

A measurement function ω gives the distance to a closed set with respect to which we aim to analyze input-to-state stability of the system (see Remark 1 below).

[1] Here we consider discrete-time systems.

[2] The converse Lyapunov theorem is constructive for control purposes in the sense that it provides an explicit way of construction of a Lyapunov function for control systems.

Definition 2. Given a measurement function ω, the function $G_\sigma(x,u): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called to be $\mathcal{K}$-bounded with respect to ω, uniformly in σ, if there exist class $\mathcal{K}$ functions κ_1 and κ_2 such that $\omega(G_\sigma(\xi,\mu)) \le \kappa_1(\omega(\xi)) + \kappa_2(|\mu|)$ for all $\xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$. □

The concept of $\mathcal{K}$-boundedness was introduced in (Lazar et al., 2013) for $\omega(\cdot) = |\cdot|$. Note that $\mathcal{K}$-boundedness is a *necessary* condition for input-to-state stability, see (Geiselhart and Wirth, 2016, Remark 3.3) and is weaker than continuity (Geiselhart and Noroozi, 2017).

In the following, we provide *necessary and sufficient conditions* for input-to-state stability of family systems (1) with respect to a measurement function ω via so-called finite-step Lyapunov functions.

3. FINITE-STEP LYAPUNOV FUNCTIONS FOR SWITCHED SYSTEMS

The aim of this section is to develop necessary and sufficient Lyapunov conditions for input-to-state stability of nonlinear switched discrete-time systems. Three forms of ISS Lyapunov functions are provided to ensure ISS of system (1) via finite-step Lyapunov functions. We first introduce the notions of input-to-state stability with respect to a measurement function.

Definition 3. The family of systems (1) is uniformly input-to-state stable with respect to a measurement function ω (ωUISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any initial condition $\xi \in \mathbb{R}^n$, any input $u \in \ell_\infty$ and any switching signal $\sigma \in \mathcal{S}$, the solution of the system satisfies the following inequality for any $k \in \mathbb{N}_0$

$$\omega(x(k,\xi,\sigma,u)) \le \beta(\omega(\xi),k) + \gamma(\|u\|). \tag{2}$$

□

Remark 1. Input-to-state stability with respect to a measurement function ω, in general, covers ISS with respect to a *closed* set $\mathcal{A}$ if we take the measurement function ω as the distance to a closed set with respect to the set $\mathcal{A}$ (i.e. $\omega(\cdot) = |\cdot|_{\mathcal{A}}$). In particular, if the measurement function is chosen as $\omega(\cdot) = |\cdot|$ (i.e. $\mathcal{A} = \{0\}$), ωISS reduces to the *standard* ISS with respect to the origin. In addition to that, ISS with respect to a measurement function, in general, subsumes a wide variety of stability properties including incremental stability, independent-input-to-output stability, partial ISS, ISS for time-varying systems. □

Definition 4. Let ω be a measurement function. Consider a family of functions $V_j: \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $j \in S$. Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$, $j \in S$,

$$\alpha_1(\omega(\xi)) \le V_j(\xi) \le \alpha_2(\omega(\xi)). \tag{3}$$

The functions V_j are called
1) max-form ωISS finite-step Lyapunov functions for system (1) if there exist $M \in \mathbb{N}_0$, $\alpha_m \in \mathcal{K}_\infty$ with $\alpha_m < \mathrm{id}$ and $\gamma_m \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ and any input signal $u \in \ell_\infty$,

$$V_{\sigma(M)}(x(M,\xi,\sigma,u)) \le \max\{\alpha_m(V_{\sigma(0)}(\xi)), \gamma_m(\|u\|)\}. \tag{4}$$

2) implication-form ωISS finite-step Lyapunov functions for system (1) if there exist $M \in \mathbb{N}_0$, a positive definite function α_i and a function $\gamma_i \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ and any input signal $u \in \ell_\infty$,

$$\begin{aligned} &V_{\sigma(0)}(\xi) \ge \gamma_i(\|u\|) \Rightarrow \\ &V_{\sigma(M)}(x(M,\xi,\sigma,u)) - V_{\sigma(0)}(\xi) \le -\alpha_i(V_{\sigma(0)}(\xi)). \end{aligned} \tag{5}$$

3) dissipative-form ωISS finite-step Lyapunov functions for system (1) if there exist $M \in \mathbb{N}_0$, functions $\alpha_d \in \mathcal{K}_\infty$ and $\gamma_d \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ and any input signal $u \in \ell_\infty$,

$$V_{\sigma(M)}(x(M,\xi,\sigma,u)) - V_{\sigma(0)}(\xi) \le -\alpha_d(V_{\sigma(0)}(\xi)) + \gamma_d(\|u\|). \tag{6}$$

□

In the following, we establish the equivalence between these three forms of (finite-step) Lyapunov functions.

Theorem 1. Consider the family of systems (1), where the system dynamics is $\mathcal{K}$-bounded. The following properties are equivalent.
1. The family of systems (1) admits a set of max-form ωISS finite-step Lyapunov functions.
2. The family of systems (1) admits a set of implication-form ωISS finite-step Lyapunov functions.
3. The family of systems (1) admits a set of dissipative-form ωISS finite-step Lyapunov functions.
4. The family of system (1) are ωUISS. □

The proof is not presented due to space constraints. Basically it follows similar steps as those for the proof of (Noroozi et al., 2018a, Theorem 7).

Remark 2. We note that in Theorem 1 we do not make any assumption on the switching times. Therefore, it can be applied to both arbitrary and constrained switching signals. In terms of constrained switchings, in the next section, we consider two case: dwell-time switching, where the active system always dwells for a certain and finite number of time steps; known switching, where the switching signal is a priori given. □

4. APPLICATION

4.1 Networked control systems under periodic scheduling

To illustrate the effectiveness of our results, we consider an application to the networked control systems (NCSs). Consider the standard NCS depicted in Fig. 1. As discussed earlier, plants and controllers exchange their information only at some given transmission instants. At each transmission instant, one control loop is granted access to the network, which is selected by the scheduling protocol. The allocation scheme of communication resources is broadly divided into static and dynamic protocols (Walsh and Ye, 2001). The former type of resource allocation is frequently performed by determining a finite length of sharing the communication channel in a periodic fashion. Here we focus on *periodic* scheduling policies.

The evolution of a plant whose controller implemented over a communication network can be described by a discrete-time switched system, where the resulting control system switches between an open-loop and closed-loop operation. We permit the plant to be unstable in open-loop, whereas we assume that dynamics of the control system in closed-loop is contractive. Therefore, each control loop resembles a switched system which contains both stable and unstable systems in the family.

The aim is to guarantee ISS of each control loop using the concept of finite-step Lyapunov functions. In other words, although the switched system contains an unstable system in the family, existing finite-step Lyapunov functions guarantee that after some steps M, they will decay and stability is preserved for the control system. The dynamics of each plant can be given by

$$x(k+1) = G_{\sigma(k)}(x(k)), \quad \sigma(k) \in \{1, 2\}, \tag{7}$$

where $x \in \mathbb{R}^n$, and the switching signal $\sigma(k)$ determines the scheduling policy. If $\sigma = 1$, system (7) operates in open-loop and while $\sigma = 2$ means that the control loop is closed, i.e. the plant and the controller communicate with each other over the communication network. Therefore, $G_1(x)$ governs the evolution of the unstable system and $G_2(x)$ does it for the stable one.

We propose a systematic way to construct a finite-step Lyapunov function for linear NCSs. Assume that dynamics G_1 and G_2 are linear. In that way, the family of systems (7) is simplified to

$$x(k+1) = A_{\sigma(k)}x(k), \quad \sigma \in \{1, 2\}. \tag{8}$$

Assume that every M time steps the plant and respective controller transmit data over the communication network. To analyze the stability of the origin, we take a finite-step Lyapunov function candidate of the form $V_i(x) = x^\top P_i x, i \in \{1, 2\}$ with symmetric, positive definite matrices P_i. Note that A_1, P_1 correspond to the unstable subsystem and A_2, P_2 deal with the stable one.

Feasibility of the following M linear matrix inequalities (LMIs) implies that there exist finite-step Lyapunov functions for the system (8).

$$\Psi_r < 0, \quad r \in [0, M-1], \tag{9}$$

where

$$\begin{aligned}\Psi_0 = &\sum_{h=0}^{M-3} (A_1^h)^\top (A_1^\top P_1 A_1 - P_1)(A_1^h) \\ &+ (A_1^{M-2})^\top (A_2^\top P_2 A_2 - P_1)(A_1^{M-2}) \\ &+ (A_2 A_1^{M-2})^\top (A_1^\top P_1 A_1 - P_2)(A_2 A_1^{M-2}),\end{aligned}$$

and

$$\begin{aligned}\Psi_1 = &\sum_{h=0}^{M-4} (A_1^h)^\top (A_1^\top P_1 A_1 - P_1)(A_1^h) \\ &+ (A_1^{M-3})^\top (A_2^\top P_2 A_2 - P_1)(A_1^{M-3}) \\ &+ (A_2 A_1^{M-3})^\top (A_1^\top P_1 A_1 - P_2)(A_2 A_1^{M-3}) \\ &+ (A_1 A_2 A_1^{M-3})^\top (A_1^\top P_1 A_1 - P_1)(A_1 A_2 A_1^{M-3}).\end{aligned}$$

The other Ψ_r's ($r \in [2, M-1]$) do not follow predefined formulations as Ψ_0 and Ψ_1. However, they can be readily expressed for each given M in a similar fashion. In addition, the Lyapunov functions can be accordingly found by solving (9) for P_i's, $i \in \{1, 2\}$.

For the proof, consider the discrete-time dynamic behavior of the system at each time step. The evolution of the Lyapunov functions in M steps can be written as

$$V_i(x(M+r)) - V_i(x(r)) = x(r)^\top \Psi_r x(r), (i \in \{1, 2\})$$

for all $\xi \in \mathbb{R}^n$ and integer $r \in [0, M-1]$.

4.2 Simulation results

Here input-to-state stability of NCS (8) with respect to the origin is verified, where the system is subject to an additive disturbance. In the first example, we consider a priori known switching scenario, while the second example addresses a dwell time-based case.

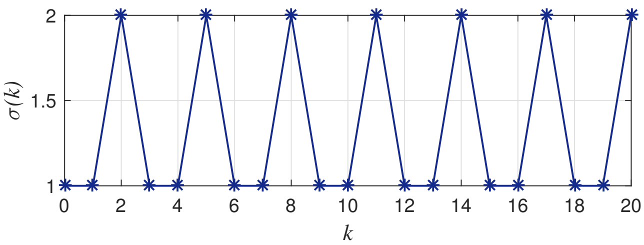

Fig. 2. Scheduling policy.

Example 1: Specify the family of systems (8) with the following matrices.

$$A_1 = \begin{bmatrix} 0.5 & 1.2 \\ 1.5 & -0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}. \tag{10}$$

The first system, which is unstable corresponds to the open-loop operation and the second one, which is stable, corresponds to the case in which the plant and the controller communicate with other. The switching signal (i.e. the scheduling policy) is given by

$$\sigma(k) = \begin{cases} 1 & \forall k \in \{3n, 3n+1\}, n \in \mathbb{N} \\ 2 & \forall k \in \{3n+2\}, n \in \mathbb{N}, \end{cases} \tag{11}$$

which is also depicted by Fig. 2. This switched policy confirms that the plant has access to its control command periodically at $k \in \{3n+2\}$. As discussed above, this motivates us to take $M = 3$ for the finite-step Lyapunov functions.

Choose $V_1(x) = 0.25x_1^2 + x_2^2$ and $V_2(x) = 0.1x_1^2 + 0.04x_2^2$ as the corresponding Lyapunov functions. In this case, the eigenvalues of matrices Ψ_0, Ψ_1 are identical and computed as $\{-0.1843, -0.7373\}$. Additionally, those of Ψ_2 are $\{-0.0737, -0.0295\}$.

By calculating V_j's at different instants, we have

$$V_1(x(3, \xi, \sigma)) - V_1(\xi) = -0.18\xi_1^2 - 0.74\xi_2^2,$$
$$V_2(x(5, \xi, \sigma)) - V_2(x(2, \xi, \sigma)) = -0.31\xi_1^2 - 1.24\xi_2^2.$$

Given the switching signal σ, state trajectories of system (8) with initial condition $\xi = (-0.5, -0.1)$ are shown in Fig. 3. Moreover, the evolution of the corresponding active V_σ is depicted by Fig. 4.

Consider system (8) subject to disturbance signal $u \in \mathbb{R}^2$ as an input, which is represented by

$$x(k+1) = A_{\sigma(k)}x(k) + u(k). \tag{12}$$

Considering system (12), we get

$$\begin{aligned}&V_1(x(3, \xi, \sigma, u)) - V_1(\xi) = -0.184\xi_1^2 - 0.737\xi_2^2 \\ &+ 0.38u_1\xi_2 + 0.35u_1\xi_1 + 1.15u_2\xi_2 + 0.08u_2\xi_1 \\ &+ 0.61u_1^2 + 1.28u_2^2 + 1.05u_1u_2 = \\ &- 0.1[(\xi_1 - 1.75u_1 - 0.385u_2)^2 + (\xi_2 - 1.9u_1 - 5.75u_2)^2] \\ &- 0.084\xi_1^2 - 0.637\xi_2^2 + 1.28u_1^2 + 4.6u_2^2 + 2.28u_1u_2 \\ &\leq -\alpha_3(|\xi|) + \alpha_4(\|u\|).\end{aligned}$$

where $\alpha_3(|\xi|) \leq 0.084\xi_1^2 + 0.637\xi_2^2$ and $1.28u_1^2 + 4.62u_2^2 + 2.28u_1u_2 \leq \alpha_4(\|u\|)$. This gives ISS of system (12) with

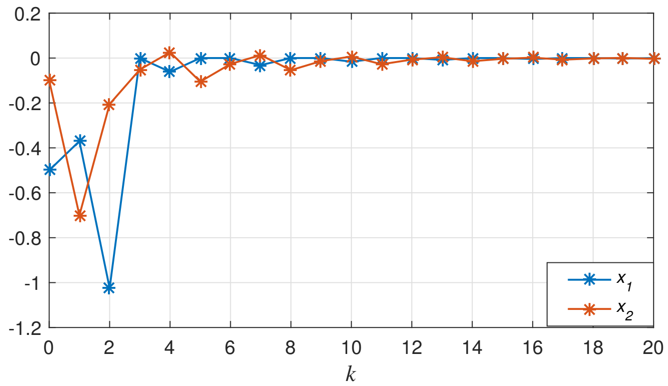

Fig. 3. State trajectories of system (8) with $u(\cdot) \equiv 0$.

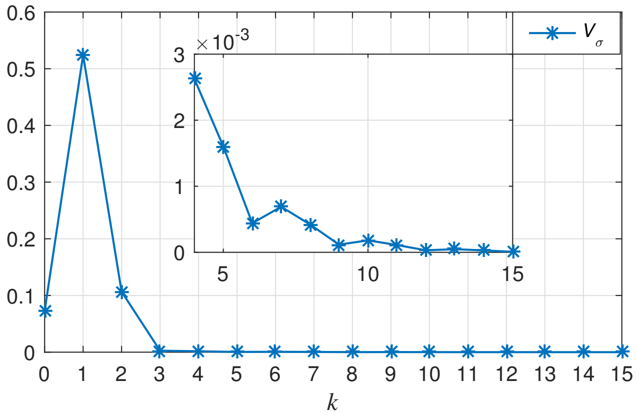

Fig. 4. Lyapunov function evolution for system (8) with $u(\cdot) \equiv 0$.

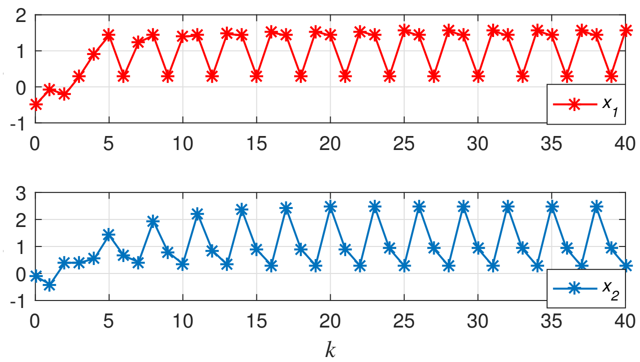

Fig. 5. State trajectories of system (12) with the input $u(\cdot) = (0.3, 0.3)$.

respect to the origin. For simulations, we pick $u(\cdot) = (0.3, 0.3)$. The state trajectories are illustrated in Fig. 5.

Example 2: Consider system (8) with the same subsystems as (10). We verify ISS under dwell time-based switchings by taking 2 time steps for the activation of subsystems between successive switchings.

The switching signals are given by

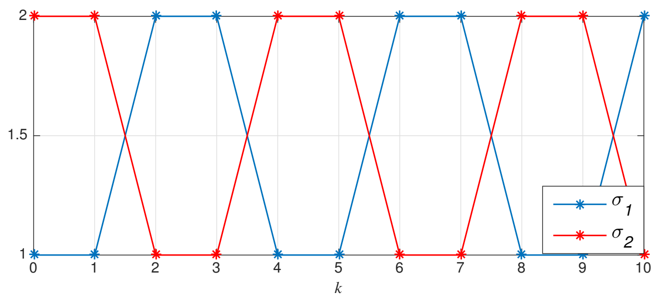

Fig. 6. Dwell time-based switching signals.

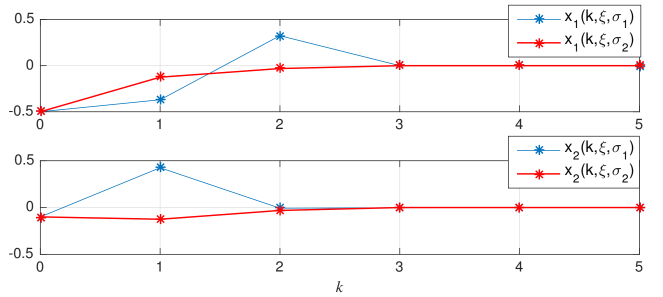

Fig. 7. State trajectories of system 8 under the dwell time-based switching and $u(\cdot) \equiv 0$.

$$\sigma_1(k) = \begin{cases} 1 & \forall k \in \{4n, 4n+1\}, n \in \mathbb{N}, \\ 2 & \forall k \in \{4n+2, 4n+3\}, n \in \mathbb{N}, \end{cases}$$

$$\sigma_2(k) = \begin{cases} 2 & \forall k \in \{4n, 4n+1\}, n \in \mathbb{N}, \\ 1 & \forall k \in \{4n+2, 4n+3\}, n \in \mathbb{N}. \end{cases}$$

The switching signals are depicted by Fig. 6. Choose $V_1(x) = 0.25x_1^2 + x_2^2$ and $V_2(x) = 0.1x_1^2 + 0.04x_2^2$ as the corresponding Lyapunov functions of the systems. By calculating V_j's at different instants for two demonstrated switched policies, we have

$$V_1(x(4, \xi, \sigma_1)) - V_1(\xi) = -0.98\xi_1^2 - 0.39\xi_2^2,$$
$$V_2(x(6, \xi, \sigma_1)) - V_2(x(2, \xi, \sigma_1)) = -0.09\xi_1^2 - 0.38\xi_2^2,$$
$$V_2(x(4, \xi, \sigma_2)) - V_2(\xi) = -2.46\xi_1^2 - 9.84\xi_2^2,$$
$$V_1(x(6, \xi, \sigma_2)) - V_1(x(2, \xi, \sigma_2)) = -0.41\xi_1^2 - 1.64\xi_2^2.$$

Therefore, the system is dwell time-based stable. The state trajectories and evolution of Lyapunov functions V_σ with $M = 4$ are, respectively, illustrated by Figs. 7 and 8 for initial condition $\xi = (-0.5, -0.1)$. Fig. 9 shows the state trajectories for $u(\cdot) = (0.3, 0.3)$.

5. CONCLUSIONS

We have provided Lyapunov characterizations of ωISS property for a nonlinear switched discrete-time system via finite-step Lyapunov functions. We have considered both arbitrary and constrained switching signals and non-ISS systems in the family. We have applied our results to linear NCSs with periodic scheduling protocols, where the stability conditions have been formulated as linear matrix inequalities. Numerical simulations have been presented to further illustrate the effectiveness of our results.

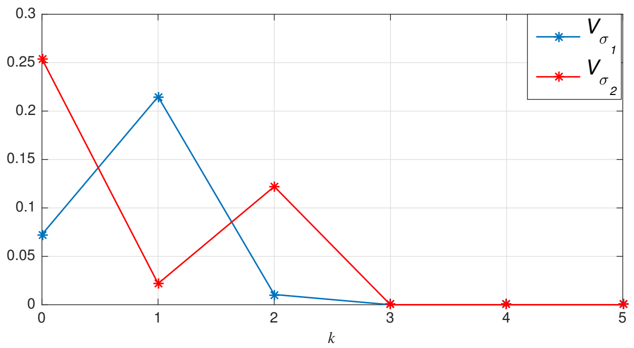

Fig. 8. Evolution of Lyapunov functions for system (8) under the dwell time-based switching and $u(\cdot) \equiv 0$.

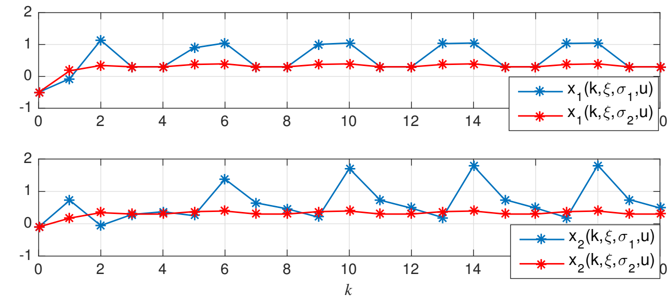

Fig. 9. State trajectories of system (12) under the dwell time-based switching and input $u(\cdot) = (0.3, 0.3)$.

In view of (Noroozi et al., 2018a), our results can be leveraged to develop non-conservative dissipativity and small-gain theorems for large-scale switched systems. Extensions to stochastic systems are also expected in lines with (Noroozi et al., 2019).

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