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Time-Varying Realization for Arbitrary Functions ⋆

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Abstract: We solve an extension of the inverse problem: Given a function, x, which differential equation does it solve. This extends the well known solution for Bohl functions, where an LTI-ODE is the solution. We first show that for functions analytic in an open interval, a regular time-variant differential polynomial annihilates the given function. Its order is determined by the highest multiplicity of a real zero of the given function. This is then extended to a class of functions that are real-analytic in $\mathbb{R}$, and further to a class of meromorphic functions, but without real poles. The solution is based on the theory of entire functions, for which essential notions are summarized. Applications in reduced-modeling and computer algebra are mentioned.

Keywords: ODE-modeling, linear systems, time-varying systems, time-series modeling, model reduction, non-parametric methods

1. INTRODUCTION

A classic problem in the theory of linear systems is the one of obtaining a state space realization of a given input-output behavior of the system. The impulse response of the system characterizes this input-output behavior. The impulse response of a linear time-invariant system (LTI) described by a finite order linear LTI ordinary differential equation (ODE), consists of a finite sum of (complex) exponentials multiplied by polynomials in t. Such a function is known as a Bohl function (Trentelman and Stoorvogel [2002]). Zeilberger calls these C-finite. In the real case, a Bohl function is a finite sum of products of polynomials, real exponentials and sines and cosines. Conversely, any Bohl function is the solution to a homogeneous linear time-invariant ODE for specific initial conditions. Or, equivalently, it is the impulse response of some nonhomogeneous linear time-invariant equation $a(\mathbf{D})y = b(\mathbf{D})u$, (where $\mathbf{D}$ denotes the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$), with $\deg b < \deg a$. An inverse problem (determining the annihilator) is associated with the previous: Given any Bohl function $x(t)$:

$$x(t) = \sum_{i=1}^{\mu} p_i(t)\, \mathrm{e}^{\lambda_i t}, \text{ such that } i \neq j \Rightarrow \lambda_i \neq \lambda_j,$$

with $p_i \in \mathbb{C}[t]$, $\lambda_i \in \mathbb{C}$, $\deg p_i = m_i$, determine the polynomials $a \in \mathbb{C}[s]$ such that x is annihilated by $a(\mathbf{D})$. It is well known that this inverse problem does not have a unique answer. If $a(\mathbf{D})$ is a solution, i.e., $a(\mathbf{D})x = 0$, then so is $c(\mathbf{D})a(\mathbf{D})$ for any polynomial $c(\cdot)$. In the ring of Bohl functions, the inverse problem specifies an ideal. However, the *monic solution of least degree* is unique. Let's denote this by $a_0(\mathbf{D})$. It follows readily that, for x as given above,

$$a_0(\mathbf{D}) = \prod_{i=1}^{\mu} (\mathbf{D} - \lambda_i)^{m_i+1}, \quad \deg a_0 = \sum_{i=1}^{\mu} (m_i + 1).$$

This paper explores how this result changes if we allow more general smooth functions, and look for *time-variant* linear ordinary differential operators that annihilate the given smooth function. Of course, the ODE should be restricted to have smooth coefficients. How smooth? Let us formulate the general inverse problem as follows:

The Problem
Given a smooth function, $x(t)$, find a linear time-variant monic differential operator,

$$a(t, \mathbf{D}) = \mathbf{D}^m + a_1(t)\mathbf{D}^{n-1} + \cdots + a_{n-1}\mathbf{D} + a_n(t),$$

with $a_i(t) \in C(\mathbb{R}, \mathbb{R})$ such that

$$a(t, \mathbf{D})x \equiv 0, \quad t \in (\alpha, \beta).$$

Define the operation of *multiplication by the independent variable* by $\mathbf{Q}$. The differential operators of the above form are then an extension of the elements from the noncommutative Weyl algebra generated by $\mathbf{Q}$ and $\mathbf{D}$, where the generators satisfy the Heisenberg commutation rule $\mathbf{DQ} - \mathbf{QD} = I$. Our idea of "realization" stems then from the fact that any continuous function can be approximated in any finite interval by a polynomial. Hence an approximate representation of a given smooth x is given by a polynomial approximation of the time-variant coefficients of the annihilating differential polynomial in the Weyl algebra, together with its initial conditions. This also means that we seek holonomic approximations of a given function. A holonomic function is one that is annihilated by a polynomial in $\mathbf{Q}$ and $\mathbf{D}$.

$$x \in \ker \mathbb{R}[\mathbf{Q}, \mathbf{D}] \quad \Leftrightarrow \quad x \text{ holonomic}.$$

The set of holonomic functions (a.k.a. $\mathbf{D}$-finite) form a ring, and are closed under integration (Zeilberger [1990]), which makes them amenable for computer algebra. In this work, we shall also allow the coefficients of the differential polynomials to be rational functions, or power series. See also Verriest [1993] and Zerz [2006].

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The problem sketched above is explored in Section 2, and completely solved for functions that are analytic in a bounded interval in Section 3. Section 4 gives an extension to infinite intervals. This requires the ideas from the theory of Weierstrass (and Hadamard) representations.

2. TIME-VARIANT ODE

In this section we answer the following questions:
1. Can one do better (in terms of degree) for Bohl functions if one allows time-varying coefficients $a_i(t)$?
2. Can one find $a(t, \mathbf{D})$ if x is non-Bohl?
3. What is the minimal order of $a(t, \mathbf{D})$? We will see that the answers to question 1 and 2 are affirmative. We then proceed with the constructive answer to question 3.

2.1 Nonvanishing functions

Given $x \in C^1((\alpha, \beta), \mathbb{R})$ such that $x(t) \neq 0$ in (α, β), then

$$a(t, \mathbf{D}) = \mathbf{D} - \frac{\dot{x}}{x} \tag{1}$$

which involves the logarithmic derivative, is well-defined and solves the problem since

$$\left(\mathbf{D} - \frac{\dot{x}}{x}\right) x = \dot{x} - \dot{x} = 0.$$

Example 1: The specific example below for a non-vanishing but smooth non-Bohl function is illustrative. It also holds for all $t \in \mathbb{R}$.

$$x(t) = \frac{1}{t^2+1} \quad \Rightarrow \quad a(t, \mathbf{D}) = \mathbf{D} + \frac{2t}{t^2+1}.$$

Example 2: Consider the continuous non-Bohl function $x(t) = a > 0$ for $t < 0$, $x(t) = a + t$ for $t > 0$. This is continuous but not differentiable at 0. If $a > 0$, it does not vanish. The corresponding annihilating differential form

$$\mathbf{D} - \frac{H(t)}{a + tH(t)} = \mathbf{D} - \frac{H(t)}{a+t},$$

where H is the Heaviside unit-step function, is not smooth. Moreover it is not uniformly bounded as the parameter $a \to 0$. Clearly, smoothness of x may be an issue.

2.2 Isolated Single Zero

If x vanishes at $t_0 \in (\alpha, \beta)$, the first order operator $a(t, \mathbf{D})$ in (1) is not well-defined. Restricted to (α, t_0), all solutions of $a(t, \mathbf{D})y = 0$ are of the form $Ax(t), A \in \mathbb{C}$. Likewise, in (t_0, β), the solutions are of the form $Bx(t)$, $B \in \mathbb{C}$. It is not necessary to assume $A = B$. Let's illustrate:

Example 3: Let $x(t) = t - 1$ so that (1) yields

$$a(t, \mathbf{D}) = \left(\mathbf{D} - \frac{1}{t-1}\right).$$

The corresponding Cauchy problem has (weak) solutions (with $H(\cdot)$ the Heaviside unit-step function):

$$[A + (B - A)H(t-1)](t-1).$$

Since two parameters pin down a particular solution, this alludes to a *higher dimensionality.* With the identity $t\delta(t) = 0$, it is readily verified that the above functions are all solutions to the non-monic differential operator $(t-1)\mathbf{D} - 1$. A non-monic ODE where the highest order coefficient can vanish is also known as a *singular ODE.*

Example 4: Consider the function $x(t) = t + t^n H(t)$, for $n \geq 2$. It has a single zero at $t = 0$, and is $(n-1)$ times differentiable. Here we obtain the singular differential polynomial

$$a(t, \mathbf{D}) = t\mathbf{D} - \frac{1 + nt^{n-1}H(t)}{1 + t^{n-1}H(t)} = t\mathbf{D} - 1 - \frac{(n-1)t^{n-1}}{1+t^{n-1}}H(t).$$

The equation $a(t, \mathbf{D})y(t) = 0$ has the general solution $y(t) = At$ for $t < 0$, and $y(t) = Bt(1 + t^{n-1})$ for $t > 0$. However, unless $A = B$, the solution is not differentiable. Solutions with $A = B$ are only $n - 1$ times differentiable. For $n = 2$, its second derivative, $\ddot{y}$, contains the singular term $(B - A)\delta(t)$ and a jump $2BH(t)$.
The next section explores this singular case in more detail. These examples also illustrate that more than finite smoothness in x will be needed to obtain nice results.

3. ANALYTIC FUNCTIONS IN (α, β)

The examples in Section 2 suggested an idea on how to generate a singular time-variant first order ODE in case the given function x has a single zero in some interval (α, β). The first main result proven here relates to arbitrary nonzero analytic $x(t)$. By the principle of permanence, this implies that the zeros of x are isolated (no cluster points). Consequently, a real analytic function has finitely many zeros in any bounded interval.

Definition 1: The linear differential polynomial

$$a(t, \mathbf{D}) = \mathbf{D}^m + a_1(t)\mathbf{D}^{n-1} + \cdots + a_{n-1}\mathbf{D} + a_n(t),$$

is *regular* in (α, β) if it is monic ($a_0(t) \equiv 1$) and the coefficients $a_i(t)$ are *continuous* in (α, β).
It is well-known that if x solves $a(t, \mathbf{D})x = 0$, then it and its first n derivatives are all *continuous.* We shall now restrict the given function x to be *real analytic.* The set of real analytic functions on an interval (α, β) is denoted by $C^\omega((\alpha, \beta), \mathbb{R})$. Recall that x is real analytic on (α, β) iff x can be extended to a complex analytic (a.k.a holomorphic) function on an open set $\mathcal{D} \subset \mathbb{C}$, which contains the real interval (α, β). This is not true in general for real analytic functions defined on all of $\mathbb{R}$. The function x in Example 1 gives a counterexample. Its Taylor series about 0 does not converge for $|x| > 1$.

Theorem 1. If $x \in C^\omega((\alpha, \beta), \mathbb{R})$ then there exists a regular second order linear differential operator $a(t, \mathbf{D})$ such that $a(t, \mathbf{D})x = 0$, iff x has no repeated real zeros in (α, β).

Proof. Let the zeros of x in (α, β) be $\alpha < t_1 < t_2 < \cdots < t_n < \beta$, and factor $x(t)$ as

$$x(t) = \underbrace{(t - t_1)(t - t_2)\cdots(t - t_n)}_{=p(t)} x_r(t),$$

where $x_r(t)$ is twice differentiable and has a fixed sign in (α, β). The (weak) first-order differential form

$$p(t)\mathbf{D} - \frac{p(t)\dot{x}_r(t)}{x_r(t)} - \dot{p}(t),$$

annihilates the given function x but is not regular. Operate on the left with the first-order differential polynomial $\mathbf{D} - \eta(t)$, where $\eta \in C((\alpha, \beta), \mathbb{R})$, to get

$$(\mathbf{D}-\eta)\left(p\mathbf{D}-\frac{p\dot{x}_r}{x_r}-\dot{p}\right)$$
$$=p\,\mathbf{D}^2-p\left(\eta+\frac{\dot{x}_r}{x_r}\right)\mathbf{D}+p\left(\frac{\dot{x}_r}{x_r}\eta-\left(\frac{\dot{x}_r}{x_r}\right)'\right)+$$
$$+\left[\dot{p}\left(\eta-\frac{\dot{x}_r}{x_r}\right)-\ddot{p}\right].$$

If the term in $[\,\cdot\,]$ were a multiple of p, we could cast out the "p" from the above differential operator to obtain a monic one. Thus motivated, we ask for the solvability for η and k in

$$\dot{p}\left(\eta-\frac{\dot{x}_r}{x_r}\right)-\ddot{p}\overset{?}{=}kp.$$

Note that by the Gauss-Lucas theorem, p and $\dot{p}$ have interlaced roots. This implies that p and $\dot{p}$ are coprime polynomials. By Bezout's theorem, polynomials q_0 and k_0 exists in $\mathbb{R}[t]$ such that

$$q_0(t)\dot{p}(t)-k_0(t)p(t)=1. \tag{2}$$

A constructive solution method is given in Verriest [2020]. The Diophantine equation

$$q(t)\dot{p}(t)-k(t)p(t)=\ddot{p}(t) \tag{3}$$

is then solved by the polynomials

$$q(t)=q_0(t)\ddot{p}(t),\quad k(t)=k_0(t)\ddot{p}(t).$$

Finally, letting

$$\eta(t)=q(t)+\frac{\dot{x}_r(t)}{x_r(t)},$$

the regular second order differential polynomial, $a(t,\mathbf{D})$ annihilating $x(t)=p(t)x_r(t)$ is given by

$$\mathbf{D}^2-\left(q_0\ddot{p}+2\frac{\dot{x}_r}{x_r}\right)\mathbf{D}+\left(\left(\frac{\dot{x}_r}{x_r}\right)q_0\ddot{p}+\left(\frac{\dot{x}_r}{x_r}\right)^2-\left(\frac{\dot{x}_r}{x_r}\right)'+k_0\ddot{p}\right).$$

□

This may explain why the theory of linear time-variant ODE's mostly centers around second-order equations, culminating in the Sturm-Liouville problem.

If x possesses real zeros of higher multiplicity, one can make use of the following lemma:

Definition 2: A smooth function is called *signed* in (α,β) if it is nowhere vanishing in the interval (α,β). Consequently, it has a fixed sign in (α,β).

Lemma 2. Any $x\in C^\omega((\alpha,\beta),\mathbb{R})$ can be factored as px_b, where p is a monic polynomial with only real zeros in (α,β) and x_b is differentiable and signed in (α,β) Then

$$\left(\mathbf{D}-\frac{\dot{x}_b}{x_b}\right)px_b=\dot{p}x_b.$$

Proof. Direct verification.

The factorization alluded to in Lemma 2 is not unique. The function $x(t)=t(t^2+1)\exp(-t)$ factors in $p_1(t)=t$ and $x_{b1}(t)=(t^2+1)\exp(-t)$, or $p_2(t)=t(t^2+1)$ and $x_{b2}(t)=\exp(-t)$ in the interval $(-1,1)$. This prompts us to define a *canonical factorization*:

The factorization of $x\in C^\omega((\alpha,\beta),\mathbb{R})$ as $x=px_b$ is canonical in (α,β) if p is monic and its extension over $\mathbb{C}$ has no roots other than those in the real interval (α,β). It follows that the cofactor x_b is signed in (α,β).

Note that by the Gauss-Lucas theorem, all roots of $\dot{p}$ lie on the real axis, and $\deg\dot{p}=\deg p-1$. In addition, if p has a root of multiplicity m at $t=t^*>1$, then $\dot{p}$ has a root at t^* of multiplicity $m-1$. This leads to:

Theorem 3. Let $x=px_b$ be a canonical factorization of x. If the highest multiplicity of a root is m, then a regular differential polynomial annihilating x is given by

$$a(t,\mathbf{D})=(\mathbf{D}^2+a_1\mathbf{D}+a_2)\left(\mathbf{D}-\frac{\dot{x}_b}{x_b}\right)^{m-1}. \tag{4}$$

Proof.

$$\left(\mathbf{D}-\frac{\dot{x}_b}{x_b}\right)^{m-1}px_b=qx_b$$

where $q=p^{(m-1)}$ has only roots on the real axis with multiplicity one. By Theorem 1 smooth functions a_1 and a_2 exist such that qx_b is annihilated by $a(t,\mathbf{D})$. □

Corollary 4. If the highest multiplicity of a zero of $x(t)\in C^\omega((\alpha,\beta),\mathbb{R})$ is m, then x is annihilated by a regular differential polynoimal of degree $m+1$.

In Example 4, $x(t)=t+t^nH(t)$ has a factorization with $p(t)=t$ and $x_b(t)=1+t^{n-1}H(t)>0$, but is not analytic. For $n=2$, a regular second order differential polynomial cannot exist since x fails to be twice differentiable.

4. EXTENSIONS: FROM (α,β) TO $\mathbb{R}$

If x has a finite number of real zeros the previous extends directly from (α,β) to $\mathbb{R}$. But when x has infinitely many zeros, the factor $p(t)$ does not make sense. Still assuming that x is analytic, the zeros cannot cluster and consequently, if $\{t_i\}$ is the sequence of zeros, $|t_n|\to\infty$. The infinite product $\prod_{n=1}^\infty\left(1-\frac{1}{t_n}\right)$ converges if $\sum_{n=1}^\infty\frac{1}{|t_n|}$ converges. It seems that the first step in the generalization of Theorem 1 will be to restrict the function x to an entire function, for which a rich theory exists. See Boas [1954].

4.1 Entire Functions

These are complex-valued functions that are analytic in all of $\mathbb{C}$. Of course, we should maintain that $x:\mathbb{R}\to\mathbb{R}$, i.e., we will only consider *real-entire* functions.

The rate of growth of an entire function is closely tied to the distribution of its zeros:

Definition 3: An entire function f has *order* ρ if

$$\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}=\rho, \tag{5}$$

where $M(r)$ is the maximum modulus of $f(z)$ for $|z|=r$.

Definition 4: If $\{z_n\}$ are the zeros of f, then the *convergence exponent* of its zeros is the infimum of positive α for which $\sum_{n=1}^\infty|z_n|^{-\alpha}$ converges.

Definition 5: The *genus of the set of zeros*, p, is the integer such that $p+1$ is the smallest *integer* for which the sum in Definition 4 converges.

For $p=1,2,\ldots$, define Weierstrass's elementary factors:

$$E_p(z)=(1-z)\exp\left(z+\frac{z^2}{2}+\cdots\frac{z^p}{p}\right),\ E_0(z)=(1-z).$$

The elementary factors are close to 1 if $|z|<1$, and large p, although $E_p(1)=0$. If $\{z_n\}$ is a sequence of complex numbers, $z_n \neq 0$ and $|z_n| \to \infty$, and $\{p_n\}$ is a sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} |z_n|^{1+p_n} < \infty,$$

then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

defines an entire function which vanishes at every z_n, and which has no other zeros in $\mathbb{C}$. If α appears m times in the sequence, then P has a zero of order m at α (Rudin [1974]). This theorem, due to Weierstrass, is so general that it is of little use. Hadamard's theorem is more appropriate here.

If $\{z_n\}_{n=1}^{\infty}$ is an infinite sequence of complex numbers, ordered by increasing modulus, of genus p (Definition 5), then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_n}\right) \tag{6}$$

is called a *canonical product* of genus p. Hadamard's factorization theorem states that an entire function of finite order, ρ, with m-fold zero at 0, factors as

$$f(z) = z^m \,\mathrm{e}^{Q(z)}\, P(z) \tag{7}$$

where $P(z)$ is the canonical product (of genus p) formed of the zeros (other than $z = 0$) of f, and $Q(z)$ is a polynomial of degree $q < \rho$.

Definition 7 The *genus of the function* f is $\max(p, q)$.

4.2 Application to the function realization problem

Let us see how we can extend Theorem 1. Let x be a real entire function (meaning that $x : \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$ and $x : \mathbb{R} \to \mathbb{R}$.) Let x have infinitely many zeros on the real axis. These real zeros do not contain a cluster point, so that they are all separated. Finally let us restrict the problem to the case where these real zeros all have multiplicity one. Let $Z(x) \subset \mathbb{C}$ denote the zero-set for x, extended as a function $x : \mathbb{C} \to \mathbb{C}$, and let $Z_{\mathbb{R}} = Z(x) \cap \mathbb{R}$ be the set of real zeros and $Z_{\mathbb{C}}$ its complement in $Z(x)$. By Hadamard's factorization theorem for entire functions of finite order (Boas [1954]),

$$f(z) = z^{m_0} \,\mathrm{e}^{Q(z)}\, P(z) \tag{8}$$

where $P(z)$ is the canonical product (of genus p) formed of the zeros (other than $z = 0$ of f. If f has a (single) zero at the origin, $m_0 = 1$, while if $0 \notin Z_{\mathbb{R}}$, then $m_0 = 0$, and $Q(z)$ is a polynomial of degree $p < \rho$. Restricted to $\mathbb{R}$, this may be factored further as

$$x(t) = \underbrace{t^{m_0} \prod_{t_n \in Z_{\mathbb{R}}} E_{p_{\mathbb{R}}}\left(\frac{t}{t_n}\right)}_{=\Pi(t)} \underbrace{\prod_{z_m \in Z_{\mathcal{C}}} E_{p_{\mathbb{C}}}\left(\frac{t}{z_m}\right) y(t)}_{=x_e(t)} \tag{9}$$

where y is entire without zeros. Consequently, x_e, having no zeros in $\mathbb{R}$, is signed. Proceeding as in Theorem 1, we try to find a smooth function η such that

$$(\mathbf{D} - \eta)\left(\Pi\mathbf{D} - \dot{\Pi} - \Pi\frac{\dot{x}_e}{x_e}\right),$$

is regular. This subproblem requires the solution of the Bezout equation

$$\left(\eta - \frac{\dot{x}_e}{x_e}\right)\dot{\Pi} - k\Pi = \ddot{\Pi}.$$

But can this be solved? Consider the subclass of real entire functions of the form

$$f(z) = C\,\mathrm{e}^{-az^2+bz}\, z^m \prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right)\mathrm{e}^{z/z_k},$$

with $a \geq 0, b \in \mathbb{R}, C \in \mathbb{R}, \sum_k \frac{1}{|z_k|^2} < \infty$, $z_k \in \mathbb{C} \setminus \{0\}$, $|\mathrm{Im}\, z_n| < \infty$. Functions in this class have only real zeros. Examples are $\sin(z), \cos(z), \exp(z), \exp(-z)$, and $\exp(-z^2)$. Functions in this class have genus less than two. This class is known as the *Laguerre-Pólya class*, denoted LP. It is known that functions in LP are uniform limits of real polynomials with only real coefficients. We can now make the proper extension for real analytic x:

Theorem 5. If x, a real entire function, has a factorization $x = x_e\Pi$, with $\Pi \in \mathrm{LP}$ and x_e has no real zeros, then $x(t)$ satisfies a regular second order ODE in all of $\mathbb{R}$ if all zeros of Π have multiplicity one.

Proof. The ring of entire functions is a Bezout domain, but not a PID. However, every *finitely* generated ideal is principal. Thus a solution exists since Π and $\dot{\Pi}$ are relatively prime. The proof follows as in Theorem 1. □

Remark: There is no division algorithm to compute η and k for entire functions. However, we can make use of the following result, due to Laguerre: "If $x \in \mathrm{LP}$ with only infinitely many single real zeros, then $\dot{x}$ is well-defined and also has real zeros which are interlaced with the zeros of x." A solution set (a, b) to the Bezout equation $ax + b\dot{x} = 1$ is given for arbitrary p by

$$a = \frac{x}{x^2 + \dot{x}^2} + p\dot{x}, \quad b = \frac{\dot{x}}{x^2 + \dot{x}^2} - px.$$

Example 5: Let $x(t) = \sin t$ and consider

$$\begin{aligned}&(\mathbf{D} - \eta(t))(\sin t\mathbf{D} - \cos t)\\ &\quad = \sin t\mathbf{D}^2 - \eta(t)\sin t\mathbf{D} + \eta(t)\cos t + \sin t.\end{aligned}$$

The obvious choice is $\eta(t) = 0$, leading - not surprisingly - to the monic operator $\mathbf{D}^2 + 1$. For $\eta(t) = -\cos t \sin t$, the resulting differential polynomial is $\mathbf{D}^2 + \cos t \sin t\mathbf{D} + \sin^2 t$.

Example 6: Consider the Bessel function, $x(t) = J_0(t)$, and recall that $\dot{J}_0(t) = -J_1(t)$ and by Bessel's ODE: $\ddot{J}_0(t) = -J_0(t) + \frac{J_1(t)}{t}$. Here

$$\begin{aligned}&(\mathbf{D} - \eta(t))(J_0(t)\mathbf{D} + J_1(t))\\ &\quad = J_0(t)\mathbf{D}^2 - \eta(t)J_0(t)\mathbf{D} + \dot{J}_1(t) - \eta(t)J_1(t).\end{aligned}$$

Solve the Bezout equation, $\eta(t)J_1(t) + k(t)J_0(t) = \dot{J}_1(t)$ to get

$$\begin{aligned}\eta(t) &= \left(\frac{J_1(t)}{J_0^2(t) + J_1^2(t)} + p(t)J_0(t)\right)\dot{J}_1(t),\\ k(t) &= \left(\frac{J_0(t)}{J_0^2(t) + J_1^2(t)} - p(t)J_1(t)\right)\dot{J}_1(t).\end{aligned}$$

For $p \equiv 0$ we get

$$a(t, \mathbf{D}) = \mathbf{D}^2 - \frac{J_1 \dot{J}_1}{J_0^2 + J_1^2}\mathbf{D} + \frac{J_0 \dot{J}_1}{J_0^2 + J_1^2}.$$

We do not retrieve the known Bessel ODE, but there is a good reason for that. Bessel's ODE for J_0 has a singular coefficient $1/t$, which is not allowed if we consider a domain containing 0.

Example 7: For $x(t) = \sin^2 t$, all the zeros are double zeros. As before,

$$\frac{\dot{x}(t)}{x(t)} = 2\cos t.$$

But $\mathbf{D}\sin t^2 = 2\sin t \cos t$ only has single zeros, and

$$\begin{aligned}&(\mathbf{D} - \eta(t))(\sin t \cos t\mathbf{D} - \cos^2 t + \sin^2 t)\\ &= \sin t \cos t\mathbf{D}^2 - \eta(t)(\cos t \sin t))\mathbf{D}\\ &\quad + \eta(t)(\cos^2 t - \sin^2 t) + 4\cos t \sin t.\end{aligned}$$

An obvious choice is $\eta = 0$, giving the combined third order differential polynomial $(\mathbf{D}^2 + 4)\mathbf{D}$. Time-invariance was to be expected since $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ is Bohl.

Example 8: Let now $x(t) = \sin t^2$. This has a double root at $t = 0$, all other roots $(\pm k\sqrt{\pi})$ for $k = 1, 2, \ldots$ being simple. In an interval not containing 0, we expect a second order ODE. Indeed, we can find

$$\mathbf{D}^2 - \frac{1}{t}\mathbf{D} + 4t^2.$$

In order to find a differential polynomial that is valid in all of $\mathbb{R}$, consider

$$\begin{aligned}&(\mathbf{D} - \eta(t))(t\cos t\mathbf{D}^2 - \mathbf{D} + 4t^3)\\ &= t\mathbf{D}^2 - \eta(t)\mathbf{D}^2 - \eta(t)t\mathbf{D}^2 + (\eta(t) + 4t^3)\mathbf{D} - 12t^3.\end{aligned}$$

Letting $\eta(t) = t\eta_0(t)$, we get the monic differential polynomials

$$\mathbf{D}^3 - t^2\eta_0(t)\mathbf{D}^2 + 4(\eta_0(t) + t^2)\mathbf{D} - 12t.$$

Its simplest form for $\eta_0(t) \equiv 0$, gives $\mathbf{D}^3 + 4t^2\mathbf{D} + 12t$. These examples also illustrate further that it may not be necessary to explicitly use Hadamard's factorization.

4.3 Meromorphic extension

A function f is meromorphic in an open set $\mathcal{D}$ if there exists a set $A \in \mathcal{D}$ such that (See Rudin [1974])

i) A has no limit point in $\mathcal{D}$
ii) f is analyic in $\mathcal{D} \setminus A$.
iii f has a pole at each point of A.

Every meromorphic function in open $\mathcal{D}$ is a quotient of two functions that are holomorphic (analytic) in $\mathcal{D}$ (Rudin [1974]). Theorem 5 can now be extended:

Theorem 6. If $x(t)$ can be factored as $x = x_m\Pi$ where x_m is a real meromorphic function, without real poles, and $\Pi \in LP$, then x satisfies a regular second order time-variant ODE if all zeros of Π have multiplicity one.

Proof. Direct and omitted. □

Example 8: Let now $x(t) = \frac{\sin \pi t}{1+t^2}$. This function has a single pole at $\pm j$ and simple zeros at all integers. The computation yields

$$(\mathbf{D} - \eta)\left(\sin t\mathbf{D} - \left(\pi\cos\pi t - \frac{2t}{1+t^2}\sin\pi t\right)\right),$$

and the choice $\eta(t) = -\frac{2t}{1+t^2}$ yields the regular annihilator

$$\mathbf{D}^2 + \frac{4t}{1+t^2}\mathbf{D} + \left(\pi^2 + \frac{2}{1+t^2}\right).$$

Similar extensions of Theorem 3 and Corollary 4 are also direct and omitted.

Finally, the problem is also solvable using an idea based on the *least common multiple* in the Weyl algebra: Suppose that $x(t)$ is partitioned as $x(t) = x_1(t) + x_2(t) + \cdots + x_n(t)$, where none of the x_i have real zeros. (Example: $t^2 - 1 = (t^2 + 1) - 2$). Then the $a_i(t) = \frac{\dot{x}_i(t)}{x_i(t)}$ are regular functions, and any linear combination over $\mathbb{R}$ will be nulled by their least common multiple:

$$\mathrm{lcm}\{(\mathbf{D} - a_1(t)), (\mathbf{D} - a_1(t)) \ldots, (\mathbf{D} - a_n(t))\}.$$

However, computation of the lcm in the noncommutative Weyl algebra is a nontrivial problem.

4.4 Higher order LTI case

So far we only looked at scalar functions x. We discuss next a related problem of modeling the projection of a solution to an LTI-system by a lower dimensional time-variant one. Thus we seek to find annihilators for Bohl functions, but represented by their LTI-differential annihilator.

Problem: Consider the LTI autonomous system $\dot{x} = Ax$, evolving in $\mathbb{R}^n$. Consider in $\mathbb{R}^n$ the subspace $\mathcal{S}$. Assume that the initial state is restricted to lie in $\mathcal{S}$. It is desired to describe the system as a time-variant one *evolving* in $\mathcal{S}$. Let the projection operation on the subspace $\mathcal{S}$ be $y = Px$. By assumption, the initial state in $\mathcal{S}$ is embedded in $\mathcal{R}^n$ by $x_0 = P^\top y_0$. For simplicity, fix on the case $P = [I_r, 0]$. The solution of the autonomous system is

$$y(t) = P\,\mathrm{e}^{At}\,P^\top y_0, \quad y_0 \in \mathcal{S} = \mathbb{R}^r.$$

Thus also for all τ

$$y(t) = P\,\mathrm{e}^{At}\,P^\top(P\,\mathrm{e}^{A\tau}\,P^\top)^{-1}y(\tau) = \Phi(t,\tau)y(\tau),$$

provided $P\,\mathrm{e}^{A\tau}\,P^\top$ has full rank for all τ. The lower order model (of dimension $r = \dim\mathcal{S}$) must satisfy $\dot{y}(t) = \mathcal{A}(t)y(t)$ for some time-variant matrix ⊣ and initial condition $y(0) = y_0$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)\Phi(t,\tau)y(\tau),$$

from which it follows that

$$\mathcal{A}(t) = PA\,\mathrm{e}^{At}\,P^\top(P\,\mathrm{e}^{At}\,P^\top)^{-1}.$$

For instance, if the solution starting in a half space remains in this half-space, then the projection to the one-dimensional system along the normal has no zero crossings, and

$$\mathcal{A}(t) = \frac{[A\,\mathrm{e}^{At}]_{11}}{[e^{At}]_{11}}.$$

This yields time variant-reduced models, but only for the undriven system. Reduced input-output behavior requires the hidden-variable characterization of the full order model.

More generally, consider the homogeneous system $\dot{x} = Ax$, with partial state $y(t) = Cx(t)$ of dimension n and $r < n$ respectively. Assume then that $y(0) = y_0$, and $x(0) = By_0$, so that for all y_0: $CBy_0 = y_0$, i.e., $CB = I_r$. It follows that

$$\dot{y}(t) = CA\,\mathrm{e}^{At}\,By_0.$$

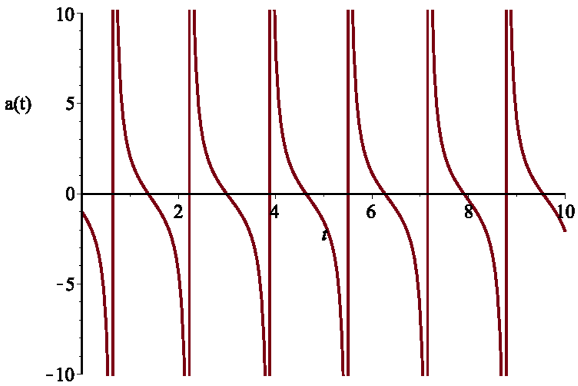

Fig. 1. Coefficient for the first order differential operator associated with the third order LTI system (Ex. 9)

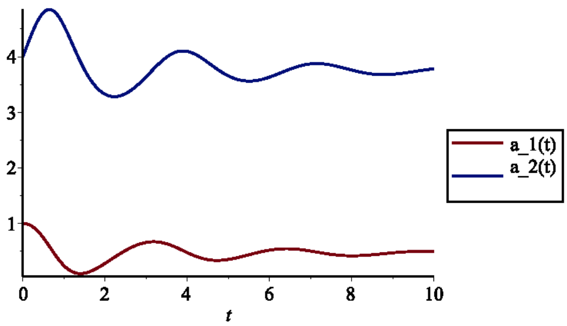

Fig. 2. Coefficients for the second order differential operator associated with the third order LTI system

On the other hand, expressing the r-th order system as $\dot{y}(t) = F(t)y(t)$, we can identify

$$CA\,\mathrm{e}^{At}\,By_0 = F(t)C\,\mathrm{e}^{At}\,By_0$$

so that

$$\mathcal{A}(t) = CA\,\mathrm{e}^{At}\,B(C\,\mathrm{e}^{At}\,B)^{-1}.$$

Successive differentiation yields the representation

$$y^{(k)} = (\overleftarrow{\mathbf{D}} + \mathcal{A})^k y, \tag{10}$$

where $\overleftarrow{\mathbf{D}}$ is the derivative operator *acting to anything on the left*: (i.e, if $x(t)$ and $y(t)$ are arbitrary differentiable functions, then $x(t)\overleftarrow{\mathbf{D}}y(t) = \dot{x}(t)y(t)$).
The vector ODE (10) generalizes the scalar ODE time-variant homogeneous ODE.
Example 9: Consider the third order system with A-matrix in the reachable canonical form

$$A = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Choosing the first state as the partial state of interest, a singular first order representation follows, the scalar $\mathcal{A}_1(t)$ is shown in figure 1. Choosing instead the fist and second state as the partial states yieds a second order time varying representation with the time-varying 2×2 dynamic matrix given in the (time-varying) reachable canonical form ,

$$\mathcal{A}_2(t) = \begin{bmatrix} -a_1(t) & -a_2(t) \\ 1 & 0 \end{bmatrix}.$$

with $a_1(t)$ and $a_2(t)$ shown in Figure 2.

5. CONCLUSION

We solved the inverse problem for obtaining differential annihilators for a class of analytic functions, and obtained the interesting property that any function in this class having only real zeros of multiplicity one can be represented as the solution to a regular time-variant ODE of order 2. In the more general case where the largest multiplicity of a real zero of the given function is m, a regular annihilator of order $m+1$ exists. This problem has potential applications in reduced order modeling of an LTI system from its impulse response, and in approximation of functions with a finite data set. Indeed, by representing x by its annihilator and initial conditions, one can then use polynomial approximations (in an open interval, by Weierstrass's theorem) of its time-varying coefficients. Hence a finite data set characterizes an approximation of the given function. It was also shown that analyticity was needed in this result. Counterexamples showed that a finite order regular annihilator may not exist if x is merely smooth. Finally, this paper shed some light onto the more obscure class of Laguerre-Pólya functions, and algebraic aspects of the class of holomorphic functions. In fact, recently renewed interest in entire functions have given impetus to the long standing Riemann conjecture (Griffin et al [2019]). In a companion paper (Verriest [2020]), an approximation to the deformed exponential, the unit solution to the scalar scale-delay functional differential equation (pantograph equation) is derived.

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